ON THE UNIQUE COUPLED COLORING OF PLANE GRAPHS

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Abstract In this paper we prove that such graphs as modulus 3-regular maximal plane graphs and open maximal outerplane graphs are unique coupled colorable.

Key words Maximal plane graph, outerplane graph, unique coupled coloring.

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1 Introduction

Throughout this paper we shall consider simple connected plane graphs with vertex set \( V(G) \), edge set \( E(G) \) and face set \( F(G) \). Let \( p(G) = |V(G)| \). For \( u \in V(G) \), let \( N_u \) and \( F_u \) denote the set of vertices adjacent with \( u \) and the set of faces incident to \( u \) respectively. Similarly, for \( f \in F(G) \), we denote by \( V_f \) and \( F_f \) respectively, the set of vertices incident to \( f \) and the set of faces adjacent with \( f \). A face \( f \) whose boundary, denoted by \( b(f) \), contains vertices \( u_1, u_2, \ldots, u_n \) is written as \( f = u_1 u_2 \ldots u_n \).

A plane graph \( G \) is \( k \)-coupled colorable if the elements of \( V(G) \cup F(G) \) can be colored with \( k \) colors such that any two distinct adjacent or incident elements receive different colors. The coupled chromatic number \( \chi_{cf}(G) \) of \( G \) is defined as the minimum integer \( k \) for which \( G \) is \( k \)-coupled colorable. Obviously, \( \chi_{cf}(G) = 2 \) iff \( E(G) = \emptyset \); \( \chi_{cf}(G) = 3 \) iff \( G \) is a nonempty forest; \( \chi_{cf}(G) \geq 4 \) iff \( G \) contains at least one cycle different from a loop.

In 1968, G. Ringel conjectured \([3]\) that \( \chi_{cf}(G) \leq 6 \) for any plane graph \( G \). Recently, O. V. Borodin \([2]\) proved this conjecture to be true. Moreover, an interesting result was due to D. Archdeacon \([1]\); If \( G \) is a bipartite plane graphs or an Eulerian plane graph, then \( \chi_{cf}(G) \leq 5 \). The purpose of the present paper is to study the unique coupled coloring of plane graphs.

Let \( \sigma \) denote a \( k \)-coupled coloring of \( G \) with a color set \( C \). We denote by \( \sigma(x) \) the color as-
signed to the element \( x \in V(G) \cup F(G) \) under \( \sigma \). Two \( k \)-coupled colorings \( \sigma_1 \) and \( \sigma_2 \) of \( G \) are said to be equivalent if they partition \( V(G) \cup F(G) \) into \( k \) same parts. If any two \( k \)-coupled coloring of \( G \) are equivalent, then \( G \) is said to be unique \( k \)-coupled colorable. By the definition, every nonempty tree \( T \) is unique \( 3 \)-coupled colorable, because \( V(T) \) is unique \( 2 \)-vertex colorable and the only one face of \( T \) must be assigned to a color different from the colors which present on \( V(T) \). For a cycle \( C_n (n \geq 3) \), if \( n \equiv 0 \, (\text{mod} \, 2) \), then \( \chi_{cf}(G) = 4 \) and \( C_n \) is unique \( 4 \)-coupled colorable. Otherwise, \( \chi_{cf}(G) = 5 \), and \( C_n \) is not unique \( 5 \)-coupled colorable.

A maximal plane graph \( G \) is said to be modulus \( 3 \)-regular if \( d_G(u) \equiv 0 \, (\text{mod} \, 3) \) for each vertex \( u \in V(G) \). For an outerplane graph \( G \), we call its unbounded face the outer face and denote it by \( f_{\text{out}}(G) \), and other faces inner faces. An outerplane graph \( G \) is said to be maximal if each of inner faces of \( G \) is a triangle. An inner face \( f \) of \( G \) is called closed if \( f \) is not adjacent to \( f_{\text{out}}(G) \). If each inner face of \( G \) is not closed, then we say that \( G \) is open.

