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ON THE UNIQUE COUPLED COLORING OF PLANE GRAPHS*

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Abstract In this paper we prove that such graphs as modulus 3-regular maximal plane graphs and open maximal outerplane graphs are unique coupled colorable.

Key words Maximal plane graph, outerplane graph, unique coupled coloring.
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1 Introduction

Throughout this paper we shall consider simple connected plane graphs with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. Let $p(G)=|V(G)|$. For $u\in V(G)$, let N_u and F_u denote the set of vertices adjacent with u and the set of faces incident to u respectively. Similarly, for $f\in F(G)$, we denote by V_f and F_f, respectively, the set of vertices incident to f and the set of faces adjacent with f. A face f whose boundary, denoted by $b(f)$, contains vertices $u_1,u_2,\cdots,u_n$ is written as $f=u_1u_2\cdots u_n$.

A plane graph G is k-coupled colorable if the elements of $V(G)\cup F(G)$ can be colored with k colors such that any two distinct adjacent or incident elements receive different colors. The coupled chromatic number $\chi_{vf}(G)$ of G is defined as the minimum integer k for which G is k-coupled colorable. Obviouly, $\chi_{vf}(G)=2$ iff $E(G)=\varnothing$; $\chi_{vf}(G)=3$ iff G is a nonempty forest; $\chi_{vf}(G)\geqslant 4$ iff G contains at least one cycle different from a loop.

In 1968, G. Ringel conjectured [3] that $\chi_{vf}(G)\leqslant 6$ for any plane graph G. Recently, O. V. Borodin [2] proved this conjecture to be true. Moreover, an interesting result was due to D. Archdeacon [1]: If G is a bipartite plane graphs or an Eulerian plane graph, then $\chi_{vf}(G)\leqslant 5$. The purpose of the present paper is to study the unique coupled coloring of plane graphs.

Let σ denote a k-coupled coloring of G with a color set C. We denote by $\sigma(x)$ the color as-

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signed to the element $x \in V(G) \cup F(G)$ under σ. Two k-coupled colorings σ_1 and σ_2 of G are said to be equivalent if they partition $V(G) \cup F(G)$ into k same parts. If any two k-coupled coloring of G are equivalent, then G is said to be unique k-coupled colorable. By the definition, every nonempty tree T is unique 3-coupled colorable, because $V(T)$ is unique 2-vertex colorable and the only one face of T must be assigned to a color different from the colors which present on $V(T)$. For a cycle $C_n (n \geqslant 3)$, if $n \equiv 0 \pmod 2$, then $\chi_{vf}(G) = 4$ and C_n is unique 4-coupled colorable. Otherwise, $\chi_{vf}(G) = 5$, and C_n is not unique 5-coupled colorable.

A maximal plane graph G is said to be modulus 3-regular if $d_G(u) \equiv 0 \pmod 3$ for each vertex $u \in V(G)$. For an outerplane graph G, we call its unbounded face the outer face and denote it by $f_{out}(G)$, and other faces inner faces. An outerplane graph G is said to be maximal if each of inner faces of G is a triangle. An inner face f of G is called closed if f is not adjacent to $f_{out}(G)$. If each inner face of G is not closed, then we say that G is open.

2 Main Results

Lemma 1 Let G^* be the dual of a plane graph G. Then $\chi_{vf}(G^*) = \chi_{vf}(G)$, and G^* is unique k-coupled colorable if and only if G is unique k-coupled colorable.

Proof By the definition of G^*, the lemma is obvious.

Theorem 2 Every modulus 3-regular maximal plane graph is unique 4-coupled colorable.

Proof Let G be a modulus 3-regular maximal plane graph. We first prove that $\chi_{vf}(G) = 4$. SInce each face f of G is a triangle, then, for any χ_{vf}-coupled coloring of G, at least four different colors are required to color the elements of $V_f \cup \{f\}$, which implies that $\chi_{vf}(G) \geqslant 4$. Now let us construct a 4-coupled coloring σ of G with a color set $C = \{1, 2, 3, 4\}$ as follows:

Step 1 Choose a vertex $u \in V(G)$ with $N_u = \{x_1, x_2, \cdots, x_m\}$, where $m \equiv 0 \pmod 3$. We color u with 1, color alternately $x_1, x_2, \cdots, x_m$ with 2, 3 and 4, and color alternately ux_1x_2, $ux_2x_3, \cdots, ux_{m-1}x_m, ux_mx_1$ with 4, 2 and 3. Afterward, we say that the coloring for the vertex u has been finished.

