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A NOTE ON CYCLES IN REGULAR MULTIPARTITE TOURNAMENTS*

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Abstract In this paper we show that if T is a regular n-partite ($n\geqslant 4$) tournament, then each vertex of T is contained in a k-cycle for all k, $3\leqslant k\leqslant n$. However, when $n=3$, the above statement is not true, a counterexample is illustrated.

Key words multipartite fournaments, regular, cycle.
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1 Introduction

We use the terminology and notation of [1]. Let $D=(V(D),A(D))$ be a digraph. If xy is an arc of D, then we say that x dominates y, denoted by $x\rightarrow y$. More generally, if A and B are two disjoint vertex set of D such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A\Rightarrow B$. The outset $N^+(x)$ of a vertex x is the set of vertices dominated by x in D, and the inset $N^-(x)$ is the set of vertices dominating x in D. The irregularity $i(D)$ is $\max|d^+(x)-d^-(y)|$ over all vertices x and y of D($x=y$ is admissible), where $d^+(x)=|N^+(x)|$ and $d^-(y)=|N^-(y)|$. If $i(D)=0$, we say D is regular. Let T be a multipartite tournament and $x\in V(T)$, we denote $V(x)$ the partite set of T to which x belongs. Guo and Volkmann[2] proved that every partite set of a strongly connected n-partite tournament has at least one vertex which lies on a k-cycle for each $k\in\{3,4,\cdots,n\}$. In this paper we show that every vertex of a regular n-partite ($n\geqslant 4$) tournament lies on a k-cycle for each $k\in\{3,4,\cdots,n\}$. However, when $n=3$, the above statement is not true. In Figure 1, we can easily check that it is a regular 3-partite tournament, but the vertex-v is not contained in any 3-cycle. Hence our result is the best possible in a sense.

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2 Main Result

Theorem Let T be a regular n-partite ($n \geqslant 4$) tournament, then each vertex of T is contained in a k-cycle for all k, $3 \leqslant k \leqslant n$.

Proof Let $V_1, V_2, \cdots, V_n$ be the partite set of T. From the regularity of T, it is not difficult to check that all the partite set of T have the same cardinality (say, s). So it is clear that $|N^+(x)| = |N^-(x)| = (n-1)s/2$ for each $x \in V(T)$.

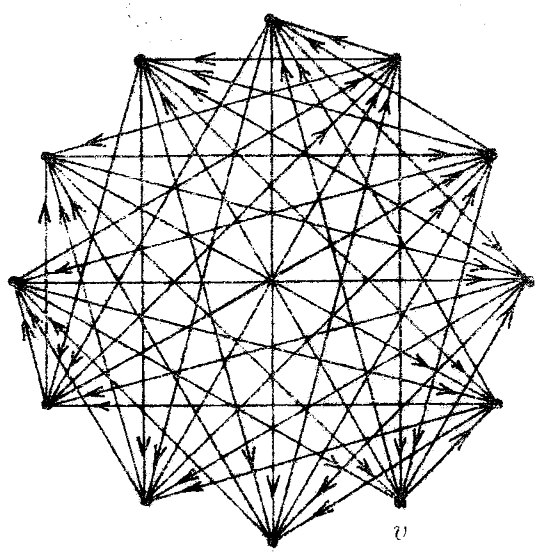

Figure 1

Let v be an arbitrary vertex of T, we first show that v is contained in a 3-cycle.

Suppose that v is not contained in any 3-cycle. Then it is clear that there are no arcs from $N^+(v)$ to $N^-(v)$. If $n \geqslant 5$, let u be a vertex in $N^-(v)$ such that $d^-_{T[N^-(v)]}(u) \leqslant (|N^-(v)|-1)/2$. Since v is not contained in any 3-cycle, $N^-(u) \backslash N^-(v) \subseteq V(v) \backslash \{v\}$. And then we have $s-1 \geqslant d^-(u) - (|N^-(v)|-1)/2$. That is, $(5-n)s \geqslant 6$, which is impossible since $n \geqslant 5$. Now suppose that $n=4$. Without loss of generality, we assume $v \in V_1$. Since T is regular, s=even. Let $s=2m$. suppose $d^-_{T[N-(v)]}(y) \geqslant m+1$ for each $y \in N^-(v)$. Let $a_i = |N^-(v) \cap V_{i+1}|$, where $i \in \{1,2,3\}$. Then we have

$$|A(T[N^-(v)])| \geqslant (a_1+a_2+a_3)(m+1) = (a_1+a_2+a_3)((a_1+a_2+a_3)/3+1).$$

