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A PROPERTY OF TOTAL COLORING OF GRAPHS *

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图的全着色性质

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摘　要

$G=(V,E)$是一个简单图，其最大度$\Delta(G)\geqslant 2$. 令$\chi_T(G)$表示 G 的全色数. 对任意$e\in E(G)$, 本文证明了:$\chi_T(G-e)\leqslant\chi_T(G)\leqslant\chi_T(G-e)+1$. 作为上述结果的推论，对边临全着色图 G 有$\chi_T(G)=\Delta(G)+2$. 从而对上述特殊的图类，全着色猜测成立.

Abstract

Let $G=(V,E)$ be a graph with maximum degree $\Delta(G)\geqslant 2$. And we use $\chi_T(G)$ to denote the total chromatic number of G. This paper proves that $\chi_T(G-e)\leqslant\chi_T(G)\leqslant\chi_T(G-e)+1$ for each $e\in E(G)$. And as a corollary, $\chi_T(G)=\Delta(G)+2$ for the edge-critical total coloring graph G, follows.

1 Introduction

Let $G=(V,E)$ be a simple graph. A proper k-total coloring of G is an assignment of k colors $1,2,\cdots,k$, to the elements of $V\bigcup E$ such that no two adjacent or incident elements receive the same color. We shall abbreviate k-total colorable. The total chromatic number, $\chi_r(G)$, of G is the minimum k for which G is k-total colorable. If $\chi_T(G)=k$, G is said to be k-total chromatic. A simple graph G with $\chi_T(G)>\Delta(G)+1$ is called an edge-critical total coloring graph if for any $e\in E$, $\chi_T(G-e)=\Delta(G-e)+1$. Other terms and symbols

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are found in [2].

A lot of practical and theoretical problems are closely in contact with a coloring problem of graphs. So the coloring problem is an important branch of graph theory. According different definition, there are many kind of coloring such as vertex–coloring, edge–coloring, total coloring etc. One of basic problems in this area is finding the chromatic number of a graph. Usually, to find the total chromatic number of a graph is more difficult than to find the vertex–chromatic number or the edge–chromatic number.

In 1965, M. Behzed suggested a well–known **total coloring conjecture:** For every simple graph G.

$$\Delta(G)+1\leqslant\chi_T(G)\leqslant\Delta(G)+2.$$

It is evident for $\chi'_T(G\geqslant\Delta(G)+1$. So far, it is shown that $\chi_T(G)\leqslant\Delta(G)+2$ is true for a complete bipartite graph, a complete balance k–partite graph[3]; a tree, a cycle, a compltet graph[7]; an outerplanar graph[7]; a graph which contains only odd cycles or a graph which contains both odd cycles and even cycles, but there are no common edges between odd and even cycles[6]. The conjecture is also true for the graphs of maximum degree $\Delta(G)\leqslant 3$[1] or $\Delta(G)\geqslant|V(G)|-4$[5].

In [6], it proves that

Theorem A Let G be an edge–critical total coloring graph. If the number of vertices of maximum degree is at least 3, then $\chi_T(G)=\Delta(G)+2$.

This note shows that the condition, the number of vertices of maximum degree is at least 3 in Theorem A, is nonessential. In other words, the total coloring conjecture is true for the edge–critical total coloring graphs.

2 Main Results

Theorem Let $G=(V,E)$ be a graph with $\Delta(G)\geqslant 2$. Then

$$\chi_T(G-e)\leqslant\chi_T(G)\leqslant\chi_T(G-e)+1$$

for each $e\in E(G)$.

Proof Since $G-e\subset G, \chi_T(G-e)\leqslant\chi_T(G)$. Hence we need only show that $\chi_T(G)\leqslant\chi_T(G-e)+1$ for each $e\in E(G)$. Let $e=uv$ and $\chi_T(G-e)=n$. Let π be a proper n–total coloring of G–e with colors $\{1,2,\cdots,n\}$. We use $\pi(b)$ to denote the color of element $b\in V\bigcup E$ and C_x to denote the set of colors which includs $\pi(x)$ and all of $\pi(e')$, where $x\in V, e'\in E, e'$ and x are incident. It is clear that $|C_x|=d_G(x)+1$. If $\pi(u)\neq\pi(v)$. Let color n+1 is presented at edge e, thus, by π, G is n+1 total colorable. If $\pi(u)=\pi(v)$ and $\chi_T(G-e)=n>\Delta(G-e)+1$, thus there is at least one color $i\in\{1,2,\cdots,n\}$ such that $i\notin C_v$. In this case, we may reassign color n+1 to the vertices which adjacent with v

