ON VERTEX–PANCYCLIC GRAPHS WITH THE DISTANCE TWO CONDITION *

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距离为二条件的顶点泛圈图

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摘 要

本文证明了顶点数为 n 的图 G，若对 G 中任意距离为 2 的点对 u，v，总满足$d(u)+d(v)\geqslant n$条件，且$G\not\simeq K_{n/2,n/2}$，则 G 中任意顶点 w，恒存有包含 w 的长度为 k 的圈，这里 $4\leqslant k\leqslant n$.

本文还给出了上面所叙述的图类中，不在 3—圈上的顶点数目的上界.

Abstract For a graph G with order n, let $d(u)+d(v)\geqslant n$ for each pair of vertices u, v a distance two apart in G. We show that each vertex of G lies on a cycle of every length from 4 to n inclusive except if $G\simeq K_{n/2,n/2}$.

Upper bound is given for the number of vertices in this type of graphs which do not lie on 3—cycles.

1. INTRODUCTION

In this paper, we consider only simple graphs. Throughout we use the terminology and notation of [2]. Hence we use $N(v)$ for the neighborhood of a vertex v, $d(v)=|N(v)|$ and $d(u,v)$ for the distance between u and v. In addition we will let $\overline{N}(v)=N(v)\bigcup\{v\}$.

A graph is said to be **pancyclic** if it contains a cycle of length l for all l such that $3\leqslant l\leqslant n$, where n is the number of vertices in the graph. In this paper, we consider the concept of pancyclicity from the point of view of a vertex. So we say that a vertex is **pancyclic** if that vertex lies on a cycle of every length from 3 to n inclusive. We will be particularly interested in vertices which are not quite pancyclic. Hence we say that a vertex is **3^-—pancyclic** if it lies on a cycle of every length from 4 to n inclusive and it does not lie on any 3—cycle in G. We say that G is **vertex pancyclic** if every vertex is pancyclic and **vertex 3^-—pancyclic** if every

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vertex is 3^-–pancyclic or pancyclic.

It is well–known Ore's condition on a graph G, i.e. if $u,v\in V$ and $uv\notin E$, then $d(u)+d(v)\geqslant n$; The other condition[3], called Fan's condition, on a 2–connected graph G is that if for all u,v with $d(u,v)=2$, $\max(d(u),d(v))\geqslant\frac{n}{2}$. We combine Ore's and Fan's condition to give **the distance two condition,** for all $u,v\in V$ with $d(u,v)=2$, $d(u)+d(v)\geqslant n$.

Pancyclic graphs was first considered by Bondy in [1]. And then the result was extended by Zhang et al in [4] to vertex–pancyclic graphs.

Theorem A. Let G be a graph of order n with Ore's condition. Then G is vertex 3^-–pancyclic unless $G\simeq K_{n/2,n/2}$

In this paper we show that if G satisfies the distance two condition, then G is vertex 3^-–pancyclic. Further we find the largest number of 3^-–pancyclic vertices in graphs satisfying this condition.

2. MAIN RESULTS

In this section we prove two results concerning the distance two condition.

Theorem 1. Let G be a graph of order n. If for each $u,v\in V$ for which $d(u,v)=2$, we have $d(u)+d(v)\geqslant n$, then G is vertex 3^-–pancyclic unless $G\simeq K_{n/2,n/2}$.

Proof. Clearly, G is 2–connected. By the result of [3], G is hamiltonian. If G is not 3^-–pancyclic. Suppose that for such largest m, $5\leqslant m\leqslant n$, there is no $(m-1)$–cycle through $x\in V$. Let $C_m=xv_1v_2\cdots v_{m-1}$ be an m–cycle, and let $x\equiv v_0\equiv v_m$. In the following, suppose that $G\not\simeq K_{n/2,n/2}$. We have:

(1) There exists i, $i\in\{0,1,\cdots,m-2\}$ such that $d_{C_m}(v_i)+d_{C_m}(v_{i+2})\leqslant m-1$, where $m\geqslant 6$ (Subscripts taken mod m).

