Completely Strong Path-Connectivity of Local Tournaments*

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ABSTRACT. Let $T = (V, A)$ be an oriented graph with n vertices. T is completely strong path-connected if for each arc $(a, b) \in A$ and k ($k = 2, \dots, n-1$), there is a path from b to a of length k (denoted by $P_k(a, b)$) and a path from a to b of length k (denotod by $P'_k(a, b)$) in T. In this paper, we prove that a connected local tournament T is completely strong path-connected iff for each arc $(a, b) \in A$, there exist $P_2(a, b)$ and $P'_2(a, b)$ in T, and $T \not\cong T_0 - D'_8$-type digraph and D_8.

1 Introduction

Let $T = (V, A)$ be an oriented graph with n vertices. If an arc $(x, y) \in A$, then we say that x dominates y, denoted by $x \to y$. If S_1 and S_2 are disjoint subsets of V such that there is a complete connection between them and all arcs between them are directed toward S_2, we say that S_1 dominates S_2, denoted by $S_1 \to S_2$. We write $x \to S_2$ (resp., $S_2 \to x$) instead of $\{x\} \to S_2$ (resp., $S_2 \to \{x\}$). For $x \in V$, we define $O(x) = \{y \mid y \in V, (x, y) \in A\}$, $I(x) = \{y \mid y \in V, (y, x) \in A\}$.

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T is arc-k-cyclic if each arc $(a, b) \in A$, there is a path from b to a of length $k-1$ in T. T is arc-pancyclic (resp., arc-antipancyclic) if for each arc $(a, b) \in A$, there is a path from b to a (resp., from a to b) of length k ($k = 2, 3, \ldots, n-1$) in T, denoted by $P_k(a, b)$, or briefly P_k (resp., $P'_k(a, b)$, P'_k). An oriented graph T is completely strong path-connected if T is arc-pancyclic and arc-antipancyclic. Other notations and terminologies not defined in this paper can be found in [3].

A local tournament T is an oriented graph such that $T[O(x)]$ and $T[I(x)]$ are tournaments for every vertex x in T. Local tournaments were first introduced by J. Bang-Jensen [1], [2]. Clearly, tournaments is a special class of local tournaments. In [1], [2], it was shown that every connected local tournament has a Hamiltonian path, and every strong local tournament has a Hamiltonian cycle. Many other results for tournaments are also shown for local tournaments. In this paper, Zhang and Wu's results in [5] and [6] are extended. We get the following main result.

Theorem. *Let $T = (V, A)$ be a connected local tournament with n vertices ($n \geq 3$). If for each arc $(a, b) \in A$, there exist $P_2(a, b)$ and $P'_2(a, b)$ in T. Then T is completely strong path-connected, except $T \simeq T_0$- or D'_8-type digraph or D_8. (see Figures 1, 2 and 3).*

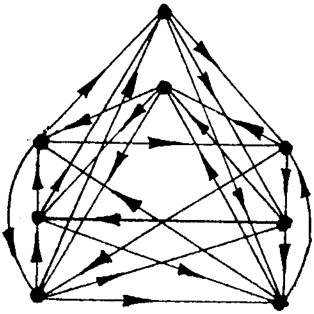


Figure 1. D_8

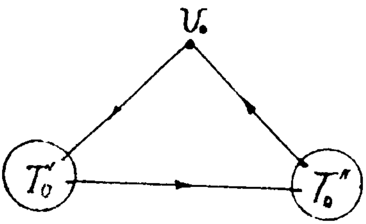


Figure 2. T_0-type digraph.
(Here T'_0, T''_0 are tournaments)

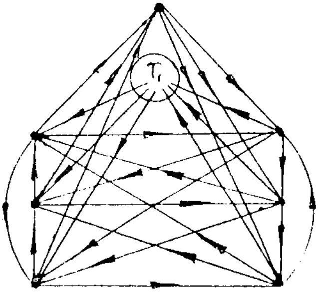


Figure 3. D'_8-type digraph.
(Here T_1 is a tournament)

Immediately we have,

Corollary. ([5], Theorem 1) *A tournament $T = (V, A)$ with n vertices is completely strong path-connected if and only if for each arc $(a, b) \in A$, there exist $P_2(a, b)$, and $P'_2(a, b)$ in T, and $T \not\cong T_0$-type digraph.*

2 The Proof of the Theorem

In order to prove the Theorem, we need the following lemmas.

Lemma 1. *Let $T = (V, A)$ be a connected local tournament. For each arc $(a, b) \in A$, there exist $P_2(a, b)$ and $P'_2(a, b)$ in T, then there exists a cycle in the induced subgraph $T[O(x_0)]$ (resp., $T[I(x_0)]$ for any $x_0 \in V$. Furthermore, $|O(x_0)| \geq 3$, (resp., $|I(x_0)| \geq 3$).*

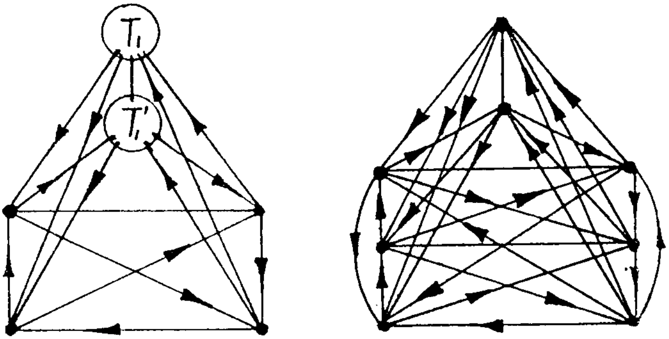
Lemma 2. *Let $T = (V, A)$ be a connected local tournament. For each arc $(a, b) \in A$, there exist $P_2(a, b)$ and $P'_2(a, b)$ in T, then there always exists a $P'_k(a, b)$ in T for each arc $(a, b) \in A$ $(k = 2, 3, \ldots, 6)$.*

By the definition of a local tournament, the proof of Lemma 1 and Lemma 2 is an analogous to the proof of Lemma 1 and Lemma 3 in [7].

Lemma 3. ([4] Theorem 1) *Except for T_6-, T_8-type digraphs and D_8 (see Figures 1 and 5), every arc-3-cyclic connected local tournament is arc-pancyclic.*

The proof of the Theorem.

Let $T = (V, A)$ be a connected local tournament of order n $(n \geq 3)$ such that for each arc $(a, b) \in A$, there exist $P_2(a, b)$ and $P'_2(a, b)$ in T. For T_6- or T_8-type digraph, it is easy to find that there exists a vertex x such that $|O(x)| = 2$. So T is not a T_6- or a T_8-type digraph by Lemma 1. Hence by Lemma 3 T is an arc-panyclic local tournament except T is isomorphic to D_8. And by Lemma 2 there always exists a $P'_k(a,b)$ in T for $k \leq 6$. Therefore it is enough to prove the following.



T_6-type digraph (where T_1, T_1' are tournaments)

T_8-type digraph

Figure 5.

The directions of the edges without arrow can be chosen arbitrary.

Proposition. *Suppose T is not isomorphic to a T_0- or D'_8-type digraph or D_8. If for each arc $(a,b) \in A$ and k $(7 \leq k \leq n-1)$, there exists a $P'_{k-1}(a,b)$ in T. Then there exists a $P'_k(a,b)$ in T.*

From now on, we shall assume that there is a $P'_{k-1}(a,b)$ in T, and denote it by $(1,2,\ldots,k)$, where $a=1$ and $b=k$. The set of vertices $\{1,2,\ldots,k\}$ of $P'_{k-1}(a,b)$ is also denoted by P'_{k-1}. Let $W = V - P'_{k-1}$. Hence $|W| \geq 1$. For any $w \in W$ we define

$$O'(w) \equiv O(w) \cap P'_{k-1}, \quad I'(w) \equiv I(w) \cap P'_{k-1}.$$

When $O'(w) \neq \emptyset$ and $I'(w) \neq \emptyset$ for $w \in W$, set

$$a(w) = \max\{i \mid i \in O'(w)\}, \quad b(w) = \min\{i \mid i \in I'(w)\}.$$

If the condition of the proposition were false, we should assume that

$$\text{There does not exist any } P'_k(a,b) \text{ in } T. \qquad (\star)$$

By the assumption above, we may obtain the following claims.

