南京大学学报数学半年刊
第15卷 第1期 JOURNAL OF NANJING UNIVERSITY Vol. 15, No. 1
1998年5月 MATHEMATICAL BIQUARTERLY May, 1998

TWO ORDER EXPONENT SET OF STRONG CONNECTED DIGRAPHS*

Miao Zhengke *Zhang Kemin*
(Dept. of math., Nanjing University, Nanjing, 210093, P. R. C.)

Abstract Let D be a strong connected digraph on n vertices, and let A be the adjacency matrix of D. Then $A+A^2$ is primitive, and the primitive exponent of $A+A^2$ is known as two order exponent of D. In this paper, we show that the two order exponent set of strong connected digraphs on n vertices is $\{1,2,\cdots,n-1\}$. Further, we also describe the characterization of strong connected digraphs on n vertices with two order exponent $n-1$.

Key words primitive matrix, primitive exponent, matrix, two order exponent.
AMS(1991) subject classifications 05C50.

1 Introduction

Let A be a $(0,1)$-matrix, J an universal matrix. A is called a primitive matrix if there exists a positive integer k such that $A^k=J$. The least such k is called the primitive exponent of A, denoted by $\gamma(A)$. Let D be a digraph. We call D primitive if there exists a positive integer k such that for all ordered pairs of vertices $i,j\in V(D)$ (not necessarily distinct), there is a directed walk from i to j with length k. The least such k is called the primitive exponent of digraph D, denoted by $\gamma(D)$.

Let $A=(a_{ij})$ be an $n\times n(0,1)$-matrix, and let D be a digraph with vertex set $V=\{1,2,\cdots,n\}$ and arc set E. If $(i,j)\in E$ iff $a_{ij}=1$, then D is called the associated digraph of A, denoted by $D(A)$, and A is called the adjacency matrix of D, denoted by $A(D)$. It is well known that $(A^k)_{ij}=1$ iff there is a directed walk with length k from i to j in $D(A)$. So $A=A(D)$ is primitive iff $D=D(A)$ is primitive, and $\gamma(A)=\gamma(D)$. In the following, we use $\gamma(A)$ and $\gamma(D)$ without distinction.

Let D be a primitive digraph on n vertices. For any $i,j\in V(D)$, the (local) exponent from i to j, denoted by $\gamma(i,j)$, is the least integer k such that there exists a directed walk of

* Reveived: Aug. 9, 1997.

length m from i to j for all $m \geqslant k$. Let $L(D)$ denote the set of the distinct length of the cycle of digraph D. Let $R \subset L(D)$ and the great common divisor of the number in R is 1. For any $i,j \in V(D)$, we define the general distance from i to j relative R, denoted by $d_R(i,j)$, to be the length of the shortest directed walk from i to j which touchs every number in R (If there exists a common vertex on P and some r-cycle, then we call P touching r).

Let $\{r_1, r_2, \cdots, r_s\}$ be a set of distinct positive integer with the great common divisor 1. We define the Frobenius number $F(r_1, r_2, \cdots, r_s)$ to be the least integer k such that every integer $m \geqslant k$ can be expressed in the form $m = c_1r_1 + c_2r_2 + \cdots + c_sr_s$, where $c_1, c_2, \cdots, c_s$ are nonnegative integers. A result due to Schur[1] shows that the Frobenius number is well defined. And the following Proposition holds.

Proposition[2] Let D be a primitive digraph, $R = \{r_1, r_2, \cdots, r_s\} \subset L(D)$, and g. c. d. $(r_1, r_2, \cdots, r_s) = 1$. Then for any $i,j \in V(D)$, $\gamma(i,j) \leqslant d_R(i,j) + F(r_1, r_2, \cdots, r_s)$.

Other definitions and notations can be found in [3] and [4].

Let D be a strong connected digraph. In [3], $A(D) + A^2(D)$ is a primitive matrix. Its exponent is called the two order exponent of D, denoted by $\gamma(2,D)$. Let $ES(2,n)$ be the two order exponent set of all strong connected digraph on n vertices. In this paper, the following results are obtained.

Theorem 1 $ES(2,n) = \{1, 2, \cdots, n-1\}$.

Theorem 2 Let D be a strong connected digraph on n vertices with two order exponent $n-1$. Then D is isomorphic to the strong subdigraph of D_0 in Figure 1.