In fact, if there exists $j\in\{0,1,2,\cdots,m-2\}$ such that $v_iv_j, v_{i+2}v_{j+1}\in E$, thus there is an $(m-1)$–cycle $v_iv_jv_{j-1}\cdots v_{i+2}v_{j+1}v_{j+2}\cdots v_i$ in G. Hence we only need to consider the following case: $d_{C_m}(v_i)+d_{C_m}(v_{i+2})=m$ and there does not exist j such that $v_iv_j,v_{i+2}v_{j+1}\in E$, $i\neq m-1$. This implies that one and only one of $\{v_iv_j,v_{i+2}v_{j+1}\}$ belongs to E. Especially, since $v_iv_{i+2}\notin E$, $i\neq m-1$. We have $v_jv_{j+3}\in E$ for each j. So for each $v\in C_m$, we can make choice of j such that $v_{j+1}=v$, $v_{j+2}\neq x$ or $v_{j+1}\neq x$, $v_{j+2}=v$. Note that if $v_jv_{j+3}\in E$, then when $x\neq v_{j+2}$($x\neq v_{j+1}$ resp.), at most one of $\{v_{j+1}v_k, v_{j+1}v_{k+1}\}$ (of $\{v_{j+2}v_k,v_{j+2}v_{k+1}\}$ resp.) belongs to E. Hence for each $v\in C_m$, $d_{C_m}(v)\leqslant\frac{m}{2}$. Thus since $d(v_i)+d(v_{i+2})=m$, it deduces that $m=$even and $N(v_i)=N(v_{i+2})=\{v_{i+1},v_{i+3},\cdots,v_{i+2k+1},\cdots\}$. Therefore $G[V(C_m)]\simeq K_{m/2,m/2}$.

Let $A_i = \{v | v \in V \setminus V(C_m), vv_i \in E\}$. It is easy to show that for each $v \in A_i, vv_{i+1}, vv_{i-1} \notin E$, otherwise there is an $(m-1)$–cycle $v_m(=x)v_1v_2\cdots v_ivv_{i+1}v_{i+4}v_{i+5}\cdots v_m$, if $i+2, i+3 \neq m$ or $v_m(=x)v_1v_2\cdots v_{i-3}v_ivv_{i+1}v_{i+2}\cdots v_m$, if $i-2, i-1 \neq m$. Hence we have: $A_i \cap A_{i-1} = \phi, A_i \cap A_{i+1} = \phi$.

Now, without loss of generality, let $|A_i| \leqslant |A_{i+1}|$. Since $d(v_i, v_{i+2}) = 2$, by the distance two condition, $d(v_i) + d(v_{i+2}) \geqslant n$. On the other hand $d(v_i) + d(v_{i+2}) \leqslant \left(\frac{1}{2} + |A_i|\right) + \left(\frac{1}{2} + n - m - |A_{i+1}|\right) = n - |A_{i+1}| + |A_i| \leqslant n$. Hence we have three asserts as follows:

α) $|A_i| = |A_{i+1}|$, especially $|A_1| = |A_2|$;

β) $A_i = A_{i+2}$. i.e. $A_{2i+1} = A_1$ and $A_{2i} = A_2$ and $A_1 \cup A_2 = V \setminus V(C_m)$;

γ) $N_{C_m}(v) = \{v_1, v_3, \cdots, v_{m-1}\}$ if $v \in A_1$

$N_{C_m}(v) = \{v_2, v_4, \cdots, V_m\}$ if $v \in A_2$

Finally, A_1 must be an independent set. It not, there are $u, v \in A_1$ with $uv \in E$. thus there is an $(m-1)$–cycle: $v_1uvv_3v_6v_7\cdots v_mv_1$. This is a contradiction. An analagous argument shows that A_2 must be an independent set. Thus it deduces that $G \simeq K_{n/2,n/2}$, a contradiction. Hence (1) is true.

(2) $m=5$. In fact, if $m \geqslant 7$ or $m=6$ with $i \neq 2$, then by (1) there always exists $i \in \{0,1,2,\cdots,m-2\}$ such that $d_{C_m}(v_i) + d_{C_m}(v_{i+2}) \leqslant m-1$, $d_{C_m}(v_{i+3}) + d_{C_m}(v_{i+5}) \leqslant m$ and $x \notin \{v_{i+1}, \cdots, v_{i+4}\}$ or $d_{C_m}(v_i) + d_{C_m}(v_{i+2}) \leqslant m-1$, $d_{C_m}(v_{i-3}) + d_{C_m}(v_{i-1}) \leqslant m$ and $x \notin \{v_{i-2}, \cdots, v_{i+1}\}$. Without loss of generality, we assume that the former. if each $v \in V \setminus V(C_m)$, then at most one of $\{vv_i, vv_{i+3}\}$ and of $\{vv_{i+2}, vv_{i+5}\}$ belong to E respectively. Otherwise there is an $(m-1)$–cycle containing x in G. Hence we have $2n \leqslant d(v_i) + d(v_{i+2}) + d(v_{i+3}) + d(v_{i+5}) \leqslant 2(n-m) + (m-1) + m = 2n-1 < 2n$, a contradiction. If $m=6$ with $i=2$. Note that $2 \leqslant d(v_i) \leqslant 3$ for all i and $d_{C_6}(v_2) + d_{C_6}(v_4) \leqslant m-1=5$. Hence, without loss of generality, we assume that $d_{C_6}(v_2) = 2$. Thus we have $d_{C_6}(v_0) + d_{C_6}(v_2) \leqslant 5 = m-1$. So in this case, we can make choice of $i=0 \neq 2$. Hence (2) is true.