2 Main Results

**Lemma 1** Let \( G^* \) be the dual of a plane graph \( G \). Then \( \chi_{cf}(G^*) = \chi_{cf}(G) \), and \( G^* \) is unique \( k \)-coupled colorable if and only if \( G \) is unique \( k \)-coupled colorable.

**Proof** By the definition of \( G^* \), the lemma is obvious.

**Theorem 2** Every modulus \( 3 \)-regular maximal plane graph is unique \( 4 \)-coupled colorable.

**Proof** Let \( G \) be a modulus \( 3 \)-regular maximal plane graph. We first prove that \( \chi_{cf}(G) = 4 \). Since each face \( f \) of \( G \) is a triangle, then, for any \( \chi_{cf} \)-coupled coloring of \( G \), at least four different colors are required to color the elements of \( V_f \cup \{ f \} \), which implies that \( \chi_{cf}(G) \geq 4 \). Now let us construct a \( 4 \)-coupled coloring \( \sigma \) of \( G \) with a color set \( C = \{ 1, 2, 3, 4 \} \) as follows:

**Step 1** Choose a vertex \( u \in V(G) \) with \( N_u = \{ x_1, x_2, \ldots, x_m \} \), where \( m \equiv 0 \, (\text{mod} \, 3) \). We color \( u \) with 1, color alternately \( x_1, x_2, \ldots, x_m \) with 2, 3 and 4, and color alternately \( ux_1, x_2, ux_2, x_3, \ldots, u x_{m-1}, x_m, u x_m x_1 \) with 4, 2 and 3. Afterward, we say that the coloring for the vertex \( u \) has been finished.

**Step 2** If the colorings for all vertices in \( V(G) \) have been finished, then a \( 4 \)-coupled coloring of \( G \) has been already formed. Otherwise, we take a \( v \in V(G) \) satisfying that

1. the coloring for \( v \) has not been finished; and
2. there are at least three vertices in \( N_v \) which are consecutively adjacent in order and has been alternately colored with three different colors in \( C \), say \( c_1, c_2, c_3 \).

**Step 3** Color \( v \) with the color \( c_4 \in C - \{ c_1, c_2, c_3 \} \). The other uncolored vertices in \( N_v \) are alternately colored with \( c_1, c_2 \) and \( c_3 \) in the same order. The faces in \( F_v \) are alternately colored with \( c_1, c_2 \) and \( c_3 \) without violating the coloring for \( N_v \cup \{ v \} \). Go to Step 2.

It is easy to see that the above procedure is feasible and will stop within finite steps. Thus we get a \( 4 \)-coupled coloring of \( G \). Therefore \( \chi_{cf}(G) = 4 \).

Next we prove that \( G \) is unique \( 4 \)-coupled colorable. Let \( \sigma \) be any \( 4 \)-coupled coloring of \( G \).
with a color set $C = \{1, 2, 3, 4\}$. Without loss of generality, we may first color a face, say $f_0$, of $G$ and its boundary vertex set $F_0 = \{x_1, x_2, x_3\}$ by $\sigma(f_0) = 4$, $\sigma(x_i) = i, i = 1, 2, 3$. Suppose that $F_0 = \{f_1, f_2, f_3\}$ such that $x_1x_2 \in b(f_1), x_2x_3 \in b(f_2)$ and $x_3x_1 \in b(f_3)$. Further let $y_1 \in V_{f_1} \setminus \{x_1, x_2\}, y_2 \in V_{f_2} \setminus \{x_2, x_3\}$ and $y_3 \in V_{f_3} \setminus \{x_3, x_1\}$. Thus, based on the coloring of $V_{f_0} \cup \{f_0\}$, we must color $f_1$ with the unique color 3, similarly, $f_2$ with 1, $f_3$ with 2, and all $y_1, y_2, y_3$ with 4. Recursively, we can color one by one the remaining vertices and faces of $G$ and the color selected in every step is unique. This implies that any two 4-coupled colorings of $G$ must partition $V(G) \cup F(G)$ into four same color classes. Thus $G$ is unique 4-coupled colorable.

