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THE LIST CHROMATIC NUMBERS OF SOME PLANAR GRAPHS

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Abstract. In this paper, the choosability of outerplanar graphs, 1-tree and strong 1-outerplanar graphs have been described completely. A precise upper bound of the list chromatic number of 1-outerplanar graphs is given, and that every 1-outerplanar graph with girth at least 4 is 3-choosable is proved.

§1 Introduction

Throughout this paper, we only consider the finite undirected simple graphs. The terms and notations can be found in [1]. Given a graph $G=(V,E)$ in which each vertex v is assigned a list $L(v)$ of possible colors. If there is a vertex coloring φ such that $\varphi(v)\in L(v)$ for all $v\in V(G)$, we call G L-colorable and also say φ is an L-coloring of G. Given an integer k, G is called k-choosable if it is L-colorable for every assignment L with $|L(v)|=k$ for all $v\in V$. Finally the list chromatic number $\chi_l(G)$ of G is the smallest k such that G is k-choosable. Clearly, every k-choosable graph G is k-colorable and so $\chi(G)\leqslant\chi_l(G)$ holds.

The study of list coloring problems was initiated by Vizing[2] in 1976 and, independently but later, by Erdös, Rubin and Taylor[3] in 1979. During the last years, some new results were found about the choosability of planar graphs. Alon and Tarsi[4] proved that every bipartite graph is 3-choosable. In 1993 Thomassen[5] proved that every planar graph is 5-choosable whereas Voigt[6] presented an example of a planar graph which is not 4-choosable. And in 1995 Thomassen[7] showed that every planar graph with girth at least 5 is 3-choosable. In this paper, we characterize the choosability of some planar graphs.

Let G be a planar garph. If there exists a face f such that each vertex of G on the boundary of f, then G is called an outerplanar graph. The edges on the boundary of f are said to be outer edges and other edges inner edges, let ε_{in} denote the number of the inner edges. If there is a vertex $v_0\in V(G)$ such that $G-v_0$ is a forest, then G is called 1-tree. If there is a $v_0\in V(G)$ such that $G-v_0$ is an outerplanar graph, then G is called 1-outerplanar

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graph. If G is neither a 1-tree nor an outerplanar graph, and for each $v\in V(G)$, $G-v$ is an outerplanar graph, then G is called a strong 1-outerplanar graph. Let W_m be a wheel with order $m+1$. Let φ be a list coloring of G and for $S\subseteq V(G)$, we denote by $C_\varphi(S)$ a set of colors used on the vertices of $N_G(S)$ under φ.

§ 2 Main Results

Let $\mathscr{P}_k$ be a collection of simple graphs satisfying the following properties:

(1) For every $G\in\mathscr{P}_k$, $\delta(G)\leqslant k$;

(2) If $G\in\mathscr{P}_k$, then for every $H\subseteq G$, $H\in\mathscr{P}_k$.

Theorem 1. If $G\in\mathscr{P}_k(k\geqslant 1)$, then $\chi_l(G)\leqslant k+1$.

Proof. It is clear by induction on $v(G)$.

In order to get the main results, we need several facts and lemmas as follows:

(1) If G consists of n components $G_1, G_2, \ldots, G_n$, then $\chi_l(G)=\max\limits_{1\leqslant i\leqslant n}\{\chi_l(G_i)\}$.

(2) $\chi_l(G)=1$ iff G is a null graph.

(3) If $H\subseteq G$, then $\chi_l(H)\leqslant\chi_l(G)$.

Without loss of generality, we always consider that G is a simple connected planar graph with at least two vertices.

Lemma 1. Let C be an even cycle, then $\chi_l(C)=2$.

Proof. It is easy to prove this.

Lemma 2. Let T be a tree and $x\in V(T)$. Then there exists a 2-list coloring of T such that the color of x can be assigned in advance.

Proof. Let us construct an effective method for giving a 2-list coloring φ of T. The procedure is described as follows:

Step 1. Let $\varphi(x)\in L(x)$ and set $U=:\{x\}$.

Step 2. For every $y\in N_T(U)\setminus U$, let $y_1\in U$ be a vertex adjacent to y. We put $\varphi(y)\in L(y)\setminus\{\varphi(y_1)\}$.

Step 3. Set $U=:N_T(U)\cup U$. If $V(T)\setminus U=\varnothing$, stop. Otherwise, go to step 2.

Since T is finite, the above procedure must stop within finite steps. Thus a 2-list coloring φ of T is formed.

Lemma 3. [8] If G is an outerplanar graph, then $\delta(G)\leqslant 2$; and when $\delta(G)=2$, G contains at least two vertices of degree 2.

