# PREFACE

The Second International Conference on Combinatorial Mathematics and Computing was held in Canberra, Australia, from August 24-28, 1987. It was jointly sponsored by the International Mathematical Union and the Combinatorial Mathematics Society of Australasia.

The first conference in the series was called the "International Conference on Combinatorial Theory", and was held in Canberra in 1977. The Proceedings of that conference were published as Volume 686 of Lecture Notes in Mathematics, published jointly by Springer-Verlag and the Australian Academy of Sciences.

At the second conference, we were fortunate to hear invited addresses by the following distinguished scholars:

Gregory Freiman, Israel
Tatsuro Ito, Japan
Richard Karp, USA
Curt Lindner, USA
Kevin Phelps, USA
Fred Piper, Great Britain
Robert Robinson, USA
Branca Vucetic, Australia
Zhang Ke Min, China.

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Of the Many people who assisted with the organization and running of the conference, I particularly wish to thank Bau Sheng, Richard Brent, Derek Holton, Chris Johnson, Joan McKay, Malcolm Newey, Cheryl Praeger, Ingrid Rinsma, and Anne Street. Thanks is also due to the members of the small army who chaired sessions or refereed papers.

Brendan McKay

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# ON THE POWER SEQUENCE OF BOOLEAN MATRICES

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A matrix A over the binary Boolean algebra $\{0,1\}$ is called a Boolean matrix. Let $B_n$ forms a finite semigroup with order $2^{n^2}$.

If $A \in B_n$, the behaviour of the sequence $A, A^2, A^3, \ldots$ depends on two parameters:

There exists a least positive integer k=k(A) such that $A^k=A^{k+t}$ for some t>0, and there exists a least positive integer p=p(A) such that $A^k=A^{k+p}$. We call the integer:

k=k(A)---the index of convergence of A,

p=p(A)---the period of oscillation of A.

There are another two parameters about this sequence.

A is primitive iff p=1, $A^l=J$. In this case, we call:

$\gamma(A)$=k(A)---the exponent of A.

The least positive integer h such that the number of 1's in $A^h$ is maximized in all powers of A. We call the integer:

h=index(A)---the index of maximum density of A.

Clearly, if A is primitive, then $\gamma(A)$=k(A)=index(A), But if A is imprimitive, k(A), index(A) are different parameters of A. For example:

$$A=\begin{bmatrix}0&0&1&0\\0&0&1&0\\0&0&0&1\\1&1&0&0\end{bmatrix},\quad A^2=\begin{bmatrix}0&0&0&1\\0&0&0&1\\1&1&0&0\\0&0&1&0\end{bmatrix},\quad A^3=\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad A^4=A.$$

So $k(A)=1$, $\text{index}(A)=3$, $p(A)=3$.

Boolean matrix A may arise from nonnegative matrix $A_o$ by replacing all positive entrices by 1, say pattern of $A_o$, or from digraph D, say associated digraph of A, which adjacency matrix is $A=(a_{ij})$. i.e. Let 1,2,...,n be the vertices of D, ij is an arc in D iff $a_{ij}=1$.

The properties of p(A) is well-known by the work of Rosenblatt in 1957. He get:

(1) If A is irreducible, then p(A)= the greatest common divisor of the distinct lengths of the cycles of the associated digraph D(A);

(2) If A is reducible the p(A)= the least common multiple of $p(A_1),\ldots, p(A_m)$, where $A_1,\ldots, A_m$ are the irreducible components of A.

So, the main interest about the powers of Boolean matrices will be the study of $\gamma(A)$, k(A) and index(A). There are three fundamental problems about those parameters:

(a) the upper bound;

(b) the exponent set or index set;

(c) the problem of (a) or (b) of especial matrix classes.

§1. $\gamma(A)$ ( the exponent of primitive Boolean matrices)

(a) the upper bound

In 1950, H. Wielandt (Math. Z. 52: 642--648) first stated the exact upper bound for $\gamma(A)$. That is:

$$\gamma(A) \leq W_n = (n-1)^2+1$$

And in 1967, A.L. Dulmage and N.S. Mendelsohn (Graph Theory and Theoretical Physics, edited F. Harary, Acad. Press, 167 --277) proved that if $A \in PBM(n)$ (the set of nxn primitive Boolean matrices), $\gamma(A)=W_n$ iff there is a permutation matrix P such that $PAP^{-1}=W(n)$,

where
$$W(n)=\begin{bmatrix} 0 & 1 & & & 0 \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

So, this problem is completely solved.