**Corollary 3** Let $G$ be a maximal plane graph. Then $\chi_{cf}(G) = 4$ if and only if $G$ is modulus 3-regular.

**Proof** The sufficiency follows directly from Theorem 2. Now let us prove the necessity. Let $G$ be a maximal plane graph that is not modulus 3-regular. Then there is a vertex $u \in V(G)$ such that $d_G(u) \equiv 0 \pmod{3}$. Consider a wheel $W(u)$, a subgraph of $G$, with the center vertex $u$ and the border vertices $v_1, v_2, \ldots, v_n$ in clockwise order, where $n = d_G(u) \geq 4$. Let $\sigma$ be any $\chi_{cf}$-coupled coloring of $G$ with a color set $C$. Set $C^0 = \{\sigma(v_i) | i = 1, 2, \ldots, n\}$. Clearly, $|C^0| \geq 2$. We consider two cases:

Case 1. If $|C^0| \geq 4$, then it follows from $\sigma(u) \in C^0$ that $\chi_{cf}(G) \geq |C^0 \cup \{\sigma(u)\}| \geq 5$, a contradiction.

Case 2. If $|C^0| \leq 3$, then since $d_G(u) \equiv 0 \pmod{3}$ there must exist a color $a \in C^0$ and an integer $k (1 \leq k \leq n)$ such that $\sigma(v_k) = \sigma(v_{k+1}) = a, \pmod{n}$. Hence we have

$$\chi_{cf}(G) \geq |\{\sigma(u), \sigma(v_k), \sigma(v_{k+1}), \sigma(uv_kv_{k+1}), \sigma(uv_{k+1}v_k)\}| = 5,$$

which yields a contradiction yet.

**Corollary 4** Let $G$ be a 2-connected 3-regular plane graph. If $|V_f| \equiv 0 \pmod{3}$ for each $f \in F(G)$, then $\chi_{cf}(G) = 4$ and $G$ is unique 4-coupled colorable.

**Proof** Note that $G$ is the dual of a maximal modulus 3-regular plane graph. By Lemma 1 and Theorem 2, the corollary follows.

**Lemma 5** ([4]) If $G$ is an outerplane graph without cut vertices, then $G$ contains at least two vertices of degree 2.

**Lemma 6** If $G$ is a maximal outerplane graph, then $\chi_{cf}(G) = 5$, and there is a 5-coupled coloring of $G$ satisfying the following property $P$:

- Some color is only used to color $f_{\text{int}}$.

**Proof** By induction on $p$. If $p = 3$, then $G = K_3$, thus the theorem holds trivially. Assume the theorem is true for $p - 1 (p \geq 4)$. Let $G$ be a maximal outerplane graph of order $p$. By Lemma 5, there is $u \in V(G)$ such that $d_G(u) = 2$. Let $x$ and $y$ be two vertices in $G$ adjacent to $u$. Since $G$ is maximal, $xy \in E(G)$. Consider the graph $H = G - u$. Clearly $H$ also is a maximal outerplane graph, and $|V(H)| = p - 1$. By the induction assumption, $H$ has a 5-coupled coloring $\lambda$ with a color set $C$. Let $f$ be the inner face of $H$ with $xy \in b(f)$ and $f \neq$
On the basis of $\lambda$, we form a 5-coupled coloring $\sigma$ of $G$ as follows:

$$\sigma(f_{\text{out}}(G)) = \lambda(f_{\text{out}}(H)),$$
$$\sigma(uxy) \in C - \{\lambda(x), \lambda(y), \lambda(f), \sigma(f_{\text{out}}(G))\},$$
$$\sigma(u) \in C - \{\lambda(x), \lambda(y), \sigma(uxy), \sigma(f_{\text{out}}(G))\}.$$

The other uncolored elements of $V(G) \cup F(G)$ are colored with the same colors as in $\lambda$. Thus $\chi_{ef}(G) \leq 5$.

On the other hand, since $G$ contains at least one inner face $f_0$ that is not closed, five different colors are required to color properly the elements in $V_{f_0} \cup \{f_0, f_{\text{out}}\}$. So $\chi_{ef}(G) \geq 5$. Therefore we have $\chi_{ef}(G) = 5$.