Theorem 2. Let G be an outerplanar graph. Then

$$\chi_l(G)=\begin{cases}2, & \text{if } G \text{ is a bipartite graph with at most one cycle;}\\ 3, & \text{Otherwise.}\end{cases}$$

Proof. Note that $\chi_l(G)\geqslant 2$ is trivial. By Lemma 3, $G\in\mathscr{P}_2$. Thus, using Theorem 1, we have $\chi_l(G)\leqslant 3$.

Now suppose that G is a bipartite graph with at most one cycle. If G contains no cycle, G is a tree and then, by Lemma 2, $\chi_l(G)=2$. Let G contain a cycle C, thus $G-E(C)$ is a forest. Since G is connected, every component of $G-E(C)$ contains exactly one vertex in $V(C)$. Now a 2-list coloring of G can be formed as follows: since C is even, by Lemma 1 we first give a 2-list coloring of C; based on this and by Lemma 2, we further color every tree of $G-E(C)$ attached at some vertex of $V(C)$. Thus $\chi_l(G)=2$.

Conversely, suppose $\chi_l(G)=2$. Thus it is obvious that G is a bipartite graph. Suppose that G contains at least two cycles, say $C_1=u_0u_1\ldots u_mu_0$ and $C_2=v_0v_1\ldots v_nv_0$. Since G is bipartite outerplanar, both C_1 and C_2 are even and one of the following cases must occur:

Case 1. $|V(C_1)\cap V(C_2)|=1$.

Let $u_0=v_0$ and set $L(u_1)=L(v_2)=\{1,3\}, L(v_1)=L(u_2)=\{2,3\}$ and $L(x)=\{1,2\}$ for all $x\in V(C_1\cup C_2)\backslash\{u_1,u_2,v_1,v_2\}$. If $\varphi(u_0)=1$, then $\varphi(u_1)=3, \varphi(u_2)=2, \varphi(u_3)=1,\ldots,$ $\varphi(u_{m-1})=2$. It follows that $C_\varphi(u_m)=L(u_m)=\{1,2\}$. Thus u_m can't be colored properly. If $\varphi(u_0)=2$, we can similarly obtain $C_\varphi(v_n)=L(v_n)=\{1,2\}$, which implies that v_n can't be colored properly. Therefore $\chi_l(G)\geqslant\chi_l(C_1\cup C_2)\geqslant 3$.

Case 2. $|V(C_1)\cap V(C_2)|=2$.

Let $u_0=v_0, u_m=v_n$. Assigning the same color-lists as in Case 1 to the vertices in $C_1\cup C_2$ and using a simliar argument, we have $\chi_l(G)\geqslant\chi_l(C_1\cup C_2)\geqslant 3$.

Case 3. $|V(C_1)\cap V(C_2)|=0$.

Since G is connected, there exists a path P in G connecting C_1 and C_2. Without loss of generality, let $P=u_0x_1x_2\ldots x_kv_0, k\geqslant 0$ and $x_1,\ldots,x_k\notin V(C_1\cup C_2)$.

3.1. If $k\equiv 1 \pmod 2$, we set $L(u_1)=L(v_2)=\{1,3\}, L(u_2)=L(v_1)=\{2,3\}$ and $L(x)=\{1,2\}$ for each $x\in V(C_1\cup C_2\cup P)\backslash\{u_1,u_2,v_1,v_2\}$. Since $|V(P)|=k+2$ is odd, we must use same color (i.e., 1 or 2) to u_0 and v_0. Thus we may identify the vertices u_0 and v_0, remove all internal vertices of P, and now this subcase is reduced to Case 1.

3.2. If $k\equiv 0 \pmod 2$, we set $L(u_1)=L(v_1)=\{1,3\}, L(u_2)=L(v_2)=\{2,3\}$ and $L(x)=\{1,2\}$ for each $x\in V(C_1\cup C_2\cup P)\backslash\{u_1,u_2,v_1,v_2\}$. Let φ be any 2-list coloring of $C_1\cup C_2\cup P$. If $\varphi(u_0)=1$, using the similar argument of Case 1, we have that u_m can't be colored properly. If $\varphi(u_0)=2$, then $\varphi(v_0)=1$ since $|V(P)|=k+2$ is even. It follows that v_m can't be colored properly. Hence $\chi_l(G)\geqslant\chi_l(C_1\cup C_2\cup P)\geqslant 3$.

Now, we always have $\chi_l(G)\geqslant 3$. This contradicts the assumption $\chi_l(G)=2$. Thus G contains at most one cycle.