(1) $O'(w) = \{1,2,\ldots,a(w)\}$ and $a(w) < k$ as $O'(w) \neq \emptyset$. Similarly, $I'(w) = \{b(w),\ldots,k\}$ and $b(w) > 1$ as $I'(w) \neq \emptyset$.

Suppose $O(w) \neq \emptyset$. If there exists an $i \in O'(w)$ with $i-1 \notin O'(w)$, by the definition of a local tournament and $\{w, i-1\} \subseteq I(i)$, then $i-1$ and w are adjacent in T. Thus $i-1 \to w$ by the definition of i. Hence there is a $P'_k(a,b) = (1,\dots,i-1,w,i,\dots,k)$ in T. This contradicts $(\star)$. So $O'(w) = \{1,2,\dots,a(w)\}$. And if $a(w) = k$, then $w \to P'_{k-1}$. Note that there exists a $P_2(w,1) = (1,x,w)$. Clearly $x \notin P'_{k-1}$. Hence $x \in W$. Thus T contains a $P'_k(a,b) = (1,x,w,3,\dots,k)$. This contradicts $(\star)$. So $a(w) < k$.

(2) For any $w \in W$, $O'(w) \neq \emptyset$ if and only if $I'(w) \neq \emptyset$.

If $O'(w) \neq \emptyset$, there is a $P_2(w,1) = (1,x,w)$. If $x \in W$, then $1 \in I'(x)$ and $b(x) = 1$. This contradicts $b(w) > 1$ by (1). Hence $x \in I'(w)$ and $I'(w) \neq \emptyset$. Similarly, if $I'(w) \neq \emptyset$, then $O'(w) \neq \emptyset$.

(3) Let $W_1 = \{w \mid w \in W, O'(w) \neq \emptyset\}$ and $W_2 = W - W_1$ then $W_2 = \emptyset$. Furthermore, $T[W]$ is a tournament and $O'(w) \neq \emptyset$, $I'(w) \neq \emptyset$ for every $w \in W$.

Since T is connected, $W_1 \neq \emptyset$. Suppose $W_2 \neq \emptyset$. Let $w_1 \in W_1$ and $w_2 \in W_2$ such that w_1 and w_2 are adjacent. Without loss of generality,we assume $w_2 \to w_1$. Since $k \to w_1$ by (1) and (2), w_2 and k are adjacent. Then $k \to w_2$ and $O'(w_2) \neq \emptyset$ by (1) and (2). This is a contradiction. Hence $W_2 = \emptyset$. i.e. $W = W_1$.

From (1) and (2), we have $W \subseteq I(1)$ and $O'(w) \neq \emptyset$, $I'(w) \neq \emptyset$ for every $w \in W$. Thus $T[W]$ is a tournament by the definition of a local tournament.

(4) $b(w) = b(w')$ and $a(w) = a(w')$ for any $w, w' \in W$.

Suppose there are $w, w' \in W$ such that $b(w) \neq b(w')$. Set $b(w_0) = \min\{b(w) \mid w \in W\}$. Let $W_3 = \{w \mid w \in W, b(w) > b(w_0)\}$ and $W_4 = W - W_3$. Then $W_3 \neq \emptyset$, $W_4 \neq \emptyset$ and $b(w_0) = b(w) \to w$ for any $w \in W_4$.

Case 1. There exist $w_3 \in W_3$ and $w_4 \in W_4$ such that $w_3 \to w_4$.

Since $b(w_4) = b(w_0) < b(w_3)$ and $b(w_3) - 1 \to w_4$, w_3 and $b(w_3) - 1$ are adjacent by $w_3 \to w_4$ and the definition of a local tournament. From (1) we have $w_3 \to b(w_3) - 1$. Thus $a(w_3) = b(w_3) - 1$. Similarly, since $b(w_4) - 1 < b(w_3) - 1 = a(w_3)$ and $w_3 \to w_4$, we have $a(w_4) = b(w_4) - 1$.

Now we need the following three Lemmas

Lemma 4. *There are no u, v, n and m in P'_{k-1} such that $u < n \leq b(w_4) - 1 < b(w_3) \leq v < m$ and $(u,v), (n,m) \in A$.*

Proof: Otherwise, it will contradict $(\star)$. □

Now, $(n,m), (u,v) \in A$ are called cis-crosswise arcs with respect to the P'_{k-1} (briefly cis-crosswise arcs) if n, m, u and v are on P'_{k-1} such that $u < n < v < m$.

Lemma 5. (a) *For each $i \in \{3,4,\dots,b(w_4)-1\}$, we have $(i,1) \in A$.* (b) *For each $j \in \{b(w_3),\dots,k-2\}$, we have $(k,j) \in A$.*

Proof: (a) Since $\{1,2,\dots,b(w_4)-1 = a(w_4)\} \subseteq O(w_4)$, $T[\{3,4,\dots,b(w_4)-1,1\}]$ is a tournament. If there is an $i_0 \in \{3,4,\dots,b(w_4)-1\}$ such that $1 \to i_0$, then $w_3 \to i_0 - 1$ by $i_0 - 1 < a(w_3)$. There is a $P_2(w_3, i_0 - 1) =$

(i_0-1, u, w_3). By the definition of $b(w_4)$, we have $u \notin W$. Hence $u \in I'(w_3)$. Thus there is a $P'_k(a,b) = (1, i_0, \dots, u-1, w_4, 2, \dots, i_0-1, u, \dots, k)$. This contradicts $(\star)$. So (a) is valid.

An analogous proof of (a), we have that (b) is true. □

Lemma 6. *If $(b(w_4)-1, b(w_3)) \in A$ and $(b(w_4)-1, b(w_3)) \neq (a,b) = (1,k)$, then there is an arc $(u,v) \in A$ such that (u,v) and $(b(w_4)-1, b(w_3))$ are cis-crosswise arcs.*

Proof: By Lemmas 1, 4 and 5, using an analogous proof of Lemma 3 in [5], Lemma 6 follows. □

Now, let's back to discuss case 1.

There are $P_2(w_3, a(w_4) = b(w_4)-1) = (b(w_4)-1, m, w_3)$ and $P_2(w_3, w_4) = (w_4, w_5, w_3)$, where $m \notin W$ and $w_5 \notin P'_{k-1}$, by the choice of w_4 and $b(w_3) > b(w_4)$. Hence we have that $b(w_3) \leq m \leq k$ and $w_5 \in W$.

If $b(w_3) - b(w_4) \geq 4$, then $a(w_3) \geq b(w_4)+3$. There is a $P'_k(a,b) = (1, \dots, b(w_4), w_4, w_5, w_3, b(w_4)+3, \dots, a(w_3), \dots, k)$. This contradicts $(\star)$. Hence $b(w_3) - b(w_4) \leq 3$.

Subcase 1.1. $(b(w_4)-1, b(w_3)) \notin A$.

First, we have $m > b(w_3)$. Let $P_2(b(w_3), w_4) = (w_4, y, b(w_3))$. If $y \in W$, then $a(y) \geq b(w_3)$. If $a(y) \geq b(w_4)+2$, then there is a $P'_k(a,b) = (1, \dots, b(w_4), w_4, y, b(w_4)+2, \dots, a(y), \dots, k)$. This contradicts $(\star)$. Hence $a(y) \leq b(w_4)+1$. Since $a(y) \geq b(w_3) \geq b(w_4)+1$, we have $a(y) = b(w_3) = b(w_4)+1$. By Lemma 1 ,there exists an $x \in O(1) - \{2, k\}$. Obviously, $x \notin W$. And $x \notin \{3, \dots, b(w_4)-1\}$ by Lemma 5. So $x \geq b(w_4)$.