On the other hand, $|A(T[N^-(v)])| = a_1a_2 + a_1a_3 + a_2a_3 \leqslant (a_1+a_2+a_3)^2/3 < (a_1+a_2+a_3) \times ((a_1+a_2+a_3)/3+1) \leqslant |A(T[N^-(v)])|$. Which is impossible. Hence there is $x \in N^-(v)$ such that $d^-_{T[N^-(v)]}(x) \leqslant m$. Since $N^-(x) \backslash N^-(v) \subseteq V(v) \backslash \{v\}$, $2m-1 = |V(v) \backslash \{v\}| \geqslant |N^-(x) \backslash N^-(v)| \geqslant (n-1)s/2 - m = 2m$, a contradiction.

All of above, when $n \geqslant 4$, v is always contained in a 3-cycle.

Let $C = v_1v_2 \cdots v_kv_1$ be a k-cycle containing v, where $v_1 = v$ and $3 \leqslant k \leqslant n-1$. Let

$$S = \{x \in V_i | V_i \cap V(C) = \varnothing, \quad 1 \leqslant i \leqslant n\},$$
$$A = \{x \in S | V(C) \Rightarrow x\},$$
$$B = \{x \in S | x \Rightarrow V(C)\},$$
$$X = S \backslash (A \cup B).$$

Clearly $S \neq \varnothing$, since $k < n$.

Case 1 $A \neq \varnothing$ or $B \neq \varnothing$.

We need only to assume $A \neq \varnothing$, otherwise we consider the converse of T. Let a be an arbitrary vertex of A. If there exists $x \in N^+(a)$ such that $x \to v_k$, then $v_1 v_2 \cdots v_{k-2} a x v_k v_1$ is a $(k+1)$-cycle containing v. If $v_k \Rightarrow N^+(a)$, then we have $d^+(v_k) \geqslant d^+(a) + |\{a, v_1\}|$, which contradicts the regularity of T. Hence there exists $y \in N^+(a)$ such that $V(y) = V(v_k)$. If $y \to v_1$, then $v_1 v_2 \cdots v_{k-1} a y v_1$ is a $(k+1)$-cycle containing v. So we have $v_1 \to y$. Moreover, if $y \to v_i$ for some $i \geqslant 3$, then $v_1 v_2 \cdots v_{i-2} a y v_i \cdots v_k v_1$ is a $(k+1)$-cycle containg v. So we have $y \to v_i$ for all $i \geqslant 3$. Note that $V(y) = V(v_k)$, if $v_k \Rightarrow N^+(y) \setminus V(C)$, then $d^+(v_k) \geqslant d^+(y) + |\{a, v_1\}| = d^+(y) + 2$, which contradicts the regularity of T. Hence there is $z \in N^+(y) \setminus V(C)$ such that $z \to v_k$. Now, if $k = 3$, then $v_1 y z v_k v_1$ is a 4-cycle containing v; If $k \geqslant 4$, then $v_1 v_2 \cdots v_{k-3} a y z v_k v_1$ is a $(k+1)$-cycle containing v.

Case 2　$A = \varnothing$ and $B = \varnothing$.

In this case we have $X = S$. Let $x \in X$, thus there are $v_i, v_j \in V(C)$ such that $v_i \to x$ and $x \to v_j$. By the definition of S, there is a $v_h \in V(C)$ such that $v_h \to x$ and $x \to v_{h+1}$. So $v_1 v_2 \cdots v_h x v_{h+1} \cdots v_k v_1$ is a $(k+1)$-cycle containing v.

This completes the proof of the Theorem.

References

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2　Guo Y and Volkmann L, Cycles in Multipartite tournaments, J. Combin. Theory, 1994, B(62): 363－366.

多步正则竞赛图中的圈

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摘要　本文证明了如下结果：若 T 为 n 步正则竞赛图($n>3$)．则 T 的每一个顶点都在 k-圈上，($2<k<n+1$)．若 $n=3$，上述结果不再正确，我们在文中给出了一个反例．

关键词　多步竞赛图，正则，圈．

分类号　O157.5.