Let $\overline{G}$ denote a graph such that $\overline{G}=C_1\cup C_2$ with $|V(C_1)\cap V(C_2)|=n-1$, where $C_i(i=1,2)$ are even cycles of length $n(\geqslant 4)$. Let G_0 be a graph containing $\overline{G}$ as an induced subgraph such that $G-E(\overline{G})$ is a forest.

Lemma 4. $\chi_l(G_0)=2$.

Proof. It is enough to prove $\chi_l(\overline{G})=2$ by Lemma 2. Let $C_1=u_1u_2\ldots u_{n-1}u_nu_1, C_2=u_1u_2\ldots u_{n-1}v_nu_1$ and $P=u_1u_2\ldots u_{n-1}$, where $u_n\neq v_n$. It is enough to consider the following three cases to form a 2-list coloring of $\overline{G}$.

Case 1. $L(u_n)=L(v_n)$.

By Lemma 1, there is a 2-list coloring φ of C_1. Since $N_{\overline{G}}(u_n)=N_{\overline{G}}(v_n)$ and $L(u_n)=L(v_n)$, then we put $\varphi(v_n)=\varphi(u_n)$. Hence there is a 2-list coloring of $\overline{G}$.

Case 2. $L(u_n)\cap L(v_n)=\varnothing$.

Suppose that $L(u_n)=\{1,2\}$ and $L(v_n)=\{3,4\}$. Let Φ denote a set of all (proper) 2-list colorings of P. Then $\Phi=\varnothing$ by Lemma 2. Moreover, we set $\Phi_{ij}=\{\varphi\in\Phi \mid \{\varphi(u_1),\varphi(u_{n-1})\}=\{i,j\}$, where $i\in L(u_1)$ and $j\in L(u_{n-1})$, or $i\in L(u_{n-1})$ and $j\in L(u_1)\}$. Now this case can be reduced to prove the following statement:

There is a $\varphi\in\Phi$ with $\{\varphi(u_1),\varphi(u_{n-1})\}\neq L(u_n)$ and $\{\varphi(u_1),\varphi(u_{n-1})\}\neq L(v_n)$. (*)

In fact, if (*) holds, then $L(u_n)\backslash C_\varphi(u_n)\neq\varnothing$ and $L(v_n)\backslash C_\varphi(v_n)\neq\varnothing$. Thus, we can put $\varphi(u_n)\in L(u_n)\backslash C_\varphi(u_n), \varphi(v_n)\in L(v_n)\backslash C_\varphi(v_n)$, and then a 2-list coloring of $\overline{G}$ is formed.

Suppose that (*) is not true. Thus $\Phi=\Phi_{12}\cup\Phi_{34}\neq\varnothing$. There are two possibilities:

2. 1. $\Phi_{12}\neq\varnothing$ and $\Phi_{34}\neq\varnothing$. Without loss of generality, we assume that $L(u_1)=\{1,3\}$ and $L(u_{n-1})=\{2,4\}$. Let $\varphi\in\Phi_{12}$ and $\psi\in\Phi_{34}$. Thus we have $\varphi(u_1)=1,\varphi(u_{n-1})=2,\psi(u_1)=3$, and $\psi(u_{n-1})=4$. Now we claim that $\varphi(u_{n-2})=4$. In fact, if $\varphi(u_{n-2})\neq 4$, we can define a new 2-list coloring φ_1 of P as follows: $\varphi_1(u_{n-1})=4, \varphi_1(u_i)=\varphi(u_i), i=1,2,\ldots,n-2$. Clearly, $\varphi_1\in\Phi_{14}$, which contradicts the fact $\Phi_{14}=\varnothing$. Similarly, we can deduce that $\psi(u_{n-2})=2$. This implies that $2,4\in L(u_{n-2})$. Noting that $|L(u_{n-2})|=2$, we have $L(u_{n-2})=\{2,4\}$. Furthermore, we must have $\varphi(u_{n-3})=2$. Otherwise we can construst a 2-list coloring φ_2 of P as follows: $\varphi_2(u_{n-1})=4,\varphi_2(u_{n-2})=2,\varphi_2(u_i)=\varphi(u_i),i=1,2,\ldots,n-3$. Thus $\varphi_2\in\Phi_{14}$, a contradiction. Using analogous argument, we get $\psi(u_{n-3})=4$. Thus $L(u_{n-3})=\{2,4\}$. Along this way, we obtain that $L(u_2)=\ldots=L(u_{n-1})=\{2,4\}$. Now let's put $\varphi^*(u_1)=1,\varphi^*(u_i)=2,i=2,4,\ldots,n-2,\varphi^*(u_j)=4,j=3,5,\ldots,n-1$. It is easily checked that φ^* is a 2-list coloring of P and then $\varphi^*\in\Phi_{14}$, a contradiction.

