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THE LOCAL EXPONENT SETS OF PRIMITIVE DIGRAPHS WITHOUT LOOP*

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Abstract Let $D=(V,E)$ be a primitive digraph. We define the local exponent of D at a vertex $u\in V$, denoted by $\exp_D(u)$, is the least integer k such that there is a directed walk of length k from u to v for each $v\in V$. Let $V=\{1,2,\cdots,n\}$. We order the vertices of V so that $\exp_D(1)\leqslant \exp_D(2)\leqslant\cdots\leqslant\exp_D(n)$. Let $\hat{E}_n(k):=\{\exp_D(k)\mid D\in PD_n(0)\}$, where $PD_n(0)$ is the set of all primitive digraphs without loop with order n. In 1990, $\hat{E}_n(n)$ is solved. In this paper, $\hat{E}_n(k)$ are completely solved for all $n(\geqslant 3)$, k with $1\leqslant k\leqslant n-1$.

Keywords primitive digraph, local exponent, gap
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Let $D=(V,E)$ be a digraph on n vertices. We permit loops but no multiple arcs in D. We use the notation $u\to v[k]$ ($u\not\to v[k]$, resp.) that there is a $u\to v$ walk (no $u\to v$ walk, resp.) with length k in D. A digraph D is primitive if there exists an integer k such that $u\to v[k]$ for every pair $u,v\in V$. The least such k is called the exponent of D, denoted $\gamma(D)$.

In 1950, H. Wielandt([10]) found that $\gamma(D)\leqslant w_n=(n-1)^2+1$ and shows there is a unique digraph $W(n)$ that attains this bound, where $W(n)=(V,E)$ is defined as follows: $V=\{v_i\mid 1\leqslant i\leqslant n\}$ and $E=\{(v_i,v_{i+1})\mid 1\leqslant i\leqslant n-1\}\cup\{(v_{n-1},v_1),(v_n,v_1)\}$. In 1964, A. L. Dulmage and N. S. Mendelsohn([2]) observe that there are gaps in the exponent set $E_n=\{\gamma(D)\mid D\in PD_n\}$, where PD_n is the set of all primitive digraphs of order n. Each gap is a set S of consecutive integers below w_n such that no $D\in PD_n$ has an exponent in S. In 1981, M. Lewin and Y. Vitek([3]) find a general method for determining all the gaps between $[\frac{1}{2}w_n]+1$ and w_n, and they conjecture that there is no gap in $\{1,2,\cdots,[\frac{1}{2}w_n]+1\}$. In 1985, Shao Jiayu([7]) proves that the conjecture is true when n is sufficiently large and gave a counterexample

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to show that the conjecture is not true when $n=11$. In 1987, Zhang Kemin[11] proves that the conjecture is true except for $n=11$. Therefore, the problems of determining the exponent set is completely solved.

Let $L(D)$ denote the set of cycle lengths of D. Let $D\in PD_n$ with $L(D)=\{s_1,s_2,\cdots,s_\lambda\}$. let $u,v\in V(D)$. The relative distance $d_{L(D)}(u,v)$ from u to v is the length of the shortest walk from u to v, which meets at least one s_i-cycle for $i=1,2,\cdots,\lambda$. The exponent from u to v, denoted by $\exp_D(u,v)$ or $\exp(u,v)$ if D is specified, is the least integer k such that $u\to v[m]$ for all $m\geqslant k$. Clearly, $\gamma(D)=\max\limits_{u,v\in V(D)}\exp_D(u,v)$.

Now let $\{s_1,s_2,\cdots,s_\lambda\}$ be a set of distinet positive integers with $\gcd(s_1,s_2,\cdots,s_\lambda)=1$. Then we define $\varphi(s_1,s_2,\cdots,s_\lambda)$ to be the least integer m such that every integer $k\geqslant m$ can be expressed in the form $k=\sum\limits_{i=1}^{\lambda}a_is_i$, where $a_i(i=1,2,\cdots,\lambda)$ are nonnegative integers. A result due to Schur shows that $\varphi(s_1,s_2,\cdots,s_\lambda)$ is well defined if $\gcd(s_1,s_2,\cdots,s_\lambda)=1$. It is known that $\varphi(s_1,s_2)=(s_1-1)(s_2-1)$ if $\gcd(s_1,s_2)=1$.

Lemma 1[7] Let $D\in PD_n$ and $L(D)=\{s_1,s_2,\cdots,s_\lambda\}$ be the set of cycle length of D. Then $\exp_D(u,v)\leqslant d_{L(D)}(u,v)+\varphi(s_1,s_2,\cdots,s_\lambda)$ for any $u,v\in V(D)$.