(a) $x = b(w_4)$. Since $x \geq 3$ and $a(w_4) = b(w_4)-1 \geq 2$, there is a $P'_k(a,b) = (1, x = b(w_4), \dots, m-1, w_4, 2, \dots, b(w_4)-1, m, \dots, k)$. This contradicts $(\star)$.

(b) $x = b(w_4)+1$. Note that $m-1 > b(w_3)-1 = b(w_4)$ and $a(y) = b(w_4)+1 = a(w_4)+2 \geq 3$. If $a(w_4) > 1$, then there is a $P'_k(a,b) = (1, x = b(w_3), \dots, m-1, w_4, y, 2, \dots, a(w_4) = b(w_4)-1, m, \dots, k)$. This contradicts $(\star)$. Hence $a(w_4) = 1$. Thus we have $I(k) \subseteq \{1, b(w_3)-1 = 2, k-1\}$ by Lemma 5 (b). And then $2 \to k$ by Lemma 1. Hence there is a $P'_k(a,b) = (1, x = b(w_3), \dots, k-1, w_3, b(w_4) = 2, k)$. This contradicts $(\star)$ too.

(c) $x > b(w_3) = b(w_4)+1$. Since $x < k$ and $1 \to x$, there is no $j_0 \in \{2, \dots, b(w_4)-1\}$ such that $j_0 \to k$ by Lemma 4. Then $I(k) \subseteq \{1, b(w_4), k-1\}$ by Lemma 5 (b). Hence $I(k) = \{1, b(w_4), k-1\}$ by Lemma 1. That is, $b(w_4) \to k$. If $k = 7$, there is a $P'_k(a,b) = (1, x, w_4, y, b(w_3), w_3, b(w_4), k)$. This contradicts $(\star)$. Hence $k > 7$. There are two distinct vertices $i, j \in P'_{k-1} - \{1, a(w_4), b(w_4) = a(w_4)+1, b(w_3) = a(w_4)+2, x, k\}$. Using two arcs $(1,x)$, $(b(w_4), k)$ and w_3, w_4, y, then there is always a $P'_k(a,b)$ in T. e.g., $1 < i < a(w_4)$ and $b(w_3) < j < x$, then there is a $P'_k(a,b) = (1, x, \dots, k-1, w_4, y, b(w_3), \dots, x-2, w_3, 3, \dots, a(w_4), b(w_4), k)$. These contradict $(\star)$.

Hence $y \notin W$ and $1 \leq y \leq a(w_4) = b(w_4) - 1$. Since $(b(w_4) - 1, b(w_3)) \notin A$, we have $y < b(w_4) - 1$. Now, there are two arcs $(y, b(w_3))$ and $(b(w_4) - 1, m)$ in T with $y < b(w_4) - 1 < b(w_3) < m$. This contradicts $(\star)$ by Lemma 4.

Subcase 1.2 $(b(w_4) - 1, b(w_3)) \in A$.

Since $b(w_3) - b(w_4) \leq 3$ and $k \geq 7$, we have $(b(w_4) - 1, b(w_3)) \neq (a, b)$. There exists an arc (u, v) such that (u, v) and $(b(w_4) - 1, b(w_3))$ are cis-crosswise arcs by Lemma 6.

Suppose $u < b(w_4) - 1 < v < b(w_3)$. For $v = b(w_4)$ or $b(w_4) + 1$ or $b(w_4) + 2$, there exists a $P'_k(a, b)$ in T respectively. e.g., we assume $v = b(w_4) + 1$. If $b(w_3) = b(w_4) + 3$, then there is a $P'_k(a, b) = (1, \ldots, u, v = b(w_4) + 1, w_4, w_5, w_3, u + 1, \ldots, b(w_4) - 1, b(w_3), \ldots, k)$. If $b(w_3) = b(w_4) + 2$, then $v = b(w_3) - 1 = a(w_3)$. Since $w_4 \rightarrow w_5$ and $w_4 \rightarrow a(w_4)$, w_5 and $a(w_4)$ are adjacent. By the definition of $b(w_4) = b(w_0)$, we have $w_5 \rightarrow a(w_4)$. Hence $a(w_5) \geq a(w_4) > u$ and $w_5 \rightarrow u + 1$. Thus there is a $P'_k(a, b) = (1, \ldots, u, v = b(w_4) + 1, w_4, w_5, u + 1, \ldots, b(w_4) - 1, b(w_3) = b(w_4) + 2, \ldots, k)$. These contradict $(\star)$.

Using an analogous method, if $b(w_4) - 1 < u < b(w_3) < v$, then there is also a $P'_k(a, b)$ in T. This contradicts $(\star)$.

Therefore no vertex of W_3 dominates any vertex of W_4. We have that $W_4 \rightarrow W_3$ since $T[W]$ is a tournament.

Case 2. $W_4 \rightarrow W_3$.

We choose $w_3 \in W_3$, $w_4 \in W$, such that $b(w_3) = \max\{b(w) \mid w \in W_3\}$. Thus $w_4 \rightarrow w_3$. Since $b(w_3) > b(w_4)$, there exists a $P'_2(w_4, w_3) = (w_4, w_6, w_3)$ with $w_6 \in W$. Now, we have the following claims.

(4.1) $b(w_4) \leq a(w_3) \leq b(w_4) + 1$.

Let $P_2(w_4, w_3) = (w_3, y, w_4)$. Since $W_4 \rightarrow W_3$, we have $y \notin W$ and $y \in P'_{k-1}$. Thus $b(w_4) \leq y \leq a(w_3)$. If $a(w_3) - 2 \geq b(w_4)$, then there is a $P'_k(a, b) = (1, 2, \ldots, a(w_3) - 2, w_4, w_3, a(w_3), \ldots, k)$. This contradicts $(\star)$. Hence $a(w_3) \leq b(w_4) + 1$.

(4.2) $(a(w_4), b(w_3)) \in A$.

Let $P_2(b(w_3), w_4) = (w_4, u, b(w_3))$ and $P_2(w_3, a(w_4)) = (a(w_4), m, w_3)$. By the choice of w_3 and w_4, we have $u, m \notin W$. Then $u \leq a(w_4)$ and $b(w_3) \leq m$. If $u < a(w_4)$ and $b(w_3) < m$, then

(a) $a(w_3) \geq a(w_4) + 2$. Since $b(w_3) - 1 \geq b(w_4)$, there is a $P'_k(a, b) = (1, \ldots, u, b(w_3), \ldots, m - 1, w_3, a(w_4) + 2, \ldots, a(w_3), \ldots, b(w_3) - 1, w_4, u + 1, \ldots, a(w_4), m, \ldots, k)$;

(b) $b(w_4) \leq b(w_3) - 2$. There is a $P'_k(a, b) = (1, \ldots, u, b(w_3), \ldots, m - 1, w_3, a(w_4) + 1, \ldots, b(w_4), \ldots, b(w_3) - 2, w_4, u + 1, \ldots, a(w_4), m, \ldots, k)$;

(c) $a(w_3) \leq a(w_4) + 1$ and $b(w_4) \geq b(w_3) - 1$. Since $a(w_3) \geq b(w_4) \geq a(w_4) + 1$ and $b(w_4) \leq b(w_3) - 1$, we have $a(w_3) = b(w_4) = a(w_4) + 1 = b(w_3) - 1$. Thus there is a $P'_k(a, b) = (1, \ldots, u, b(w_3), \ldots, m - 1, w_4, w_3, u +$

$1, \dots, b(w_3) - 2 = a(w_4), m, \dots, k)$.