2. 2. $\Phi_{12}=\varnothing$, or $\Phi_{34}=\varnothing$, say $\Phi_{12}=\varnothing$. We claim that $L(u_1)=L(u_{n-1})=\{3,4\}$. Otherwise there exists a color $a(\notin\{3,4\})$ belonging to $L(u_1)$ or $L(u_{n-1})$. By Lemma 2, we can obtain a 2-list coloring φ of P such that $\varphi(u_1)$ or $\varphi(u_{n-1})$ equals a. So $\varphi\notin\Phi_{34}$, a contradiction. Therefore (*) is proved.

Case 3. $|L(u_n)\cap L(v_n)|=1$.

In this case, we suppose that $L(u_n)=\{1,2\}$ and $L(v_n)=\{1,3\}$. We can also prove the claim (*). Suppose that (*) is not true. Thus $\Phi=\Phi_{12}\cup\Phi_{13}\neq\varnothing$. There are two possibilities:

3. 1. $\Phi_{12}\neq\varnothing$ and $\Phi_{13}\neq\varnothing$. First, we claim that $1\in L(u_1)\cap L(u_{n-1})$. Otherwise we can suppose $L(u_1)=\{\alpha,\beta\}$ and $L(u_{n-1})=\{a,b\}$ where $1\notin\{\alpha,a,b\}$. By Lemma 2, we can obtain a 2-list coloring φ of P such that $\varphi(u_1)=\alpha$ and $\varphi(u_{n-1})\in\{a,b\}$. So $\varphi\notin\Phi_{12}\cup\Phi_{13}$, a contradiction. Hence we assume that $L(u_1)=\{1,2\}$ and $L(u_{n-1})=\{1,3\}$. Analogous to Case 2. 1, there is a 2-list coloring φ^* of P such that $\varphi^*\in\Phi_{11}$, a contradiction.

3.2. $\Phi_{12}=\emptyset$, or $\Phi_{13}=\emptyset$, say $\Phi_{12}=\emptyset$. Using an analogous argument as Case 2.2, we can also gain a contradiction. Hence (*) is proved.

Up to now, we have proved $\chi_l(\overline{G})\leqslant 2$. But $\chi_l(\overline{G})\geqslant 2$ is trivial. Therefore $\chi_l(\overline{G})=2$. The lemma is proved.

Theorem 3. Let G be a 1-tree. Then

$$\chi_l(G)=\begin{cases}2, & \text{if } G \text{ is a bipartite graph with (i) at most one cycle } C \text{ or} \\ & \quad \text{(ii) } G \text{ belongs to } \{G_0\} \text{ as above;} \\ 3, & \text{otherwise.}\end{cases}$$

Proof. Note that $\chi_l(G)\geqslant 2$ is trivial. Since any subgraph of a 1-tree is also a 1-tree, $G\in\mathscr{P}_2$. Thus we have $\chi_l(G)\leqslant 3$ by Theorem 1.

Since (i) implies that G is bipartite outerplanar, using Theorem 2, $\chi_l(G)=2$. For (ii), $\chi_l(G)=2$ by Lemma 4.

Conversely, suppose $\chi_l(G)=2$. Thus it is obvious that G is bipartite by contradiction. Since G is a 1-tree, one of the following cases must occur:

Case 1. G contains two cycles C_1 and C_2 such that there are $v_1,v_2\in V(C_1)\backslash V(C_2)$, $u_1,u_2\in V(C_2)\backslash V(C_1)$ with $v_1v_2, u_1u_2\in E(G)$. Since G is 1-tree, $V(C_1)\cap V(C_2)\neq\emptyset$. Let $C_1=v_1v_2\ldots v_nw_kw_{k-1}\ldots w_1v_1$, and $C_2=u_1u_2\ldots u_mw_kw_{k-1}\ldots w_1u_1$ where $k\geqslant 0$. Set $L(u_1)=L(v_2)=\{1,3\}$, $L(u_2)=L(v_1)=\{2,3\}$ and $L(x)=\{1,2\}$ for all $x\in V(C_1\cup C_2)\backslash\{u_1,u_2,v_1,v_2\}$. Note that C_1,C_2 are even. If $\varphi(w_1)=1$, then $\varphi(u_1)=3$, $\varphi(u_2)=2$, $\varphi(u_3)=1,\ldots,\varphi(w_3)=2$. It follows that $C_\varphi(w_2)=L(w_2)=\{1,2\}$. Thus w_2 can't be colored properly. If $\varphi(w_1)=2$, then $\varphi(v_1)=3$, $\varphi(v_2)=1$, $\varphi(v_3)=2,\ldots,\varphi(w_3)=1$. Also it follows that $C_\varphi(w_2)=L(w_2)=\{1,2\}$ and w_2 can't be colored properly. Therefore $\chi_l(G)\geqslant\chi_l(C_1\cup C_2)\geqslant 3$.