Step 2 If the colorings for all vertices in $V(G)$ have been finished, then a 4-coupled coloring of G has been already formed. Otherwise, we take a $v \in V(G)$ satisfying that

(1) the coloring for v has not been finished; and

(2) there are at least three vertices in N_v which are consecutively adjacent in order and has been alternately colored with three different colors in C, say c_1, c_2, c_3.

Step 3 Color v with the color $c_4 \in C - \{c_1, c_2, c_3\}$. The other uncolored vertices in N_v are alternately colored with c_1, c_2 and c_3 in the same order. The faces in F_v are alternately colored with c_1, c_2 and c_3 without violating the coloring for $N_v \cup \{v\}$. Go to Step 2.

It is easy to see that the above procedure is feasible and will stop within finite steps. Thus we get a 4-coupled coloring of G. Therefore $\chi_{vf}(G) = 4$.

Next we prove that G is unique 4-coupled colorable. Let σ be any 4-coupled coloring of G

with a color set $C=\{1,2,3,4\}$. Without loss of generality, we may first color a face, say f_0, of G and its boundary vertex set $F_{f_0}=\{x_1,x_2,x_3\}$ by $\sigma(f_0)=4, \sigma(x_i)=i, i=1,2,3$. Suppose that $F_{f_0}=\{f_1,f_2,f_3\}$ such that $x_1x_2\in b(f_1)$, $x_2x_3\in b(f_2)$ and $x_3x_1\in b(f_3)$. Further let $y_1\in V_{f_1}\backslash\{x_1,x_2\}$, $y_2\in V_{f_2}\backslash\{x_2,x_3\}$ and $y_3\in V_{f_3}\backslash\{x_3,x_1\}$. Thus, based on the coloring of $V_{f_0}\cup\{f_0\}$, we must color f_1 with the unique color 3, similarly, f_2 with 1, f_2 with 2, and all y_1, y_2, y_3 with 4. Recursively, we can color one by one the remaining vertices and faces of G and the color selected in every step is unique. This implies that any two 4-coupled colorings of G must partition $V(G)\cup F(G)$ into four same color classes. Thus G is unique 4-coupled colorable.

Corollary 3 Let G be a maximal plane graph. Then $\chi_{vf}(G)=4$ if and only if G is modulus 3-regular.

Proof The sufficiency follows directly from Theorem 2. Now let us prove the necessity. Let G be a maximal plane graph that is not modulus 3-regular. Then there is a vertex $u\in V(G)$ such that $d_G(u)\not\equiv 0 \pmod 3$. Consider a wheel $W(u)$, a subgraph of G, with the center vertex u and the border vertices $v_1, v_2, \cdots, v_n$ in clockwise order, where $n=d_G(u)\geqslant 4$. Let σ be any χ_{vf}-coupled coloring of G with a color set C. Set $C^0=\{\sigma(v_i)\,|\,i=1,2,\cdots,n\}$. Clearly, $|C^0|\geqslant 2$. We consider two cases:

Case 1. If $|C^0|\geqslant 4$, then it follows from $\sigma(u)\notin C^0$ that $\chi_{vf}(G)\geqslant|C^0\cup\{\sigma(u)\}|\geqslant 5$, a contradiction.

Case 2. If $|C^0|\leqslant 3$, then since $d_G(u)\not\equiv 0 \pmod 3$ there must exist a color $\alpha\in C^0$ and an integer $k(1\leqslant k\leqslant n)$ such that $\sigma(v_k)=\sigma(v_{k+2})=\alpha$, (mod n). Hence we have

$$\chi_{vf}(G) \geqslant |\{\sigma(u),\sigma(v_k),\sigma(v_{k+1}),\sigma(uv_kv_{k+1}),\sigma(uv_{k+1}v_{k+2})\}| = 5,$$

which yields a contradiction yet.

Corollary 4 Let G be a 2-connected 3-regular plane graph. If $|V_f|\equiv 0 \pmod 3$ for each $f\in F(G)$, then $\chi_{vf}(G)=4$ and G is unique 4-coupled colorable.

Proof Note that G is the dual of a maximal modulus 3-regular plane graph. By Lemma 1 and Theorem 2, the corollary follows.

Lemma 5([4]) If G is an outerplane graph without cut vertices, then G contains at least two vertices of degree 2.

Lemma 6 If G is a maximal outerplane graph, then $\chi_{vf}(G)=5$, and there is a 5-coupled coloring of G satisfying the following property P:

P: Some color is only used to color f_{out}.