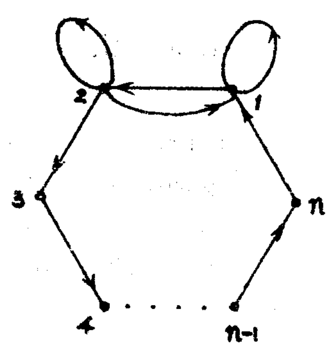

Figure 1 D_0

2 The Determination of $ES(2,n)$

Lemma 1 Let D be a strong connected digraph on n vertices with diameter d. Then
$\gamma(2,D) \leqslant d \leqslant n-1$.

Proof Let A be the adjacency matrix of D. Because D is a strong connected digraph with diameter d, there exists a directed walk from i to j for any $i,j \in V(D)$ which length is less than or equal to d. Since $(A^k)_{ij} = 1$ iff there exists a directed walk from i to j, then $E + A + A^2 + \cdots + A^d = J$. By D is strong connected, A is no zero-row. So is A^d. Hence

$$(A + A^2)^d = A^d(E + A + \cdots + A^d) = A^dJ = J.$$

This shows that $\gamma(A + A^2) \leqslant d \leqslant n-1$, i. e. $\gamma(2,D) \leqslant d \leqslant n-1$.

Lemma 2 Suppose that C_k is cycle of length $k \geqslant 1$. Then $\gamma(2,C_k) = k-1$.

Proof Let A be the adjacency matrix of C_k. Then

$$A=\begin{pmatrix}0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1\\ 1 & & & 0\end{pmatrix}.$$

By directly calculation

$$A^t=\begin{pmatrix}0 & E_{k-t}\\ E_t & 0\end{pmatrix},\qquad 1\leqslant t\leqslant k-1.$$

Hence $(A+A^2)^{k-1}=A^{k-1}(E+A+\cdots+A^{k-1})=J$, and

$$(A+A^2)^{k-2}=A^{k-2}(E+A+\cdots+A^{k-2})$$

$$=\begin{pmatrix}1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1\\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1\\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 & 1\\ 1 & 1 & 0 & \cdots & 1 & \vdots & \vdots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 1 & 1 & 1 & 1\\ 1 & 1 & \cdots & 1 & 0 & 1 & 1 & 1\end{pmatrix}\neq J \qquad (*)$$

So $\gamma(2,C_k)=k-1$.

Lemma 3 $\{1,2,\cdots,n-1\}\subseteq ES(2,n)$.

Proof It is obvious that $1\in ES(2,n)$ since $\gamma(2,K_n)=1$. When $k\in\{2,3,\cdots,n-1\}$, we check the digraph D shown in Figure 2.

It is easy to see that

$$A(D)=\begin{pmatrix}0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0\\ \cdots & & \cdots & & \cdots & & & \\ 0 & 0 & 0 & \cdots & 1 & 1 & \cdots & 1\\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \cdots & & 0 & \ddots & & \cdots & & \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}.$$

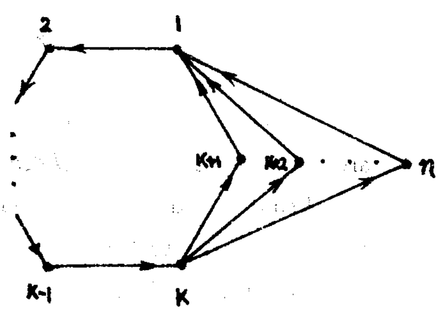

Figure 2 D

For any $i,j\in V(D)$, if $i\leqslant k$ or $j\leqslant k$, by Lemma 2, there exists a directed walk of length k from i to j in the assocated digraph of $A(D)+A^2(D)$. If $i>k$ and $j>k$, it is easy to check that there exists a directed walk of length k from i to j in the associated digraph of $A(D)+A^2(D)$. So $\gamma(2,D)\leqslant k$. On the other hand, there is no directed walk of length $k-1$ from 1 to $k-1$ in the associated digraph of $A(D)+A^2(D)$. Hence $\gamma(2,D)>k-1$. Thus $\gamma(2,D)=k$, i.e. $k\in ES(2,n)$.

By Lemmas 1 and 3, the proof of Theorem 1 is completed.

3 The Description of Extreme Digraph

Lemma 4 Let D be a strong connected digraph on n vertices. If $\gamma(2,D)=n-1$, then D is Hamiltonian.

Proof Assume that D is not Hamiltonian. Since D is strong connected, each vertex of D is on some cycle. For any $i,j\in V(D)$,

(1) i and j are on the same r-cycle. By Lemma 2, in the associated digraph of $A(D)+A^2(D)$, $\gamma(i,j)\leqslant r-1\leqslant n-2$. So there exists a directed walk of length $n-2$ from i to j.