Case 2. G contains three cycles C_1,C_2 and C_3 such that $|V(C_1)|=|V(C_2)|=|V(C_3)|=n(n\geqslant 4)$ and $C_1\cap C_2\cap C_3$ is a path $P=v_1v_2\ldots v_{n-1}$ of the length $n-2$. Let $\{x_i\}=V(C_i)\backslash V(P)$, $i=1,2,3$. Set $L(x_1)=\{1,3\}$, $L(x_2)=\{2.4\}$, $L(x_3)=\{1,4\}$, $L(v_1)=\{1,2\}$, $L(v_2)=\{2,3\}$, and $L(x)=\{3,4\}$ for all $x\in V(C_1\cup C_2\cup C_3)\backslash\{x_1,x_2,x_3,v_1,v_2\}$. Since G is bipartite, C_i, $(i=1,2,3)$ are even. We have $|V(P)|\equiv 1 \pmod 2$. It is easily checked that any 2-list coloring φ of P must satisfy $\{\varphi(v_1),\varphi(v_{n-1})\}=\{1,3\}$ or $\{2,4\}$ or $\{1,4\}$. So there is an $i\in\{1,2,3\}$ such that $L(x_i)=C_\varphi(x_i)$. It follows that $\chi_l(G)\geqslant\chi_l(C_1\cup C_2\cup C_3)\geqslant 3$, which contradicts the assumption $\chi_l(G)=2$. Thus the theorem is proved.

Lemma 5. [8] If G is a 1-outerplanar graph, then $\delta(G)\leqslant 3$.

Theorem 4. If G is a 1-outerplanar graph, then $2\leqslant\chi_l(G)\leqslant 4$.

Proof. $\chi_l(G)\geqslant 2$ is trivial. Since any subgraph of 1-outerplanar graph is a 1-outerplanar graph, $G\in\mathscr{P}_3$ by Lemma 5. Thus using Theorem 1, we have $\chi_l(G)\leqslant 4$.

Corollary 4.1. If G is a 1-outerplanar graph, but neither outerplanar nor 1-tree, then $3\leqslant\chi_l(G)\leqslant 4$

Proof. Suppose $\chi_l(G)=2$, then G is a bipartite graph. We claim G satisfies the condition (i) or (ii) of Theorem 3. Otherwise G contains two cycle C_1 and C_2 and the following cases must occur since G is 1-outerplanar.

Case 1. $V(C_1)\cap V(C_2)=\varnothing$. Analogous to Case 3 of Theorem 2, we can deduce $\chi_l(G)\geqslant 3$.

Case 2. $V(C_1)\cap V(C_2)\neq\varnothing$. Analogous to Case 1 and Case 2 of Theorem 3, also we can deduce $\chi_l(G)\geqslant 3$.

Thus G satisfies the condition (i) or (ii) of Theorem 3, which implies that G is an outerplanar graph or a 1-tree, a contradiction.

Lemma 6. [9] G is an outerplanar graph iff G doesn't contain the subdivision of K_4 or $K_{2,3}$.

Lemma 7. If G is a strong 1-outerplanar graph with $\delta(G)=3$, then $\kappa(G)=3$.

Proof. First we prove $\kappa(G)\geqslant 2$. Suppose that there is $x\in V(G)$ such that $G-x$ has components $G'_1, G'_2, \ldots, G'_m$. Let $H'_1=G[V(G'_1)\cup\{x\}], H'_2=G[V(\bigcup_{i=2}^{m}G'_i)\cup\{x\}]$. Since G is strong 1-outerplanar, H'_1 and H'_2 are outerplanar graphs. Thus $H_1\cup H_2=G$ is outerplanar, a contradiction. So $\kappa(G)\geqslant 2$. Now suppose that there are $u,v\in V(G)$ such that $G-\{u,v\}$ has components $G_1, G_2, \ldots, G_n$. Let $H_i=G[V(G_i)\cup\{u,v\}], i=1,2$. Since G is strong 1-outerplanar, H_1 and H_2 are outerplanar. Thus $\delta(H_i)\leqslant 2(i=1,2)$ by Lemma 3. Since $\delta(G)=3$, $d_{H_i}(u)\leqslant 2$ or $d_{H_i}(v)\leqslant 2$, and $d_{H_i}(x)\geqslant 3$ for all $x\in V(G)\backslash\{u,v\}(i=1,2)$. Suppose $uv\in E(G)$, it is impossible since the outerplanar graph H_1 only contains two adjacent vertices of degree$\leqslant 2$. So we have $uv\notin E(G)$ and one of the following cases must occur:

Case 1. $\kappa(H_1)\geqslant 2$. It is clear that there are two internally vertex-disjoint (u,v)-paths $P_1, P_2\subseteq H_1$ with $\nu(P_i)\geqslant 3$.