These contradict $(\star)$. So $u = a(w_4)$ or $m = b(w_3)$. Thus (4. 2) is valid.

(4.3) $a(w_3) = b(w_3) - 1$ and $a(w_4) = b(w_4) - 1$.

If $a(w_3) < b(w_3) - 1$, then i and w_3 are nonadjacent for each $i \in \{a(w_3) + 1, \dots, b(w_3) - 1\}$. Since $\{a(w_3) + 1, \dots, b(w_3) - 1, k\} \subseteq I(w_4)$, k and j are adjacent for each $j \in \{a(w_3) + 1, \dots, b(w_3) - 1\}$. Since the definition of local tournaments and $k \rightarrow w_3$, we have $\{a(w_3) + 1, \dots, b(w_3) - 1\} \rightarrow k$. If $b(w_3) < k$, then there is a $P'_k(a, b) = (1, \dots, a(w_4), b(w_3), \dots, k - 1, w_3, a(w_4) + 1, \dots, b(w_3) - 1, k)$ by (4.2) and $a(w_4) + 1 \le b(w_4) \le a(w_3)$. This contradicts $(\star)$. So $b(w_3) = k$. Since $1, k - 1 \in I(k), 1$ and $k - 1 = b(w_3) - 1$ are adjacent, and then $1 \rightarrow b(w_3) - 1$ by $w_3 \rightarrow 1$. Now, we consider the following two subcases:

(a) $a(w_3) < b(w_3) - 2 = k - 2$.

If $a(w_3) \ge 3$, then there is a $P'_k(a, b) = (1, b(w_3) - 1, w_4, w_3, 3, \dots, b(w_3) - 2, k)$. This contradicts $(\star)$. Hence $a(w_3) \le 2$. Since $a(w_3) \ge b(w_4) > a(w_4) \ge 1$, we have $a(w_3) = 2$. Then $b(w_3) - 3 > a(w_3)$ by $k \ge 7$. Hence $b(w_3) - 3 \rightarrow k$ and there is a $P'_k(a, b) = (1, b(w_3) - 1 = k - 1, w_4, w_3, a(w_3) = 2, \dots, k - 3 = b(w_3) - 3, k)$. This also contradicts $(\star)$.

(b) $a(w_3) = b(w_3) - 2 = k - 2$.

Since $k \ge 7$, we have $a(w_3) \ge 5$ and $b(w_4) \ge a(w_3) - 1 \ge b(w_4) - 1 \ge 4$ by (4.1). If $b(w_4) = a(w_4) + 1$, then $a(w_4) \ge 3$. When $a(w_3) = b(w_4) + 1$, there is a $P'_k(a, b) = (1, b(w_3) - 1 = k - 1, w_4, w_6, w_3, 2, \dots, k - 4 = a(w_4), b(w_3) = k)$ by (4.2). When $a(w_3) = b(w_4)$, there is a $P'_k(a, b) = (1, b(w_3) - 1 = k - 1, w_4, w_3, 2, \dots, k - 3 = a(w_4), b(w_3) = k)$. These contradict $(\star)$. So $b(w_4) \ge a(w_4) + 2$. Since $a(w_4) \rightarrow b(w_3) = k$ and $a(w_4) \rightarrow a(w_4) + 1, k$ and $a(w_4) + 1$ are adjacent, and then $a(w_4) + 1 \rightarrow k$ by $k \rightarrow w_4$ and w_4 and $a(w_4) + 1$ are nonadjacent. Similarly, we can get that $\{a(w_4) + 1, \dots, b(w_4) - 1\} \rightarrow k$ and $1 \rightarrow \{a(w_4) + 1, \dots, b(w_4) - 1\}$ since $\{1, a(w_4) + 1, \dots, b(w_4) - 1\} \subseteq O'(w_3)$. If $b(w_4) - 2 \ge a(w_4) + 1$, then there is a $P'_k(a, b) = (1, b(w_4) - 1, \dots, a(w_3) = k - 2, w_4, w_3, 2, \dots, b(w_4) - 2, k)$ by (4.1) and $b(w_4) \ge 4$. If $b(w_4) = a(w_4) + 2$, then there is a $P'_k(a, b) = (1, a(w_4) + 1, \dots, k - 1, w_4, 2, \dots, a(w_4), b(w_3) = k)$ by (4.2) and $a(w_4) \ge 2$. These contradict $(\star)$. So $a(w_3) = b(w_3) - 1$.

Similarly, we can prove that $a(w_4) = b(w_4) - 1$. (4.3) is valid.

Now, by (4.1), (4.2) and (4.3), we have that $b(w_3) - b(w_4) \le 2$, $a(w_3) = b(w_3) - 1, a(w_4) = b(w_4) - 1$ and $(b(w_4) - 1, b(w_3)) \in A$. Using an analogous proof of subcase 1.2, there is a $P'_k(a, b)$ in T. This contradicts $(\star)$.

Up to now, we prove that $b(w) = b(w')$ for any $w, w' \in W$. Similarly, we can prove that $a(w) = a(w')$ for any $w, w' \in W$. So (4) is valid.

We denote $a_0 = a(w)$ and $b_0 = b(w)$ for any $w \in W$. Then $O'(w) = \{1, 2, \dots, a_0\}$ and $I'(w) = \{b_0, \dots, k\}$ for any $w \in W$, and then $T[\{1, \dots, a_0\}]$ and $T[\{b_0, b_0 + 1, \dots, k\}]$ both are tournaments. Clearly for any $i \in \{a_0 +$

$1, \ldots, b_0 - 1\}$ and any $w \in W$, i and w are nonadjacent.

Now, we shall use the following lemmas and symbols.

For $1 \leq t \leq a_0$ and $b_0 \leq j \leq k$, let $R(t) = \{i \mid (t, i) \in A, b_0 \leq i \leq k\}$ and $L(j) = \{i \mid (i, j) \in A, 1 \leq i \leq a_0\}$. Since there exist $P_2(w, t), P_2(j, w)$ for any $w \in W$ and $1 \leq t \leq a_0$, $b_0 \leq j \leq k$, it is easy to check $R(t) \neq \emptyset$, $L(j) \neq \emptyset$. Hence we can define,

$\psi(t) = \max\{R(t)\}$, $\psi_1(t) = \min\{R(t)\}$, $\varphi_1(j) = \max\{L(j)\}$ and $\varphi(j) = \min\{L(j)\}$.

Then $b_0 \leq \psi_1(t) \leq \psi(t) \leq k$, $1 \leq \varphi(j) \leq \varphi_1(j) \leq a_0$, and $(t, \psi(t))$, $(t, \psi_1(t))$, $(\varphi(j), j), (\varphi_1(j), j) \in A$ for any $1 \leq t \leq a_0$ and $b_0 \leq j \leq k$.

Lemma 7. *If there are $\alpha < \gamma < \delta$ in P'_{k-1} such that $1 \leq \alpha \leq a_0 - 1$, $\alpha + 1 < \gamma$, $b_0 + 1 \leq \delta$ and $(\alpha, \gamma), (\gamma - 1, \delta) \in A$, then T contains a $P'_k(a, b)$ in T.*

Proof: Let α, γ and δ satisfy the condition of Lemma 7. Then there is a $P'_k(a, b) = (1, \ldots, \alpha, \gamma, \ldots, \delta - 1w, \alpha + 1, \ldots, \gamma - 1, \delta, \ldots, k)$. □

Lemma 8. ([2], Corollary 3.13) *Let $P_1 = (x_1, \ldots, x_m)$ and $P_2 = (y_1, \ldots, y_t)$ with $m \geq 2$ and $t \geq 3$ be paths in a connected local tournament T. If there exist i, j with $1 \leq i < j \leq m$ such that $x_i = y_1$, $x_j = y_t$ and $V(P_1) \cap (v(P_2) - \{y_1, y_t\}) = \emptyset$. Then T has an (x_1, x_m)-path P such that $V(P) = V(P_1) \cup V(P_2)$.*

(5) $b_0 = a_0 + 1$

Suppose $b_0 > a_0 + 1$. If $b_0 = k$, then $\psi(i) = k$ for each $i \in \{1, 2, \ldots, a_0\}$. That is, $\{1, 2, \ldots, a_0\} \to k$. Let $P_2(a_0, a_0 + 1) = (a_0 + 1, x, a_0)$. Obviously, $x \notin W$. If $x \in \{1, 2, \ldots, a_0 - 1\}$, then $a_0 + 1$ and w are adjacent by $w \to x$. This is a contradiction. So $x \notin \{1, 2, \ldots, a_0 - 1\}$. Similarly, $x \notin \{a_0 + 1, \ldots, b_0 - 1\}$. Thus $x = b_0 = k$. i.e, $k = x \to a_0$. This contradicts $a_0 \to k$. Hence $b_0 \leq k - 1$. Similarly, we have $a_0 \geq 2$.