(2) i and j are not on a same cycle. Let $r=\min\{k|i \text{ is on some } k\text{-cycle}\}$ and $s=\min\{k|j \text{ is on some } k\text{-cycle}\}$. If $r\leqslant n-2$ or $s\leqslant n-2$, without loss of generality, say $r\leqslant n-2$. Then, in the associated digraph of $A(D)+A^2(D)$, $\gamma(i,j)\leqslant r-1+[(n-r)/2]\leqslant(n+r-1)/2\leqslant(2n-3)/2<n-1$. So there exists a directed wallk of length $n-2$ from i to j. If $r=s=n-1$, then D has a subdigraph H which is shown in Figure 3.

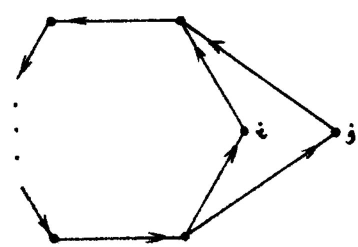

Figure 3 H

So there exists a directed walk of length $n-2$ from i to j in the associated digraph of $A(D)+A^2(D)$.

To sum up,

$\gamma(2,D)\leqslant n-2$. This contradicts to $\gamma(2,D)=n-1$. Hence D is Hamiltonian.

Lemma 5 Let D consist of an n-cycle C_n and an arc, $L(D)=\{k,n\}$. If $2<k<n$, then $\gamma(2,D)\leqslant n-2$.

Proof Without loss of generality, let the arc be $(k,1)$ and $C_n=(123\cdots n1)$. Then the associated digraph of $A(D)+A^2(D)$ is in Figure 4. Let $S=\{(1,n-2),(2,n-1),(3,n),(4,1),\cdots,(n-1,n-4),(n,n-3)\}$ be a subset of $V(D)\times V(D)$. For any $i,j\in V(D)$,

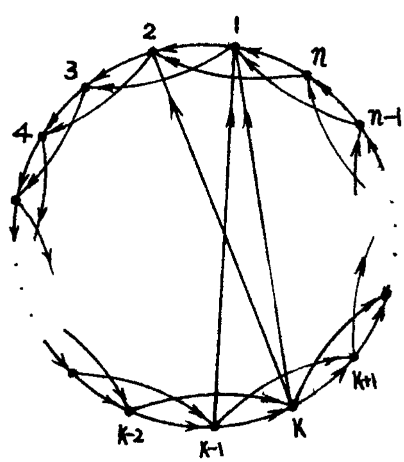

Figure 4

(1) $(i,j)\notin S$. By C_n being a subdigraph of D and (*) in the proof of Lemma 2, there exists a directed walk of length $n-2$ from i to j.

(2) $(i,j)\in S$. It is obvious that there are some cycles with length $[k/2],[k/2]+1,\cdots,k$

in the associated digraph of $A(D)+A^2(D)$, and vertices $1,2,\cdots,k$ are on these cycles. Since $k>2$, then there exists a directed walk of length $[(n-3)/2]$ from i to j which touches each number of $[k/2],[k/2]+1,\cdots,k$. By the Proposition, and $F([k/2],[k/2]+1,\cdots,k)\leqslant[k/2]$ we have $\gamma(i,j)\leqslant[(n-3)/2]+[k/2]$. If $2<k<n-1$, $\gamma(i,j)\leqslant(n-2)/2+(k+1)/2\leqslant(n-2+n-2+1)/2<n-1$. So there exists a directed walk of length $n-2$ from i to j. If $k=n-1$ and n is odd, $\gamma(i,j)\leqslant[(n-3)/2]+[(n-1)/2]=(n-3)/2+(n-1)/2=n-2$. So there exists a directed walk of length $n-2$ from i to j. If $k=n-1$ and n is even, in Figure 4, there exists directed walks: $1\to3\to\cdots\to n-1\to2\to4\to\cdots\to n-2$, $2\to4\to\cdots\to n-2\to1\to3\to\cdots\to n-1$, $3\to5\to\cdots\to n-1\to2\to4\to\cdots\to n-2\to n$, $4\to6\to\cdots\to n-2\to1\to3\to\cdots n-1\to1$, $5\to7\to\cdots\to n-1\to2\to4\to\cdots\to n-2\to n\to2$, $\cdots$, $n-3\to n-1\to2\to4\to\cdots\to n\to2\to4\to\cdots\to n-6$, $n-2\to1\to3\to\cdots\to n-1\to1\to3\to\cdots\to n-5$, $n-1\to2\to4\to\cdots\to n-2\to n\to2\to4\to\cdots\to n-4$, $n\to2\to4\to\cdots\to n-2\to1\to3\to\cdots\to n-3$, and all of their lengths are $n-2$.