Proof By induction on p. If $p=3$, then $G\equiv K_3$, thus the theorem holds trivially. Assume the theorem is true for $p-1(p\geqslant 4)$. Let G be a maximal outerplane graph of order p. By Lemma 5, there is $u\in V(G)$ such that $d_G(u)=2$. Let x and y be two vertices in G adjacent to u. Since G is maximal, $xy\in E(G)$. Consider the graph $H=G-u$. Clearly H also is a maximal outerplane graph, and $|V(H)|=p-1$. By the induction assumption, H has a 5-coupled coloring λ with a color set C. Let f be the inner face of H with $xy\in b(f)$ and $f\neq$

$f_{out}(H)$. On the basis of λ, we form a 5-coupled coloring σ of G as follows:

$$\sigma(f_{out}(G)) = \lambda(f_{out}(H)),$$
$$\sigma(uxy) \in C - \{\lambda(x), \lambda(y), \lambda(f), \sigma(f_{out}(G))\},$$
$$\sigma(u) \in C - \{\lambda(x), \lambda(y), \sigma(uxy), \sigma(f_{out}(G))\}.$$

The other uncolored elements of $V(G) \cup F(G)$ are colored with the same colors as in λ. Thus $\chi_{vf}(G) \leqslant 5$.

On the other hand, since G contains at least one inner face f_0 that is not closed, five different colors are required to color properly the elements in $V_{f_0} \cup \{f_0, f_{out}\}$. So $\chi_{vf}(G) \geqslant 5$. Therefore we have $\chi_{vf}(G) = 5$.

Theorem 7 Let G be a maximal outerplane graph. Then G is unique 5-coupled colorable if and only if G is open.

Proof Let G be a maximal outerplane graph. By Lemma 6, $\chi_{vf}(G) = 5$. If G is not open, that is to say that G contains a closed inner face $\bar{f}$, we first can give, by Lemma 6, a 5-coupled coloring σ_1 of G which satisfies property P. Then we form another 5-coupled coloring σ_2 of G as follows: $\sigma_2(\bar{f}) = \sigma_2(f_{out}) = \sigma_1(f_{out})$; $\sigma_2(x) = \sigma_1(x)$ for each $x \in (V(G) \cup F(G)) \backslash \{\bar{f}, f_{out}\}$. Clearly σ_1 and σ_2 are not equivalent, which implies that G is not unique 5-coupled coloarable.

Conversely, assume that G is open. By the definition, each inner face of G is adjacent to f_{out}. This implies that for any 5-coupled coloring of G, there must exist a color which is only used to color f_{out}. Further, as analogous to the proof of Theorem 2, we can show that $V(G) \cup F(G) \backslash \{f_{out}\}$ is unique 4-coupled colorable. Hence G is unique 5-coupled colorable.

Remark Theorem 7 is not true for non-mximal outerplane graphs. For example, we consider a θ-graph $\bar{G}$(obtained by joining an edge between two nonadjacent vertices in a cycle of length more than 5). It is easily seen that $\bar{G}$ is an open outerplane graph and $\chi_{vf}(\bar{G}) = 5$. However, $\bar{G}$ is not unique 5-coupled colorable because $\bar{G}$ is not maximal.

Corollary 8 Every fan $F_p (p \geqslant 3)$ is unique 5-coupled colorable.

Proof Since F_p is a maximal open outerplane graph, by Theorem 7, the corollary follows.

Theorem 9 Let $W_p (p \geqslant 4)$ be a wheel of order p. Then W_p is unique χ_{vf}-coupled colorable if and only if $p \equiv 1 \pmod 3$.

Proof Let u be the center vertex of W_p, and write the vertices of $V(W_p) \backslash \{u\}$ as $x_1, x_2, \cdots, x_{p-1}$ in some order. If $p \equiv 1 \pmod 3$, we can form a 4-coupled coloring σ with a color set $C = \{1, 2, 3, 4\}$ as follows: $\sigma(u) = \sigma(x_1 x_2 \cdots x_{p-1}) = 4$; $x_1, x_2, \cdots, x_{p-1}$ are alternately colored with 1, 2 and 3; $ux_1x_2, x_{p-2}x_2x_3, \cdots, ux_{p-2}x_{p-1}, ux_{p-1}x_1$ are alternately colored with 3, 1 and 2. On the other hand, $\chi_{vf}(W_p) \geqslant 4$ is trivial. Thus $\chi_{vf}(W_p) = 4$. Further, it is easily seen that any two 4-coupled colorings of W_p are equivalent. Hence W_p is unique 4-coupled colorable.

If $p \not\equiv 1 \pmod 3$, it is easy to prove $\chi_{vf}(W_p) = 5$. Moreover, in this case, we can construct two 5-coupled colorings of W_p which are not equivalent. Therefore W_p is not unique 5-coupled colorable.

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关于平面图的唯一点面着色性

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摘要　本文证明了模 3 正则极大平面图和每个内面均与外面相邻的极大外平面图具有唯一的点面着色性.

关键词　极大平面图，外平面图，唯一点面着色性.

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