**Theorem 7**  Let $G$ be a maximal outerplane graph. Then $G$ is unique 5-coupled colorable if and only if $G$ is open.

**Proof**  Let $G$ be a maximal outerplane graph. By Lemma 6, $\chi_{ef}(G) = 5$. If $G$ is not open, that is to say that $G$ contains a closed inner face $\overline{f}$, we first can give, by Lemma 6, a 5-coupled coloring $\sigma_1$ of $G$ which satisfies property P. Then we form another 5-coupled coloring $\sigma_2$ of $G$ as follows: $\sigma_2(\overline{f}) = \sigma_2(f_{\text{out}}) = \sigma_1(f_{\text{out}}); \sigma_2(x) = \sigma_1(x)$ for each $x \in (V(G) \cup F(G)) \setminus \{\overline{f}, f_{\text{out}}\}$. Clearly $\sigma_1$ and $\sigma_2$ are not equivalent, which implies that $G$ is not unique 5-coupled colorable.

Conversely, assume that $G$ is open. By the definition, each inner face of $G$ is adjacent to $f_{\text{out}}$. This implies that for any 5-coupled coloring of $G$, there must exist a color which is only used to color $f_{\text{out}}$. Further, as analogous to the proof of Theorem 2, we can show that $V(G) \cup F(G) \setminus \{f_{\text{out}}\}$ is unique 4-coupled colorable. Hence $G$ is unique 5-coupled colorable.

**Remark**  Theorem 7 is not true for non-maximal outerplane graphs. For example, we consider a $\theta$-graph $\overline{G}$ (obtained by joining an edge between two nonadjacent vertices in a cycle of length more than 5). It is easily seen that $\overline{G}$ is an open outerplane graph and $\chi_{ef}(\overline{G}) = 5$. However, $\overline{G}$ is not unique 5-coupled colorable because $\overline{G}$ is not maximal.

**Corollary 8**  Every fan $F_\rho(\rho \geq 3)$ is unique 5-coupled colorable.

**Proof**  Since $F_\rho$ is a maximal open outerplane graph, by Theorem 7, the corollary follows.

**Theorem 9**  Let $W_\rho(\rho \geq 4)$ be a wheel of order $\rho$. Then $W_\rho$ is unique $\chi_{ef}$-coupled colorable if and only if $\rho = 1 \pmod{3}$.

**Proof**  Let $u$ be the center vertex of $W_\rho$, and write the vertices of $V(W_\rho) \setminus \{u\}$ as $x_1, x_2, \ldots, x_{\rho-1}$ in some order. If $\rho = 1 \pmod{3}$, we can form a 4-coupled coloring $\sigma$ with a color set $C = \{1, 2, 3, 4\}$ as follows: $\sigma(u) = \sigma(x_1, x_2, \ldots, x_{\rho-1}) = 4; x_1, x_2, \ldots, x_{\rho-1}$ are alternately colored with 1, 2 and 3; $ux_1x_2, x_{\rho-2}x_2x_3, \ldots, ux_{\rho-2}x_{\rho-1}x_1$ are alternately colored with 3, 1 and 2. On the other hand, $\chi_{ef}(W_\rho) \geq 4$ is trivial. Thus $\chi_{ef}(W_\rho) = 4$. Further, it is easily seen that any two 4-coupled colorings of $W_\rho$ are equivalent. Hence $W_\rho$ is unique 4-coupled colorable.

If $\rho \not\equiv 1 \pmod{3}$, it is easy to prove $\chi_{ef}(W_\rho) = 5$. Moreover, in this case, we can con-
struct two 5-coupled colorings of $W$, which are not equivalent. Therefore $W_r$ is not unique 5-
coupled colorable.

References

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关于平面图的唯一点面着色性

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摘要 本文证明了模 3 正则极大平面图和每个内面均与外面相邻的极大外平面图具有唯一
的点面着色性。
关键词 极大平面图，外平面图，唯一点面着色性。
分类号 O157. 5.