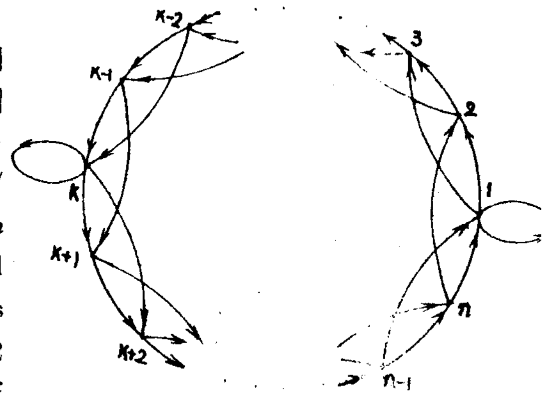

Figure 5

To sum (1) and (2), there exists a directed walk of length $n-2$ from i to j. Hence $\gamma(2,D)\leqslant n-2$.

Lemma 6　Let D be shown in Figure 5. If $2<k<n$, Then $\gamma(D)\leqslant n-2$.

Proof　Let $S=\{(1,n-2),(2,n-1),(3,n),(4,1),\cdots,(n-1,n-4),(n,n-3)\}$ be a subset of $V(D)\times V(D)$. For any $i,j\in V(D)$, (1) $(i,j)\notin S$. By $C_n\subset D$ and (*) in the proof of Lemma 2, there exists a directed walk of length $n-2$ from i to j. (2) $(i,j)\in S$. By $2<k<n$, there exists a directed walk of length $n-3$ from i to j which touches loop. So there exists a directed walk of length $n-2$ from i to j. Hence $\gamma(2,D)\leqslant n-2$.

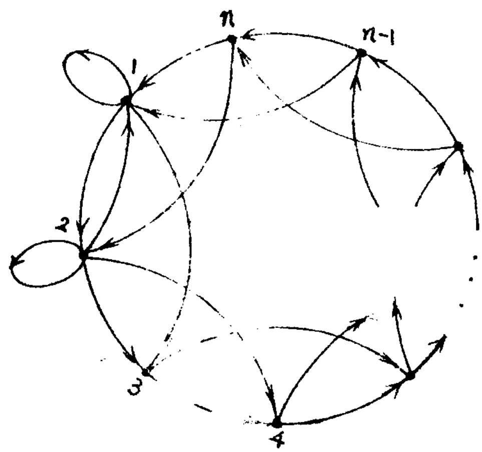

Figure 6

Proof of Theorem 2　Since $D'\subset D_0$, $\gamma(2,D_0)\leqslant\gamma(2,D')\leqslant n-1$. In the associated digraph of $A(D_0)+A^2(D_0)$ (in Figure 6), there is no directed walk of length $n-2$ from 3 to n. In fact, the length of the longest directed walk which is not touching loop from 3 to n is $n-3$. But the length of the shortest directed walk which is touching loop from 3 to n is $n-1$. So $\gamma(2,D_0)\geqslant n-1$. Thus $\gamma(2,D')=n-1$.

On the other hand, since $\gamma(2,D)=n-1$, by Lemmas 4, 5 and 6, D consists of C_n or adding at most an arc and two loops but the arc and the two loops is on two adjacency vertex. So D is isomorphic to the strong subdigraph of D_0 in Figure 1.

References

1 Kemeny J G, and Snell J L, Finite Markov Chain, Von Nostrand, princeton, 1960

2 Shao Jiayu, On the conjecture about the exponent set of primitive matrices, Linear Algebra and Applications, 1985, 64:91～123

3 Kim Ki Hang, Boolean Matrix Theory and Applications, Marcel Dekker, New York, 1982

4 Bondy J A, and Murty, U S R, Graph Theory with Applications, Macmillan, London, 1976

强连通有向图的二阶指数集

苗正科　　　　张克民

（南京大学数学系，210093，南京）

摘要　设 D 是一个 n 阶强连通有向图，A 为 D 的邻接矩阵，则 $A+A^2$ 是一个本原矩阵，称其本原指数为 D 的二阶指数．本文证明了 n 阶强连通有向图的二阶指数集为 $\{1,2,\cdots,n-1\}$，并刻划了二阶指数等于 $n-1$ 的 n 阶强连通有向图的特征．

关键词　本原矩阵，本原指数，矩阵，二阶指数．

分类号　O157.5，O151.21.