Case 2. $\kappa(H_1)=1$. First, we claim that neither u nor v is a cut-vertex of H_1. Otherwise suppose H_1-u is disconnected, which follows that $G-u$ also is disconnected. This is a contradiction with $\kappa(G)\geqslant 2$. Then, we claim H_1 exactly contains two blocks B_1 and B_2 with $u\in V(B_1)$ and $v\in V(B_2)$. If not, there is a block B of H_1 with $V(B)\cap\{u,v\}=\varnothing$ by the above claim. There is a cut-vertex y of H_1 such that $G-y$ is disconnected where $y\in V(B)$, a contradiction. Let $V(B_1)\cap V(B_2)=z$. We can assume $\delta_{B_1}(z)\geqslant 2$ by $\delta_G(z)\geqslant 3$. If $uz\in E(H_1)$, the outerplanar graph B_1 with $\nu(B_1)\geqslant 3$ contains only two adjacent vertices of degree$\leqslant 2$, which is impossible. Hence there are two internally vertex-disjoint (u,z)-paths $P_1, P_2\subseteq B_1\subseteq H_1$ with $\nu(P_i)\geqslant 3(i=1,2)$.

Similarly, we can also prove there are two internally vertex-disjoint paths $P_3, P_4\subseteq H_2$ with $\nu(P_j)\geqslant 3(j=3,4)$. It follows that $G-w$ contains a subdivision of $K_{2,3}$ with $w\in V(P_4)\backslash\{u,v\}$, which contradicts Lemma 6. Therefore the lemma is proved.

Theorem 5. Let G be a strong 1-outerplanar graph with $\delta(G)=3$. Then G is isomorphic to a wheel or a triangular prism.

Proof. Since $\delta(G)=3, \nu(G)\geqslant 4$. Take any $v\in V(G)$ such that $d_G(v)=\triangle(G)$. By Lemma 7, $G-v$ is a 2-connected outerplanar graph. Thus $G-v$ contains a Hamilton cycle $C=$

$v_1v_2\ldots v_mv_1$ $(m=\nu(G-v)\geqslant 3)$ as a boundary of the exterior face. If $m=3$, then $vv_i\in E(G), i=1,2,3$ because $d_G(v)=3$. So G is a 3-wheel. Otherwise $m\geqslant 4$, we have one of the following cases:

Case 1. $\triangle(G)=m$.

If C contains an inner edge $v_iv_j(j>i)$, there is a $v_k\in V(C)$ with $i<k<j$ such that $G-v_k$ contains a subdivision of K_4. This contradicts the definition of G by Lemma 6. Thus G doesn't contain any inner edge. In this case G is isomorphic to a m-wheel.

Case 2. $3\leqslant\triangle(G)\leqslant m-1$.

Since $\delta(G)=3$ and $\triangle(G)\leqslant m-1$, C must contain an inner edge $v_iv_j(j>i)$. Let $C_1=v_iv_{i+1}\ldots v_{j-1}v_jv_i$ and $C_2=v_jv_{j+1}\ldots v_m\ldots v_{i-1}v_iv_j$. We consider several subcases below.

2.1. $\triangle(G)\geqslant 5$.

In fact, we have $\max\{|V(C_1)\cap N_G(v)|, |V(C_2)\cap N_G(v)|\}\geqslant\lceil\frac{\Delta(G)}{2}\rceil\geqslant 3$, say $|V(C_1)\cap N_G(v)|\geqslant 3$. Thus, for any $v_0\in V(C_2)\backslash\{v_i,v_j\}, G-v_0$ contains a subdivision of K_4, a contradiction.

2.2. $\triangle(G)=3$.