Let $P_2(a_0, a_0 + 1) = (a_0 + 1, t, a_0)$. Using an analogous proof as above, we have $t \notin W \cup \{1, 2, \ldots, a_0 - 1, a_0 + 1, \ldots, b_0 - 1\}$. That is, $b_0 \leq t \leq k$. If $t = b_0$, then we have $b_0 \to a_0$, $\varphi(b_0) < a_0$ and $\psi(a_0) > b_0$. $\varphi(b_0)$ and $b_0 - 1$ are adjacent by $\varphi(b_0) \to b_0$ and $b_0 - 1 \to b_0$. Since $b_0 - 1$ and w are nonadjacent and $w \to \varphi(b_0)$, we have $\varphi(b_0) \to b_0 - 1$. Similarly, we can obtain $\varphi(b_0) \to \{a_0 + 1, \ldots, b_0 - 1\}$. Let $\alpha = \varphi(b_0)$, $\gamma = a_0 + 1$ and $\delta = \psi(a_0)$. Then there is a $P'_k(a, b)$ in T by Lemma 7. This contradicts $(\star)$. Hence $t > b_0$. Similarly, letting $P_2(b_0 = 1, b_0) = (b_0, y, b_0 - 1)$, we have $1 \leq y < a_0$.

If $b_0 > a_0 + 2$, then t and $a_0 + 2$ are adjacent by $a_0 + 1 \to t$ and $a_0 + 1 \to a_0 + 2$. If $t \to a_0 + 2$, then it will deduce that $a_0 + 2$ and w are adjacent by $t \to w$, a contradiction. Hence $a_0 + 2 \to t$. Similarly, we have $\{a_0 + 1, \ldots, b_0 - 1\} \to t$. Let $\alpha = y (< a_0)$, $\gamma = b_0 - 1$ and $\delta = t (> b_0)$. There is a $P'_k(a, b)$ in T by Lemma 7. This contradicts $(\star)$. Hence $b_0 = a_0 + 2$.

a_0+1 and $t-1$ are adjacent since $a_0+1 \to t$ and $t-1 \to t$. Thus $a_0+1 \to t-1$ by $t-1 \to w$ and w and a_0+1 are nonadjacent. Similarly, we have

$$a_0+1 \to \{b_0+1, \ldots, t-1, t\} \qquad (\star\star)$$

Now, we consider the following four cases.

Case 1. $a_0 > 2$ and $k \geq b_0+2$.

If $\varphi(b_0) < a_0$, letting $\alpha = \varphi(b_0)$, $\gamma = b_0$ and $\delta = t$, then there is a $P'_k(a,b)$ in T by Lemma 7 and $(b_0-1, t) = (a_0+1, t) \in A$. This contradicts $(\star)$. Hence $\varphi(b_0) = a_0$. That is, $a_0 \to b_0$. Since $1, a_0 \in O(w)$, 1 and a_0 are adjacent. Suppose $(1, a_0) \in A$. If $\psi(a_0-1) > b_0$, letting $\alpha = 1$, $\gamma = a_0$ and $\delta = \psi(a_0-1)$, then there is a $P'_(a,b)$ in T by Lemma 7. This contradicts $(\star)$. So $\psi(a_0-1) = b_0$. i.e., $a_0-1 \to \psi(a_0-1) = b_0$. Now, letting $\alpha = a_0-1$, $\gamma = b_0$ and $\delta = t$, there is a $P'_k(a,b)$ in T by $(\star\star)$ and Lemma 7. This contradicts $(\star)$ too. Hence in the following we always assume that $(a_0, 1) \in A$.

(5.1) $\{1, 2, \ldots, a_0-1\} \to a_0+1$ and $a_0+1 \to k$.

$1 \to a_0+1$ since a_0+1 and w are nonadjacent and $1, a_0+1 \in O(a_0)$. Furthermore, $2 \to a_0+1$ by $1 \to 2$. Similarly, we have that $\{1, 2, \ldots, a_0-1\} \to a_0+1$ and $a_0+1 \to k$ by $1 \to a_0+1$ and $1 \to k$.

(5.2) $b_0 \to 1$ and $\{b_0+2, \ldots, k\} \to b_0$.

Since $a_0 \to 1$ and $a_0 \to b_0$, 1 and b_0 are adjacent. If $1 \to b_0$, then, letting $\alpha = 1$, $\gamma = b_0$ and $\delta = t$, there is a $P'_k(a,b)$ in T by $(\star\star)$ and Lemma 7. This contradicts $(\star)$. Hence $b_0 \to 1$.

If there exists a $j \in \{b_0+2, \ldots, k\}$ such that $b_0 \to j$, then T contains a $P'_k(a,b) = (1, \ldots, a_0-1, a_0+1 = b_0-1, b_0+1, \ldots, j-1, w, a_0, b_0, j, \ldots, k\}$ by (5.1) and $(\star\star)$. This contradicts $(\star)$. Hence $\{b_0+2, \ldots, k\} \to b_0$.

(5.3) $a_0 = 3$, $b_0 = 5$ and $(a_0-1, b_0) \notin A$.

If $\psi(a_0-1) = b_0$, then there is a $P'_k(a,b) = (1, \ldots, a_0-1, \psi(a_0-1) = b_0, \ldots, k-1, w, a_0, a_0+1, k)$ by (5.1) . This contradicts $(\star)$. So $\psi(a_0-1) > b_0$ and $(a_0-1, b_0) \notin A$.

By $a_0-1 \to \psi(a_0-1) > b_0$, Lemma 7 and $(\star)$, we have $a_0 \to 2$. If $a_0 \geq 4$, then there is a $P'_k(a,b) = (1, a_0+1, \ldots, \psi(a_0-1)-1, w, a_0, 2, \ldots, a_0-1, \psi(a_0-1), \ldots, k)$ by (5.1). This contradicts $(\star)$. Hence $a_0 \leq 3$, and then $a_0 = 3$ by $a_0 > 2$. Thus $b_0 = a_0+2 = 5$.

(5.4) $k = b_0+2 = 7$

Suppose $k > b_0+2$. When $\varphi(b_0+1) \in \{1, 2\}$, there is a $P'_k(a,b) = (1, \ldots, \varphi(b_0+1),\ b_0+1, \ldots, k-1, b_0, w, \varphi(b_0+1)+1, \ldots, a_0+1, k)$ by (5.1) and (5.2). When $\varphi(bo+1) = a_0 = 3$, letting $\alpha = 1$, $\gamma = a_0+1$ and $\delta = b_0+1$, there is a $P'_k(a,b)$ in T by Lemma 7 and (5.1). These contradict $(\star)$. Hence $k = b_0+2 = 7$ by $k \geq b_0+2$.

(5.5) $k \to a_0$

a_0 and k are adjacent since $a_0 \to b_0$ and $k \to b_0$ by (5.2). If $a_0 \to k$, letting $\alpha = 1$, $\gamma = a_0+1$ and $\delta = k$, then there is a $P_k'(a,b)$ in T by Lemma 7. This contradicts ($\star$). So $k \to a_0$.