and have color i upon π, color i to v, color $n+1$ to uv. So there is a proper $n+1$ total coloring of G. i.e. $\chi_T(G) \leqslant n+1 = \chi_T(G-e)+1$ Similarly, if $\pi(u)=\pi(v)$ and $d_{G-e}(u) < \Delta(G-e)$ or $\pi(u)=\pi(v)$ and $d_{G-e}(v) < \Delta(G-e)$, we may also prove that $\chi_T(G) \leqslant \chi_T(G-e)+1$. Hence in the following, without loss of generality, we always assume that $\pi(u)=\pi(v)=1$, $\chi_T(G-e)=\Delta(G-e)+1$ and $d_{G-e}(u)=d_{G-e}(v)=\Delta(G-e)$. Let

$$N_G(v)=\{u,v_2,v_3,\cdots,v_n\}, \text{ thus } N_{G-e}(v)=\{v_2,v_3,\cdots,v_n\},$$

where $N_G(v)$ denotes the neighbour set of v in G. And we suppose that $\pi(vv_i)=i$ $(i=2,3,\cdots,n)$.

Case 1 $1 \notin C_{v_{i_0}}$, $i_0 \in \{2,3,\cdots,n\}$. We may reassign color i_0 to v, color 1 to vv_{i_0}, color $n+1$ to uv and the vertices which adjacent with v and have color i_0 upon π;

Case 2 $1 \in C_{v_i}$, $\forall i \in \{2,3,\cdots,n\}$ and $\{\pi(v_i) | i=2,3,\cdots,n\} \subset \{2,3,\cdots,n\}$. We may assume that $2 \notin \{\pi(v_i) | i=2,3,\cdots n\}$. Thus we may reassign color 1 to vv_2, color 2 to v and color $n+1$ to uv and the edges which have color 1 upon π;

Case 3 $1 \in C_{v_i}$, $\forall i \in \{2,3,\cdots,n\}$ and $\{\pi(v_i) | i=2,3,\cdots,n\} = \{2,3,\cdots,n\}$. Let $\pi(v_i)=\alpha_i$ $(i=2,3,\cdots,n)$, thus $\alpha_j \neq j$, $\pi(v_i) \neq \alpha_j$ if $i \neq j$ and for each i_0, there is a j_0 such that $\alpha_{j_0}=i_0$ $(i_0, j_0=2,3,\cdots,n)$. In the following, we consider the edge $v_{i_0}v_{j_0}$ with $\alpha_{j_0}=i_0$.

Subcase 3.1 $v_{i_0}v_{j_0} \notin E(G)$ or $v_{i_0}v_{j_0} \in E(G)$ and $\pi(v_{i_0}v_{j_0}) \neq 1$. Thus we assign color $n+1$ to uv. And we may reassign color α_{j_0} to v, color 1 to vv_{i_0}, color $n+1$ to v_{j_0} and the edge which has color 1 upon π and adjacents with v_{j_0};

Subcase 3.2 $v_{i_0}v_{j_0} \in E(G)$ and $\pi(v_{i_0}v_{j_0})=1$. Thus we may reassign color $n+1$ to v_{j_0}, Color 1 to color i_0 to $v_{i_0}v_{j_0}$ and v, color $n+1$ to uv.

Hence all of cases as above, by π, we can always get a proper $n+1$ total coloring in G. i.e. $\chi_T(G) \leqslant \chi_T(G-e)+1$. This completes the proof.

Corollary 1 Let $G=(V,E)$ be an edge-critical total coloring graph, then $\chi_T(G)=\Delta(G)+2$.

Proof It is clear for $\Delta(G)=1$. If $\Delta(G) \geqslant 2$. By the definition of the edge-critical total coloring graph, $\chi_T(G) \geqslant \Delta(G)+2$ and $\chi_T(G-e)=\Delta(G-e)+1$ for each $e \in E(G)$. On the other hand, by Theorem, we have: $\chi_T(G) \leqslant \chi_T(G-e)+1=\Delta(G-e)+2 \leqslant \Delta(G)+2$. Hence $\chi_T(G)=\Delta(G)+2$.

Corollary 2 Let G be an edge-critical total coloring graph, then there are at least two vertices of maximum degree in G. Further if there are only two vertices u,v of maximum degree in G and $\Delta(G) \geqslant 2$, then u,v are non-adjacent in G.

Proof It is clear for $\Delta(G)=1$. If $\Delta(G) \geqslant 2$ and there is only a vertex u such that

$d_G(u) = \Delta(G)$, taking $uv \in E(G)$, or if $\Delta(G) \geqslant 2$ and there are only two vertices u,v such that $d_G(u) = d_G(v) = \Delta(G)$ and $uv \in (G)$. Thus by the definition of an edge–critical total coloring graph, we have: $\chi_T(G - uv) = \Delta(G - uv) + 1 = \Delta(G) - 1 + 1 = \Delta(G)$. And then, by Theorem, we have: $\chi_T(G) \leqslant \chi_T(G - uv) + 1 = \Delta(G) + 1$. This is a contrary. The proof is completed.

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