Since $\delta(G)=3$, G is a 3-regular planar graph and thus $\nu(G)\equiv 0 \pmod 2$. By the above discussion we may assume $\nu(G)\geqslant 6$. If $\nu(G)=6$, we have obviously $N_G(v)=V(C)\backslash\{v_i, v_j\}$. So G becomes isomorphic to a triangular prism. If $\nu(G)\geqslant 8$, $\varepsilon_{in}\geqslant\frac{\nu(G)-1-3}{2}\geqslant 2$. Thus G contains the other inner edge v_sv_t such that $s,t\notin\{i,j\}$ by $\triangle(G)=3$. Let $\{v_s,v_t\}\subseteq V(C_1)$, where $t>s$. G contains a cycle $C_3=v_sv_{s+1}\ldots v_{t-1}v_tv_s$ with $V(C_2)\cap V(C_3)=\varnothing$. By Lemma 7, there are three vertex-disjoint paths from v to C_3. Since $G-v$ is outerplanar, there is $v_k\in N_G(v)$ such that $v_k\in V(C_3)\backslash\{v_s,v_t\}$. Similarly, there is $v_l\in V(C_2)\backslash\{v_i,v_j\}$. Suppose that $v_h=N_G(v)\backslash\{v_k,v_l\}\notin V(C_2)\cup V(C_3)$, then $G-v_i$ contains a subdivision of $K_{2,3}$, a contradiction. So $v_h\in V(C_2)\cup V(C_3)$. We can assume $v_h\in V(C_2)$. Then $G-v_s$ contains a subdivision of $K_{2,3}$, a contradiction too.

2.3. $\triangle(G)=4$.

First it is easily seen that $|N_G(v)\cap V(C_1)|\leqslant 2$ and $|N_G(v)\cap V(C_2)|\leqslant 2$ since any subdivision of $K_4\nsubseteq G$. Thus there must exist $v_k,v_l\in(N_G(v)\cap V(C_1))\backslash\{v_i,v_j\}$ and $v_s,v_t\in(N_G(v)\cap V(C_2))\backslash\{v_i,v_j\}$. So $G-\{v_k\}$ contains a subdivision of K_4, a contradiction.

Up to now, the theorem is proved.

Lemma 8. If G is a triangular prism, then $\chi_l(G)=3$.

Proof. Let $C_1=u_1u_2u_3u_1$ and $C_2=v_1v_2v_3v_1$ are two 3-cycles of G with $u_iv_i\in E(G), i=1,2,3$. One of the following cases must occur:

Case 1. $L(u_1)\cap L(v_2)\neq\varnothing$.

A 3-list coloring φ of G can be formed as follows: $\varphi(u_1)=\varphi(v_2)\in L(u_1)\cap L(v_2)$, $\varphi(v_3)\in L(v_3)\backslash\{\varphi(v_2)\}$, $\varphi(u_3)\in L(u_3)\backslash\{\varphi(v_3),\varphi(u_1)\}$, $\varphi(v_1)\in L(v_1)\backslash\{\varphi(u_1),\varphi(v_3)\}$, $\varphi(u_2)\in L(u_2)\backslash\{\varphi(u_1),\varphi(u_3)\}$.

Case 2. $L(u_1)\cap L(v_2)=\varnothing$. In this case we have two possibilities.

2.1. $L(u_1)\backslash L(v_1)\neq\varnothing$.

A 3-list coloring φ of G can be formed as follows: $\varphi(u_1)\in L(u_1)\backslash L(v_1)$, $\varphi(u_2)\in L(u_2)\backslash\{\varphi(u_1)\}$, $\varphi(u_3)\in L(u_3)\backslash\{\varphi(u_1),\varphi(u_2)\}$, $\varphi(v_2)\in L(v_2)\backslash\{\varphi(u_2)\}$, $\varphi(v_3)\in L(v_3)\backslash\{\varphi(v_2),\varphi(u_3)\}$, $\varphi(v_1)\in L(v_1)\backslash\{\varphi(v_2),\varphi(v_3)\}$.

2.2. $L(u_1)=L(v_1)$, thus $L(v_2)\backslash L(v_1)\neq\varnothing$.

A 3-list coloring φ of G can be formed as follows: $\varphi(v_2)\in L(v_2)\backslash L(v_1)$, $\varphi(v_3)\in L(v_3)\backslash\{\varphi(v_2)\}$, $\varphi(u_3)\in L(u_3)\backslash\{\varphi(v_3)\}$, $\varphi(u_2)\in L(u_2)\backslash\{\varphi(v_2),\varphi(u_3)\}$, $\varphi(u_1)\in L(u_1)\backslash\{\varphi(v_2),\varphi(u_3)\}$, $\varphi(v_1)\in L(v_1)\backslash\{\varphi(u_1),\varphi(v_3)\}$.

Up to now, we have proved $\chi_l(G)\leqslant 3$. But $\chi_l(G)\geqslant\chi(C_1)=3$ is trivial, therefore we have $\chi_l(G)=3$.