Now, 1 and b_0+1 are adjacent since $1, b_0+1 = k-1 \in I(k)$. We consider the following two cases.

(a) $b_0+1 \to 1$.

a_0 and b_0+1 are adjacent by $a_0 \to 1$. If $a_0 \to b_0+1$, then there is a $P_k'(a,b) = (1, a_0+1, b_0, w, 2, a_0 = 3, b_0+1, k)$. This contradicts ($\star$). Hence $b_0+1 \to a_0$.

Let $P_2'(b_0, b_0+1) = (b_0 u, b_0+1)$. Obviously, $u \notin W$. Since $b_0+1 \to 1$, $a_0 \to b_0$ $a_0+1 \to b_0$ and $b_0+1 \to k$, we have $u = 2$. i.e., $b_0 \to u = 2 \to b_0+1$. Let $P_2'(1,2) = (1,z,2)$. Obviously, $z \notin W$. Since $a_0 \to 1$, $2 \to a_0+1$ by (5.1), $b_0 \to 1$ by (5.2) and $b_0+1 \to 1$, we have $z = k$. i.e., $k = z \to 2$. Suppose $|W| > 1$. Clearly T contains a $P_k'(a,b)$. This contradicts ($\star$). Hence $|W| = 1$.

Now, by $a_0+1 \to b_0+1$, $a_0 \to b_0$, $a_0 \to 1$, (5.1)$\sim$(5.5) and (a), we have that $T \simeq D_8'$. This contradicts the assumption of the Theorem.

(b) $1 \to b_0+1$.

Since $1 \to 2$ and $1 \to b_0+1$, we have 2 and b_0+1 are adjacent. If $2 \to b_0+1$, then 2 and b_0 are adjacent and $b_0 \to 2 = a_0-1$ by (5.3). Thus there is a $P_k'(a,b) = (1, b_0+1, w, a_0, b_0, 2 = a_0-1, a_0+1, k)$ by (5.1) and (5.2) . This contradicts ($\star$). So $b_0+1 \to 2$.

Let $P_2'(b_0+1, k) = (b_0+1, y, k)$. Obviously, $y \notin W$. Note that $1 \to b_0+1$, $k \to a_0$ by (5.5), $a_0+1 \to b_0+1$ by ($\star\star$) and $b_0 \to b_0+1$. We have $y = 2$. i.e., $2 \to k$.

We easily check that $|W| = 1$ and 3 and 6, 2 and 5 are nonadjacent. Otherwise, T contains a $P_k'(a,b)$. e.g., $(3,6) \in A$, there is a $P_k'(a,b) = (1, a_0+1, b_0, w, 2, 3, 6 = b_0+1, k)$ by (5.1). These contradict ($\star$).

Now, by $a_0+1 \to b_0+1$, $a_0 \to b_0$, $a_0 \to 1$, (5.1)$\sim$(5.5) and (b), we have that $T \simeq D_8$. This contradicts the assumption of the Theorem.

Case 2. $a_0 > 2$ and $k = b_0+1$.

Since $b_0 < t \le k$, we have $t = k = b_0+1$. i.e., $b_0 - 1 = a_0+1 \to t = k$. Since $k = b_0+1 = a_0+3 \ge 7$, we have $a_0 \ge 4$. If there exists a $j_0 \in \{1, 2, \ldots, a_0-1\}$ such that $j_0 \to b_0$, letting $\alpha = j_0$, $\gamma = b_0$ and $\delta = k$, then there is a $P_k'(a,b)$ by Lemma 7. This contradicts ($\star$). Hence $(j, b_0) \notin A$ for each $j \in \{1, 2, \ldots, a_0-1\}$. Then $I(b_0) \subseteq \{a_0, a_0+1 = b_0-1\}$ by $b_0 \to W$. This contradicts Lemma 1.

Case 3. $a_0 = 2$ and $k \ge b_0+2$.

Consider the converse $\overleftarrow{T}$ of T, thus we change case 3 in T for case 2 in T. So this case is impossible.

Case 4. $a_0 = 2$ and $k = b_0 + 1$.

In this case $k = b_0 + 1 = a_0 + 2 + 1 = 5$, this contradicts $k \geq 7$.

Up to now, we have proved that $b_0 = a_0 + 1$. (5) is valid.

(6) Under the condition $b_0 = a_0 + 1$, we can obtain the following claims.

For convenience, let $s = a_0 + 1 = b_0$. Then $a_0 = s - 1$, $1 < s \leq k$ and $T[\{1, 2, \dots, s-1\}]$ and $T[\{s, \dots, k\}]$ both are tournaments.

(6.1) $3 \leq s \leq k - 1$

T is a T_0-type digraph when $s = 2$ or k. e.g.,$s = 2$, then $O'(w) = \{1\}$ and $I'(w) = \{s = 2, \dots, k\}$ for any $w \in W$. Hence $\varphi(i) = 1$ for each $i \in I'(W)$. i.e., $1 \to \{2, \dots, k\}$. Let $T_0' = T[\{2, \dots, k\}]$, $T_0'' = T[W]$ and $v_0 = 1$. Thus $T_0' \to T_0'' \to v_0 \to T_0'$. Since T_0' and T_0'' both are tournaments, $T \simeq T_0$-type digraph. This contradicts the assumption of the Theorem. So $3 \leq s \leq k - 1$.

(6.2) $\varphi(s) < s - 1$, $\psi(s-1) > s$ can not hold simultaneously.

Suppose $\varphi(s) < s - 1$, $\psi(s-1) > s$. We may choose $\alpha = \varphi(s)$, $\gamma = s$ and $\delta = \psi(s-1)$. Then there is a $P_k'(a, b)$ in T by Lemma 7. This contradicts $(\star)$. Hence we have $\varphi(s) = s - 1$ or $\psi(s-1) = s$. We may assume, without loss of generality, $\psi(s-1) = s$. Otherwise, we consider the converse $\overleftarrow{T}$ of T. Then

$$1 \leq \varphi(j) \leq \varphi_1(j) \leq s - 2 \tag{$\star\star\star$}$$

for each $j \in \{s+1, \dots, k\}$ by the definition of $\psi(s-1)$. We may define $\tilde{n} = \max\{\varphi_1(j) \mid s+1 \leq j \leq k\}$, $\tilde{m} = \min\{j \mid \varphi_1(j) = \tilde{n}, s+1 \leq j \leq k\}$. Then $\tilde{n} \leq s - 2, s < \tilde{m}, (\tilde{n}, \tilde{m}) = (\varphi_1(\tilde{m}), \tilde{m}) \in A$ and $(\tilde{n}, \tilde{m}) \neq (1, k)$. In fact, if $(\tilde{n}, \tilde{m}) = (1, k)$ then every vertex in $\{1, 2, \dots, s-2\}$ does not dominate every vertex in $\{s+1, \dots, k\}$ except for an arc $(1, k)$. If $k > s+1$, then $\varphi(s+1) \leq s - 2$ by $(\star\star\star)$. i.e., $\varphi(s+1) \to s + 1$, a contradiction. So $k = s + 1$. Since $k \geq 7$, we have $s \geq 6$. Since $(i, k) \notin A$ for each $i \in \{2, \dots, s-2\}$ and $\psi(s-1) = s$, we have $I(k) \subseteq \{1, k - 1 = s\}$. This contradicts Lemma 1. Hence $(\tilde{n}, \tilde{m}) \neq (1, k) = (a, b)$.

(6.3) (1°) For each $j \in \{\tilde{n}+1, \dots, s-1\}$ and $i \in \{s+1, \dots, k\}$, we have $(j, i) \notin A$;

(2°) $\{\tilde{n}+1, \dots, s-1\} \to s$;

(3°) $\tilde{n} + 1 \to \{1, 2, \dots, \tilde{n} - 1\}$ as $\tilde{n} \geq 2$.