The local exponent of D at vertex $u\in V$, denoted by $\exp_D(u)$ or $\exp(u)$ if D is specified, is the least integer k such that $u\to v[k]$ for each $v\in V$. Clearly, $\exp_D(u)=\max\limits_{v\in V(D)}\exp_D(u,v)$ and $\gamma(D)=\max\limits_{u\in V(D)}\exp_D(u)$. Let $V=\{1,2,\cdots,n\}$. Then the vertices can be ordered so that $\exp_D(1)\leqslant\exp_D(2)\leqslant\cdots\leqslant\exp_D(n)=\gamma(D)$.

Lemma 2[1] Let $D\in PD_n$. Then $\exp_D(k)\leqslant\exp_D(k-1)+1$.

Corollary 1 Let $D\in PD_n$ with some loops. Then $\exp_D(k)\leqslant n-2+k$.

Proof Clearly, $\exp_D(1)\leqslant n-1$. Thus $\exp_D(k)\leqslant\exp_D(1)+k-1\leqslant n-2+k$ by Lemma 2.

Let $PD_n(0)$ be the set of all primitive digraphs without loop with order n. Let $L(n)=\{(p,q)\mid 2\leqslant p<q\leqslant n,p+q>n,\ \gcd(p,q)=1\}$. Let $E_n(k)=\{\exp_D(k)\mid D\in PD_n\}$ and $E_n(k)=\{\exp_D(k)\mid D\in PD_n(0)\}$ for each $k(1\leqslant k\leqslant n)$. Clearly, $E_n(n)=E_n$. In 1998, $E_n(1)$ is solved by [9]. Recently, $E_n(k)$ is solved by [6] for all k with $2\leqslant k\leqslant n-1$. Also $\hat{E}_n(n)$ is solved by [4] and [5]. We list these in the following:

Theorem 1[9] For all $n\geqslant 2$, we have $E_n(1)=\{1,2,\cdots,\frac{1}{2}(n^2-3n+4)\}\cup\bigcup\limits_{(p,q)\in L(n)}\{m\mid(p-1)(q-1)\leqslant m\leqslant p(q-2)+n-q+1\}$.

Theorem 2[6] Let n,k be integers with $2\leqslant k\leqslant n-1$. Then

$$E_n(k)=\{1,2,\cdots,[\frac{1}{2}(n-2)^2]+k\}\cup\bigcup_{(p,q)\in L(n)}\{m\mid m(p,q,k)\leqslant m\leqslant M(p,q,k)\}$$

for all n,k with $2\leqslant k\leqslant n-1$ except $n=11$ and $9\leqslant k\leqslant 10$. And

$$E_{11}(k)=(\{1,2,\cdots,40+k\}\setminus\{37+k\})\cup\bigcup_{(p,q)\in L(11)}\{m\mid m(p,q,k)\leqslant m\leqslant M(p,q,k)\}$$

for $9\leqslant k\leqslant 10$. Where $m(p,q,k)=(p-1)(q-1)+\max\{k+q-n-1,0\}$ and

$$M(p,q,k)=\begin{cases} p(q-2)+k+(n-q), & \text{if } 1\leqslant k\leqslant p+q-n+1, \\ p(q-2)+k+(n-q-1), & \text{if } p+q-n+2\leqslant k\leqslant p+q-n+3, \\ p(q-2)+k+(n-q-2), & \text{if } p+q-n+4\leqslant k\leqslant p+q-n+5, \\ \vdots & \vdots \\ p(q-2)+k+1, & \text{if } n+p-q-2\leqslant k\leqslant n+p-q-1, \\ p(q-2)+k, & \text{if } n+p-q\leqslant k\leqslant n. \end{cases}$$

Theorem 3[4],[5]　$\hat{E}_n(n)=E_n\backslash\{1\}$ for $n\geqslant 4$ and $\hat{E}_3(3)=E_3\backslash\{1,3\}$.

In this paper, we describe $\hat{E}_n(k)$ by $E_n(k)$ for all n,k with $1\leqslant k\leqslant n-1$.