Lemma 9. For a wheel $W_m (m\geqslant 3)$, we have

$$\chi_l(W_m)=\begin{cases}3, & \text{if } m\equiv 0 \pmod 2;\\ 4, & \text{otherwise.}\end{cases}$$

Proof. If $m\equiv 1 \pmod 2$, $\chi_l(W_m)=\chi(W_m)=4$ by Theorem 4. Otherwise, let w be the center of W_m, W_m-w be a cycle $C=u_1u_2\ldots u_mu_1$. For any $u,v\in V(W_m)$, $L(u)=L(v)$, then a 3-list coloring φ of G can be formed as follows: $\varphi(v)=c(v)$ for $v\in V(G)$ where c is a 3-coloring of W_m. Otherwise, there are $u,v\in V(W_m)$ such that $L(u)\neq L(v)$ and $uv\in E(G)$. We can assume that $L(w)\neq L(u_1)$. A 3-list coloring φ of W_m can be formed as follows: $\varphi(w)\in L(w)\backslash L(u_1)$, $\varphi(u_2)\in L(u_2)\backslash\{\varphi(w)\}$, $\varphi(u_i)\in L(u_i)\backslash\{\varphi(u_{i-1}),\varphi(w)\}$, $i=1,\ldots,m$. Since $N_G(u_1)=\{u_2,u_m,w\}$ and $\varphi(w)\notin L(u_1)$, we have $L(u_1)\backslash C_\varphi(u_1)\neq\varnothing$. So let $\varphi(u_1)\in L(u_1)\backslash C_\varphi(u_1)$. Therefore the lemma is proved.

Theorem 6. Let G be a strong 1-outerplanar graph, then

$$\chi_l(G)=\begin{cases}4, & \text{if } G \text{ is } W_m \text{ with } m \text{ odd};\\ 3, & \text{otherwise.}\end{cases}$$

Proof. Since G is a strong 1-outerplanar graph, $3\leqslant\chi_l(G)\leqslant 4$ by Corollary 4.1. Thus by Theorems 2 and 5, Lemmas 5, 8 and 9, the theorem is easily proved.

Lemma 10. [8] If G is an outerplanar graph with $\delta(G)=2$, then at least one of the following cases is true:

(1) There are two adjacent vertices of degree 2.

(2) There is a vertex v of degree 2 on a 3-cycle.

Theorem 7. Let G be a 1-outerplanar graph with $g(G)\geqslant 4$, where $g(G)$ denotes the girth of G. Then $\chi_l(G)\leqslant 3$.

Proof. We first prove that $\delta(G)\leqslant 2$. In fact, if $\delta(G)\geqslant 3$, then by Lemma 5, we have $\delta(G)=3$. By virtue of the definition of 1-outerplanar graph, there is a vertex $x\in V(G)$ such that $G-x$ is an outerplanar graph. Obviously, $\delta(G-x)\geqslant\delta(G)-1=2$, thus by Lemma 3, $\delta(G)=2$. Since $g(G-x)\geqslant g(G)\geqslant 4$, it follows easily that Case 1 of Lemma 10 holds only. This means that $G-x$ contains two adjacent vertices u and v of degree 2. If

$ux, vx \in E(G)$, we have $g(G)=3$, which is impossible. Hence at most one of ux and vx belongs to $E(G)$. Therefore $\delta(G)=2$, a contradiction. Next, that any subgraph H of G is still a 1-outerplanar graph with $g(H) \geqslant g(G) \geqslant 4$ deduces $G \in \mathscr{P}_2$. By Theorem 1, we have $\chi_l(G) \leqslant 3$.

References

1 Bondy, J. A., Murty, U. S. R., Graph Theory with Applications, The Macmillian Press Limit, new York, 1976.

2 Vizing, V. G., Coloring the vertices of a graph in prescribed colors, Diskret Analiz, 1976, 29: 3～10. (in Russian)

3 Erdös, P., Rubin, A. L. and Taylor, H., Choosability in graphs, Congr. Numer., 1980, 26: 122～157.

4 Alon, N., Tarsi, M., Colorings and orientations of graphs, Combinatorica, 1992, 12(2): 125～134.

5 Thomassen, C., Every planar graph is 5-choosable, J. Combin. Theory Ser. B, 1994, 62(1): 180～181.

6 Voigt, M., List colourings of planar graphs, Discrete Math., 1993, 120: 215～219.

7 Thomassen, C., 3-list coloring planar graphs of girth 5, J. Combin. Theory Ser. B, 1995, 64(1): 101～107.

8 Wang Weifan, Equitable colorings and total colorings of graphs. [Doctoral Thesis], Najing: Nanjing University, 1997.

9 Tian Feng and Ma Zhongfan, Graphs and Network Flows, Science Press, Beijing, 1987.

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