By the definition of $\tilde{n}$ and $\psi(s-1) = s$, we easily check that (1°) and (2°) are valid. By $\tilde{n} \to \tilde{m}$, Lemma 7 and $(\star)$, we have $\tilde{n} + 1 \to \{1, 2, \dots, \tilde{n} - 1\}$.

(6.4) $k \to \{s-1, s, \dots, k-2\}$ as $k \geq s + 2$.

If there exists a $j_0 \in \{s, \dots, k-2\}$ such that $j_0 \to k$, letting $\alpha = \varphi(j_0+1)$, $\gamma = j_0 + 1$ and $\delta = k$, then there is a $P_k'(a, b)$ in T by Lemma 7 and $\varphi(j_0 + 1) \leq s - 2$. This contradicts $(\star)$. Hence $k \to \{s, \dots, k-2\}$. Since

$k \to s$ and $s-1 \to s$, we have k and $s-1$ are adjacent and $k \to s-1$ by (6.3).

(6.5) $\{s+2, \dots, k-1, k\} \to \{\tilde{n}+1, \dots, s\}$ if $k \geq s+2$; $s+1 \to \{\tilde{n}+1, \dots, s-1\}$ if $k \geq s+3$.

k and $s-2$ are adjacent since $k \to s-1$ by (6.4). If $s-2 \geq \tilde{n}+1$, then $k \to s-2$ by (6.3) (1°). Similarly, we have $k \to s-3, \dots, k \to \tilde{n}+1$. Thus $k \to \{\tilde{n}+1, \dots, s-1, s, \dots, k-2\}$ by (6.4). Hence $T[\{\tilde{n}+1, \dots, s-1, s, \dots, k-2, k\}]$ is a tournament. Then by (6.3) we have $\{s+2, \dots, k-2, k\} \to \{n+1, \dots, s-1\}$ when $k \geq s+2$, and $s+1 \to \{\tilde{n}+1, \dots, s-1\}$ when $k \geq s+3$. Since $\varphi(s+1) \leq s-2$, we have $\{s+2, \dots, k-1, k\} \to s$ by Lemma 7 and $(\star)$. Since $s-1 \to s$ and $k-1 \to s$, we have $k-1 \to s-1$ by (6.3). Similarly, $k-1 \to \{\tilde{n}+1, \dots, s-1, s\}$. Hence (6.5) is valid.

(6.6) $1 < \varphi_1(k) \leq \tilde{n}$

Note that $I(k) \subseteq \{1, 2, \dots, \tilde{n}, k-1\}$ by (6.3) and (6.4). Then there exists an $i_0 \in I(k) - \{1, k-1\}$ with $1 < i_0 \leq \tilde{n}$ by Lemma 1. Hence $1 < i_0 \leq \varphi_1(k) \leq \tilde{n}$.

In the following we consider two cases.

Case 1. $O(1) \cap \{s, s+1, \dots, k-1\} \neq \emptyset$.

Let $p = \max\{j \mid O(1) \cap \{s, s+1, \dots, k-1\}\}$. Then $s \leq p \leq k-1$.

Subcase 1.1 $|W| \geq 2$. (let $w, w' \in W$ and $w \neq w'$).

(6.7) $p = s$. That is, $1 \to s$ and $(1, j) \notin A$ for each $j \in \{s+1, \dots, k-1\}$.

Suppose $p > s$. Since $k \geq 7$, there exists an $i \in P'_{k-1} - \{1, \varphi_1(k), s-1, s, p, k\}$. If $1 < i < \varphi_1(k)$, then there is a $P'_k(a,b) = (1, p, \dots, k-1, w, \varphi_1(k)+1, \dots, p-1, w', 3, \dots, \varphi_1(k), k)$. Similarly, T contains a $P'_k(a,b)$ when $\varphi_1(k) < i < s-1$ or $s < i < p$ or $p < i < k$. These contradict $(\star)$. Hence (6.7) is valid.

(6.8) $T[V_1]$ is a strong tournament, where $V_1 = \{\tilde{n}+1, \dots, s-1\}$.

Since $V_1 \subseteq O(w)$, $T[V_1]$ is a tournament. If $T[V_1]$ is not strong, then $|V_1| \geq 2$ and $\tilde{n}+1 \to s-1$. Let $P_2(\tilde{n}+1, s-1) = (s-1, q, \tilde{n}+1)$ in T. Obviously, $q \notin W$. Since $T[V_1]$ is not strong, we have $q \notin V_1$. $q \notin \{s, s+1, \dots, k\}$ and $q \notin \{1, 2, \dots, \tilde{n}-1\}$ by (6.3). Hence $q = \tilde{n}$ and $s-1 \to q = \tilde{n}$. Let $P_1 = (1, s, \dots, \tilde{m}-1, w, 2, \dots, \tilde{n}, \tilde{m}, \dots, k)$ and $P_2 = (w, \tilde{n}+1, \dots, s-1, \tilde{n})$. Then by Lemma 8 there is a $P'_k(a,b)$ in T. This contradicts $(\star)$. So $T[V_1]$ is a strong tournament.

(6.9) $\tilde{n} = 2$ and $2 \to k$.

If $\tilde{n} > 2$, then $\tilde{n}+1 \to 2$ by (6.3). We may assume that $(\tilde{n}+1, h, \dots, \tilde{n}+1)$ is a Hamiltonian cycle in $T[V_1]$ by (6.8). Thus there is a $P'_k(a,b) = (1, s, \dots, \tilde{m}-1, w, h, \dots, \tilde{n}+1, 2, \dots, \tilde{n}, \tilde{m}, \dots, k)$. This contradicts $(\star)$. So $\tilde{n} \leq 2$. Thus $\tilde{n} = 2$ and $\varphi_1(k) = 2$ by (6.6). So $2 \to k$.

(6.10) $k = s+1$.

Suppose $k-1 \geq s+1$. If there exists a $j_0 \in \{s, \dots, k-3\}$ such that $j_0 \to k-1$, then, letting $\alpha = \varphi(j_0+1)$, $\gamma = j_0+1$ and $\delta = k-1$, T contains a $P'_k(a,b)$ in T by (6.2) and Lemma 7. This contradicts $(\star)$. Hence $k-1 \to \{s, \dots, k-3\}$. By (6.3), (6.7) and (6.9), $I(k-1) \subseteq \{\tilde{n} = 2, k-2\}$. This contradicts Lemma 1. So $k = s+1$.

(6.11) $O(1) = \{2, s = k-1, k\}$

Otherwise, there exists a $y \in O(1) - \{2, s = k-1, k\}$. Then $y \in V_1$. Let $(y, \dots, h, y)$ be a Hamiltonian cycle in $T[V_1]$. Then there is a $P'_k(a,b) = (1, y, \dots, h, s, w, 2 = \varphi_1(k), k)$ by (6.3). This contradicts $(\star)$. Hence $O(1) = \{2, s = k-1, k\}$.

(6.12) $2 \to s$.

Let $P'_2(1,s) = (1, z, s)$, then $z \in O(1) = \{2, s = k-1, k\}$. Since $s = k-1 \to k$, we have $z = 2$. Hence $2 = z \to s$.

So far, since $O(1) = \{2, s = k-1, k\}$, $2 \to s$ and $2 \to k$ by (6.9), there exists no $P'_2(1,2)$ in T. This contradicts the assumption of the Theorem.

Subcase 1.2 $|W| = 1$.

Using an analogous method of subcase 1.1, we can get $p \neq s$. Hence we may assume that $s+1 \leq p \leq k-1$.

(6.13) $k = s+2$ and $p = s+1$.