(b) the exponent set $E_n=\left\{\gamma(A) \mid A \in PBM(n)\right\}$

Clearly, $E_n \subseteq \left\{1,2,\ldots,W_n\right\}$. In 1964, A.L. Dulmage and N. S. Mendelsohn ( Illinois J. Math. 8, 642--656) discovered 'gap'. Each gap is a set S of consecutive integers below $W_n$ such that no $A \in PBM(n)$ has an exponent in S. In 1981, M. Lewin and Y. Vitek ( Illinois J. Math. 25(1), 87--98) found the genneral method for determining all gaps between $[\frac{1}{2}W_n]+1$ and $W_n$, further conjectured that there are no gaps below $[\frac{1}{2}W_n]+1$, where $[x]$ denotes the greatest integer $\leq x$.

Recently, we prove that:

' Lewin and Vitek conjecture is true except for n=11. Namely, there are no gaps below $[\frac{1}{2}W_n]+1$ except 48 when n=11'.

So, the problem of determining the exponent set is completed solved.

(c) the exponent set of the especial matrix classes

This problem isn't especial cases of (b). It has itself meaning and has especial difficulty.

There are a lot of works concerning this problem. For examples, in 1967, J.W. Moon and N.J. Pullman (JCT 3,1--9), for primitive n-tournament matrices $T=[t_{ij}]$ (i.e. $t_{ij}=0$, $t_{ij}+t_{ji}=1$, $1\leq i,j\leq n, i\neq j$ ).

$E_n(T)=\{3,4,\ldots,n+2\}$ $(n\geq 6)$

In 1986, Shao Jia Yu (Scientia Sinica A9, 931--939), for symmetric primitive matrices S

$E_n(S)=\{1, 2, \ldots, 2n-2\}/D_o$ .

where $D_o$= all of the odd numbers in $\{n, n+1, \ldots, 2n-2\}$.

In 1974, M. Lewin (Math. Z. 137, 21--30), for Doubly stochastic matrices $D=[d_{ij}]$ (i.e. $\sum_i d_{ij}=1$, $\sum_j d_{ij}=1$ ). It's upper bound is

$$\begin{cases} \lceil n^2/4\rceil +1 & \text{if } n\equiv 0 \pmod 4 \text{ or } n=5,\ 6 \\ \lceil n^4/4\rceil & \text{otherwise} \end{cases}$$

where $\lceil x\rceil$ denotes the least integer $\geq x$.

In 1980, R.A. Brualdi and J. Ross (Math. Oper. 5, 229--241). for nearly reducible matrices. It's upper bound is $n^2-4n+6$.

Conjecture: $\{\gamma(A) \mid A$ is symmetric primitive matrices with its trace $=0\}=\{2, 3,\ldots, 2n-4\}/S_1$

where $S_1$= all of the add numbers in $\{n-2, n-1,\ldots, 2n-5\}$.

§2. k(A) (the index of convergence of Boolean matrix)

Let BM(n,p) (resp. IBM(n,p)) be a set of nxn (resp.

irreducible) Boolean matrices with period p.

Clearly, PBM(n)=IBM(n,1).

(a) the upper bound

Let $k(n,p)=\mathrm{Max}\{k(A) \mid A\in BM(n,p)\}$ and

$\bar{k}(n,p)=\mathrm{Max}\{k(A) \mid A\in IBM(n,p)\}$.

In 1970, S. Schwarz ( Czech. Math. J. 20(95) 703--714) got that:

$$\bar{k}(n,p)\leq k(n,p)=\begin{cases} pW_r+s & \text{if } r>1 \\ \mathrm{Max}\{1,s\} & \text{if } r=1 \end{cases}$$

where $n=pr+s$, $0\leq s<p$ and $W_r=(r-1)^2+1$.

In 1987, Shao Jia Yu and Li Qiao proved that:

$$\bar{k}(n,p)=\begin{cases} pW_r+s & \text{if } r>1 \\ \mathrm{Max}\{1,s\} & \text{if } r=1 \end{cases}$$

So, $k(n,p)=\bar{k}(n,p)$.

And they characterize the matrices with $\bar{k}(n,p)$, but the set of $\{A\in IBM(n,p) \mid k(A)=\bar{k}(n,p)\}$ is very complex.

(b) the index of convergence set $\bar{K}(n,p)=\{k(A) \mid A\in IBM(n,p)\}$ .

Clearly, $\bar{K}(n,p)\subseteq\{1, 2,\ldots, \bar{k}(n,p)\}$. Note that $E_n=\bar{K}(n,1)$. Since there exist gaps in the $\{1, 2,\ldots,\bar{k}(n,1)=W_n\}$, does there exist any gap in the $\{1, 2,\ldots, \bar{k}(n,p)\}$ ?

The answer is in the affirmative. Shao Jia Yu and Li Qiao got that:

' Let $n=rp+s$, $0\leq s<p$, if $k\notin E_r$ for all $k_1\leq k\leq k_2$, then $m\notin\bar{K}(n,p)$ for all $k_1p\leq m\leq k_2p$. In particular $k\in E_r \Longleftrightarrow kp\in\bar{K}(n,p)$'.