(3) $m \neq 5$. i.e. there always exists a 4–cycle containing x.

Case 1. There is a 3–cycle $xuvx$.

Since G is 2–connected, without loss of generality, there is a vertex $w \in V$ with $vw \in E$. If $d(x,w) = 1$, then there is a 4–cycle $xuvwx$. If $d(x,w) = 2$, then, by the distance two condition, $|N(x) \cap N(w)| \geqslant 2$. Let $y \in \{N(x) \cap N(w)\} \setminus \{v\}$, thus there is a 4–cycle $xywvx$.

Case 2. There is no 3–cycle containing x. In other words, $N(X)$ is an independent set. Let $u,v\in N(x)$, thus, by the distance two condition, $d(u)+d(w)\geqslant n$. So $|N(u)\bigcap N(v)|\geqslant 2$. Let $y\in\{N(u)\bigcap N(v)\}\backslash\{x\}$, then there is a 4–cycle $xuyvx$.

Therefore, (3) is true.

Combining (2) and (3), the proof of Theorem is completed. #

Consider the class of graphs $G\not\supseteq K_{n/2,n/2}$ satisfying the distance two condition. Let M be the maximum number of 3^-–pancyclic vertices in a graph with the above property.

Theorem 2. If $n\geqslant 6$, then

$$M=\begin{cases}\max\left\{2,\left[\frac{1}{2}(-2+\sqrt{6+2n})\right]\right\} & \text{if } n \text{ is odd}\\ \frac{n}{2}-2 & \text{if } n \text{ is even}\end{cases}$$

Proof Consider the graph G_0 with $V(G_0)=A\bigcup\bigcup_{i=1}^{r}B_i\bigcup D$, where $G_0[A]=K_r^c$, $G_0[B_i]=K_{r+1}^c$ for all i and $G_0[D]=K_{r^2+2r-2}$. Further $N(v_i)=B_i$ for each $v_i\in A$ and each vertex of $\bigcup_{i=1}^{r}B_i$ is joined to every vertex in D. We note that $d(v_i)=r+1$ for all $v_i\in A$, $d(w_i)=r^2+2r-1$ for all $w_i\in\bigcup_{i=1}^{r}B_i$, $d(z)=2r^2+3r-3$ for all $z\in D$ and $n=|V(G_0)|=2r^2+4r-2$. It is easily checked that G_0 satisfies the dstance two condition and the vertices in A are 3^-–pancyclic. Now

$$r=\left[\frac{1}{2}(-2+\sqrt{8+2n})\right],\quad \text{so } M\geqslant\left[\frac{1}{2}(-2+\sqrt{8+2n})\right] \text{ if } n \text{ is even.}$$

When n is odd, we consider G'_0 which is dedcued by substituting K_{r^2+2r-2} in G_0 with K_{r^2+2r-1}. Thus $n=|V(G_0')|=2r^2+4r-1$. An analagous argument shows that $M\geqslant\left[\frac{1}{2}(-2+\sqrt{6+2n})\right]$, if n is odd.

Now, let $G\not\supseteq K_{n/2,n/2}$ satisfy the distance two condition. Let $R=\{v_i,i=1,2,\cdots,r\}$ be the set of 3^-–pacyclic vertices in G. Note that G is hamiltonian. Hence there is a 3–cycle containing u in G if $d(u)>\frac{n}{2}$. So we have $2\leqslant d(v_i)\leqslant\frac{n}{2}$, $i=1,2,\cdots,r$. The remainder of the proof proceeds in two steps.

(1) R is an independent set in G.