If $k \geq s+3$, then $k-1 \to \{\tilde{n}+1, \dots, s-2, s-1, s\}$ by (6.5). Since $\tilde{n}+2 \leq s$ by the definition of $\tilde{n}$, we have $k-1 \to \tilde{n}+2$. If $\tilde{n} \geq 3$, then $\tilde{n}+1 \to 2$ by (6.3). Thus there is a $P'_k(a,b) = (1, p, \dots, k-1, \tilde{n}+2, \dots, p-1, w, \varphi_1(k)+1, \dots, \tilde{n}+1, 2, \dots, \varphi_1(k), k)$. This contradicts $(\star)$. Hence $\tilde{n} = 2$. And then $2 = \varphi_1(k) \to k$ since $2 \leq \varphi_1(k) \leq \tilde{n}$. Thus there is a $P'_k(a,b) = (1, p, \dots, k-1, \tilde{n}+1 = 3, \dots, p-1, w, 2 = \varphi_1(k), k)$. This contradicts $(\star)$. So $k = s+2$ and $p = s+1$ since $s+1 \leq p \leq k-1$.

(6.14) $2 \to s$ and $\tilde{n} \to s-1$ as $\tilde{n} \geq 3$.

Since $\tilde{n}+1 \to s$ and $\tilde{n}+1 \to 2$ by (6.3), 2 and s are adjacent. If $s \to 2$, then there is a $P'_k(a,b) = (1, p = s+1, w, \varphi_1(k)+1, \dots, s, 2, \dots, \varphi_1(k), k)$. This contradicts $(\star)$. So $2 \to s$.

If $s-1 \to \tilde{n}$, letting $P_1 = (1, 2, s, \dots, \tilde{m}-1, w, 3, \dots, \tilde{n}, \tilde{m}, \dots, k)$ (Note that $s+1 \leq \tilde{m} \leq k = s+2$) and $P_2 = (w, \tilde{n}+1, \dots, s-1, \tilde{n})$, than by Lemma 8 there is a $P'_k(a,b)$ in T. This contradicts $(\star)$. Hence $\tilde{n} \to s-1$.

(6.15) $T[V_1]$ is a strong tournament, where $V_1 = \{\tilde{n}+1, \dots, s-1\}$ and $\tilde{n} \geq 3$.

If not, then $|V_1| \geq 2$ and $\tilde{n}+1 \to s-1$. Let $P_2(\tilde{n}+1, s-1) = (s-1, q, \tilde{n}+1)$. Obviously, $q \notin W$. By (6.3), $q \notin \{1, 2, \dots, \tilde{n}-1\} \cup \{s, s+1, s+2 = k\}$. Since $T[V_1]$ is not strong, we have $q \notin V_1$. Then $q = \tilde{n}$. i.e., $s-1 \to q = \tilde{n}$. This contradicts (6.14).

(6.16) $V_1 \to \{1, 2\}$ and $s \to 1$ as $\tilde{n} \geq 3$.

Let $(y, \ldots, h, y)$ be a Hamiltonian cycle in $T[V_1]$. If there exists a $y \in V_1$ such that $1 \to y$, then there is a $P'_k(a,b) = (1, y, \ldots, h, s, \ldots, \tilde{m} - 1, w, 2, \ldots, \tilde{n}, \tilde{m}, \ldots, k)$ by (6.3). This contradicts $(\star)$. So $V_1 \to 1$. Similarly, we have $V_1 \to 2$.

Since $1 \to p = s+1$ and $s \to s+1$, 1 and s are adjacent. If $1 \to s$, then there is a $P'_k(a,b) = (1, s, \ldots, \tilde{m} - 1, w, \tilde{n} + 1, \ldots, s-1, 2, \ldots, \tilde{n}, \tilde{m}, \ldots, k)$. This contradicts $(\star)$. So $s \to 1$.

(6.17) $\tilde{n} = 3$.

In fact, if $\tilde{n} \geq 4$, then $\tilde{n}+1 \to 3$ by (6.3). Let $(h, \ldots, \tilde{n}+1, h)$ be a Hamiltonian cycle in $T[V_1]$. There is a $P'_k(a,b) = (1, 2, s, \ldots, \tilde{m} - 1, w, h, \ldots, \tilde{n} + 1, 3, \ldots, \tilde{n}, \tilde{m}, \ldots, k)$ by (6.14). This contradicts $(\star)$.

If $\tilde{n} = 2$, then $\varphi_1(k) = 2$ and $2 = \varphi_1(k) \to k$. Since $2, s+1 = p \in O(1)$, 2 and $s+1$ are adjacent. If $2 \to s+1$, then by $\tilde{n}+1 = 3$, $s+1 \in O(2)$, $\tilde{n}+1$ and $s+1$ are adjacent. By (6.3) we have $s+1 \to \tilde{n}+1$. There is a $P'_k(a,b) = (1, p = s+1, \tilde{n}+1, \ldots, s, w, 2 = \varphi_1(k), k)$. This contradicts $(\star)$. Hence $s+1 \to 2$. By (6.3) we have that $I(s+1) \subseteq \{1, s\}$. This contradicts Lemma 1. So $\tilde{n} = 3$.

(6.18) $3 \to V_1$.

Suppose there exists a $y \in V_1$ such that $y \to 3$. Let $(h, \ldots, x, h)$ be a Hamiltonian cycle in $T[V_1]$. Then there is a $P'_k(a,b) = (1, 2, s, \ldots, \tilde{m} - 1, w, h, \ldots, y, 3 = \tilde{n}, \tilde{m}, \ldots, k)$ by (6.14). This contradicts $(\star)$. So $3 \to V_1$.

For $2 \leq \varphi_1(k) \leq n = 3$, we consider the following two cases

(a) $\varphi_1(k) = 3$. (that is $3 = \tilde{n} \to k$)

Since $\tilde{n}, s+1 = k-1 \in I(k)$, $\tilde{n}$ and $s+1$ are adjacent. If $\tilde{n} \to s+1$, then, by $s+1, \tilde{n}+1 \in O(\tilde{n})$, $s+1$ and $\tilde{n}+1$ are adjacent. By (6. 3) we have $s+1 \to \tilde{n}+1$. There is a $P'_k(a,b) = (1, p = s+1, \tilde{n}+1, \ldots, s, w, 2, 3 = \tilde{n}, k)$. This contradicts $(\star)$. So $s+1 \to \tilde{n}$.

By (6.3) and Lemma 1, we have $I(s+1) = \{1, 2, s\}$. Then $2 \to s+1$.

Since $s, \tilde{n} = 3 \in O(2)$ by (6.14), s and $\tilde{n}$ are adjacent. If $s \to \tilde{n}$, then there is a $P'_k(a,b) = (1, 2, s+1, w, \tilde{n}+1, \ldots, s, \tilde{n}, k)$. This contradicts $(\star)$. So $3 = \tilde{n} \to s$.

By $(\star)$ and $2 \to s+1$, we have $3 \to 1$.

Since $2, k \in O(1)$, 2 and k are adjacent. If $2 \to k$, then there is a $P'_k(a,b) = (1, s+1, \tilde{n}, \ldots, s, w, 2, k)$. This contradicts $(\star)$. So $k \to 2$.

If there exists a $y \in V_1$ such that $s+1$ and y are adjacent, then $s+1 \to y$ by (6.3). Let $(y, \ldots, h, y)$ be a Hamiltonian cycle in $T[V_1]$. There is a $P'_k(a,b) = (1, 2, s+1, y, \ldots, h, s, w, 3 = \tilde{n}, k)$ by (6.3). This contradicts $(\star)$. Hence y and $s+1$ are nonadjacent for each $y \in V_1$.

So far, by (6.3)$\sim$(6.5), (6.13)$\sim$(6.18) and (a), we have that $T \simeq D'_8$-type digraph. This contradicts the assumption of the Theorem.

(b) $\varphi_1(k) < \tilde{n} = 3$, then $2 = \varphi_1(k) \to k$.