But the problem of determining the index set $\overline{K}(n,p)$ is still open.

(c) the index of convergence set of the especial matrix classes.

There are only few results about this problem.

Using Moon's results (Topics on tournaments, Th. 22), it is easy to get that for n-tournament matrices T, the index of convergence set $\overline{K}(n,T)$ is:

$$\overline{K}(n,T)=\begin{cases}\{1\} & n=1\\ \{2\} & n=2\\ \{1,\ 3\} & n=3\\ \{1,4,\ 9\} & n=4\\ \{1,2,4,\ 5,6,7,9\} & n=5\\ \{1,2,\ldots,8,9\} & n=6\\ \{1,2,\ldots,n+2\} & n\geqslant 7\end{cases}$$

Recently, Shao Jia Yu and Li Qiao got that:

$\{k(A) \mid A \in \text{symmetric IBM}(n,2)\} = \{1,2,\ldots,n-2\}$.

There are many open questions of determining the index set of the especial matrix classes.

§3. index(A) (the index of maximum desity of A )

(a) the upper bound

By §2, we know for any $A \in BM(n,p)$, $k(A)= O(n^2)$. But the order of infinity of the upper bound of index(A) is great than of any power function.

For example, for any $i>1$, there exists an prime number

$x_i$ with $2^{i-1} < x_i < 2^i$. Let $n_i = x_i + 1$,

$$A_i = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & & 0 \\ & 0 & & & & \ddots & 1 \\ 1 & 1 & 0 & & \cdots & & 0 \end{bmatrix}_{n_i x n_i}$$

$$B_i = \begin{bmatrix} 0 & A_i & & & \\ 0 & A_i^i & \ddots & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & A_i \\ A_i & & & & A_i^i \end{bmatrix}_{x_i n_i x x_i n_i}$$

and $$C = \begin{bmatrix} B_2 & & & 0 \\ & B_3 & & \\ & & \ddots & \\ 0 & & & B_1 \end{bmatrix}_{nxn}$$

where $n = x_2 n_2 + x_3 n_3 + \cdots + x_1 n_1$.

By direct cheek, it is easy to get that:

$$k(C) = x_1 = O(n^2), \quad index(C) = x_2 x_3 \cdots x_1 > n^{(\log n - 4)/4} \succ n^m$$

where m is any positive integer.

So, for the general Boolean matrices, we have:

Question: What is the sharp order of infinity of the upper bound of index(A)?

Let $index(n,p) = Max\{ index(A) \mid A \in IBM(n,p) \}$. Shao Jia Yu and Li qiao obtained:

$$index(n,p) = p\left\lceil \frac{k(n,p)}{p} \right\rceil = \begin{cases} pW_r & \text{if } r>1,\ s=0 \\ pW_r + p & \text{if } r>1,\ 0<s<p \\ p & \text{if } r=1,\ 0<s<p \\ 1 & \text{if } r=1,\ s=0 \end{cases}$$

where $n=rp+s$, $0\leq s<p$. Moreover, if $k(A)=\bar{k}(n,p)$, then

$$\text{index}(A)=\text{index}(n,p).$$

(b) the index of maximum density set $\text{Index}(n,p)=\{\text{index}(A) \mid A\in IBM(n,p)\}$.

As $\bar{K}(n,p)$, Shao Jia Yu and Li Qiao got

' Let $n=pr+s$, $0\leq s<p$, if $k\notin E_r$ for all $k_1\leq k\leq k_2$ then $m\notin \text{Index}(n,p)$ for all $k_1p<m\leq k_2p$.'

A deep-going problem is seems very difficult.

(c) the index of maximum density set of the especial matrix classes.

For n-tournament matrices T, we got the index of maximum density set Index(n,T) is

$$\text{Index}(n,T)=\begin{cases}\{1\} & \text{if } n=1,2,3\\ \{1,9\} & \text{if } n=4\\ \{1,4,6,7,9\} & \text{if } n=5\\ \{1,2,\ldots,8,9\}/\{2\} & \text{if } n=6\\ \{1,2,\ldots,n+2\}/\{2\} & \text{if } n=7,8,\ldots,15\\ \{1,2,\ldots,n+2\} & \text{if } n\geq 16.\end{cases}$$

Shao Jia Yu and Li Qiao got that:

$$\{\text{index}(A) \mid A\in \text{symmetric } IBM(n,2)\} = \begin{cases}\{1,2,\ldots,n-2\} & \text{if } n \text{ is even}\\ \{2,3,\ldots,n-1\} & \text{if } n \text{ is odd.}\end{cases}$$

For this problem, numerous matrix classes are not investigated.