Case 1. There are two vertices, say v_1, v_2, in R such that $B = N(v_1) \cap N(v_2) \neq \phi$. Let $A = N(v_1) \setminus B$, $C = N(v_2) \setminus B$ and $D = V \setminus \{\bar{N}(v_1) \cup \bar{N}(v_2)\}$. By the distance two condition, $d(v_1) + d(v_2) \geqslant n$. Thus we have $d(v_1) = d(v_2) = \frac{n}{2}$ and n = even. If $A \neq \phi$, then $d(a) \leqslant |C| + |D| + 1 = \frac{n}{2} - 1$ for all $a \in A$ and $d(b) \leqslant |D| + 2 < \frac{n}{2}$ for all $b \in B$. Thus we have $d(a) + d(b) < n$, a contradiction. Hence $A = \phi$. An analagous argument shows that $C = \phi$. And then by the distance two condition, $d(b_1) + d(b_2) \geqslant n$ for all $b_1 \neq b_2 \in B$. Thus $B \vee D \subseteq G$. Since $G \not\supseteq K_{n/2,n/2}$, $K_{n/2,n/2} + e \subseteq G$, where the ends of edge e belong to one of bipartition of $K_{n/2,n/2}$ respectively. Hence $r \leqslant \frac{n}{2} - 2 \leqslant M$.

Case 2. $N(v_i) \cap N(v_j) = \phi$ for all $v_i \neq v_j \in R$. Let $u_i \in N(v_i)$, $i = 1,2,\cdots,r$. If $u_i u_j \in E$, without loss of generality, we assume that $d(v_i) \leqslant d(v_j)$. Since $d(u_i) \leqslant n - d(v_i) - r + 1$, $i = 1,2,\cdots,r$. Thus $d(v_i) + d(u_j) \leqslant d(v_i) + (n - d(v_j) - r + 1) < n$. On the other hand, by the distance two condition, $d(v_i) + d(u_j) \geqslant n$, a contradiction. Hence $N(v_i) \cap N(u_j) = \phi$ for all $i \neq j \in \{1,2,\cdots,r\}$. Since G is 2–connected, there is a vertex $z \in D = V \setminus \bigcup_{i=1}^{r} \bar{N}(v_i)$. Let $s = \sum_{i=1}^{r} d(v_i)$ and $d = |D|$. Thus

$$r + s + d = n \leqslant d(z) + d(v_i) \leqslant [s + (d-1)] + d(v_i)$$

for all $v_i \in R$. So $d(v_i) \geqslant r + 1$ and $s \geqslant r(r+1)$. Further, for $u_i, u'_i \in N(v_i)$,

$$r + s + d = n \leqslant d(u_i) + d(u'_i) \leqslant 2d + 2.$$

So $d \geqslant r + s - 2 \geqslant r^2 + 2r - 2$, $n = r + s + d \geqslant 2r^2 + 4r - 2$ if n is even and $n \geqslant 2r^2 + 4r - 1$ if n is odd. This implies $r \leqslant \left[\frac{1}{2}(-2 + \sqrt{6 + 2n})\right]$ if n is odd and $r \leqslant \left[\frac{1}{2}(-2 + \sqrt{8 + 2n})\right]$ if n is even.

(2) There are two vertices $v_1, v_2 \in R$ with $v_1 v_2 \in E$. Clearly, $N(v_1) \cap N(v_2) = \phi$ and $N(v_1)$, $N(v_2)$ are independent sets in G. Without loss of generality, we assume that $|N(v_1)| \leqslant |N(v_2)|$. Let $D = V \setminus \{N(v_1) \cup N(v_2)\}$. For each $u \in N(v_2) \setminus \{v_1\}$, $d(u, v_1) = 2$, so $n \leqslant d(u) + d(v_1) \leqslant 1 + |D| + |N(v_1) \setminus \{v_2\}| + |N(v_1)| \leqslant 1 + |D| + |N(v_1) \setminus \{v_2\}| + |N(v_2)| \leqslant n$. Thus we have three asserts as follows:

α) $|N(v_1) \setminus \{v_2\}| = |N(v_2) \setminus \{v_1\}|$

β) $(N(v_2) \setminus \{v_1\}) \vee (D \vee [N(v_1) \setminus \{v_2\}]) \subseteq G$.

An analagous argument shows that:

γ) $(N(v_1) \setminus \{v_2\}) \vee D \subseteq G$.

So there is no 3^-–pancyclic vertex in G except v_1, v_2. i.e. $r = 2$.

Combining (1), (2) and G_0, G_0', the proof of Theorem is completed.#

References

[1] J.A. Bondy, Pancyclic graph I, JCT B(11) 1971, 80–84.

[2] J.A. Bondy and U.S.R. Murty, Graph theory with applications, Macmillan, London, 1976.

[3] Geng–hua Fan, New sufficient conditions for cycles in graphs, JCT B(37) 1984, 221–227.

[4] Zhang Ke Min, D. A. Holton and Sheng Bau, On vertex–pancyclic graphs, to appear.