Lemma 3[8]　The local exponent set of primitive simple graph of order n is $\{2,3,\cdots,e_1(n,k)\}$ for any integers n, k with $1\leqslant k\leqslant n-1$ and $n\geqslant 3$, where

$$e_1(n,k)=\begin{cases} n-2, & \text{if } k=1,2 \text{ and } n \text{ is even}, \\ n-1, & \text{if } k=1,2 \text{ and } n \text{ is odd}, \\ n-4+k, & \text{if } 3\leqslant k\leqslant n-1. \end{cases}$$

Let $D\in PD_n(0)$. Then $u\to u[1]$ for any $u\in V(D)$. Thus $\exp_D(k)\geqslant 2$. Since $PD_n(0)\subset PD_n$, $\hat{E}_n(k)\subset E_n(k)$. By Corollary 1, $\{m\mid m\in E_n(k),\ m\geqslant n-1+k\}\subset\hat{E}_n(k)$. Hence for determining $\hat{E}_n(k)$, by Lemma 3, it is enough to determine the following

(1) $n-1\in\hat{E}_n(1)$ for $n(\geqslant 3)$ is even;

(2) $n-1$, $n\in\hat{E}_n(2)$ for $n(\geqslant 3)$ is even;

(3) $n\in\hat{E}_n(2)$ for $n(\geqslant 3)$ is odd;

(4) $n-3+k$, $n-2+k\in\hat{E}_n(k)$ for any integers $n(\geqslant 4)$, k with $3\leqslant k\leqslant n-1$.

Let $D=(V,E)$ be a digraph, and choose a vertex $u\in V$. For $i\geqslant 1$, let $R_i(u):=\{v\in V\mid u\to v[i]\}$. We define $R_0(u):=\{u\}$.

Lemma 4　$n-2+k\in\hat{E}_n(k)$ for any integers $n(\geqslant 3)$, k with $1\leqslant k\leqslant n-1$.

Proof　Let D be a digraph which consists of $C_n=(v_1,v_2,\cdots,v_n,v_1)$ and further ares (v_1,v_n) and (v_1,v_{n-1}). Then it is easy to see that D is strong and $L(D)=\{2,3,n\}$. So $D\in PD_n(0)$ since D is primitive iff D is strong and $\gcd\{x\mid x\in L(D)\}=1$. Thus for any $v_y\in V(D)$, $\exp_D(v_1,v_y)\leqslant d_{L(D)}(v_1,v_y)+\varphi(2,3,n)\leqslant n-3+2=n-1$ by Lemma 1. Hence $\exp_D(v_1)\leqslant n-1$. If $v_1\to v_{n-2}[n-2]$, then there exist nonnegative integers k_1,k_2 and k_3 such that $n-2=n-3+2k_1+3k_2+nk_3$, i. e. $\varphi(2,3,n)=1$, a contradiction. So $\exp_D(v_1)=n-1$. Since $R_{n-i+1}(v_i)=\{v_1\}$ for $2\leqslant i\leqslant n$, $\exp_D(v_i)=\exp_D(v_1)+n-i+1$ for $2\leqslant i\leqslant n$. Hence

$$\exp_D(k)=n-1+k-1=n-2+k \text{ for } 1\leqslant k\leqslant n-1.$$

Lemma 5　$n-3+k\in\hat{E}_n(k)$ for any integers $n(\geqslant 4)$, k with $3\leqslant k\leqslant n-1$.

Proof　Let D be a digraph which consists of $C_n=(v_1,v_2,\cdots,v_n,v_1)$ and further arcs (v_1,v_n) and (v_n,v_{n-2}). Then it is easy to see that D is strong and $L(D)=\{2,3,n\}$. So $D\in PD_n(0)$ since D is primitive iff D is strong and $\gcd\{x\mid x\in L(D)\}=1$. Thus for any $v_y\in V(D)$, $\exp_D(v_n,v_y)\leqslant d_{L(D)}(v_n,v_y)+\varphi(2,3,n)\leqslant n-3+2=n-1$, by Lemma 1. Hence $\exp_D(v_n)\leqslant n-1$. If $v_n\to v_{n-3}[n-2]$, then there exist nonnegative integers k_1,k_2 and k_3 such that n

$-2=n-3+2k_1+3k_2+nk_3$, i. e. $\varphi(2,3,n)=1$, a contradiction. So $\exp_D(v_n v_n)=n-1$. Since $R^*_{n-i}(v_i)=\{v_n\}$ for $2\leqslant i\leqslant n-1$, $\exp_D(v_n)<\exp_D(v_{n-1})<\cdots<\exp_D(v_3)<\exp_D(v_2)$. Since $R_1(v_1)\supset\{v_n\}$, $\exp_D(v_1)\leqslant\exp_D(v_n)+1=n$. If $v_1\to v_{n-3}[n-1]$, then there exists nonnegative integers l_1,l_2 and l_3 such that $n-1=2+n-4+2l_1+3l_2+nl_3$ or $n-1=n-4+2l_1+nl_3$ if $n\geqslant 4$, i. e. $2l_1+3l_2+nl_3=1$ or $2l_1+nl_3=3$ if $n\geqslant 4$. This contradicts to $\varphi(2,3,n)=2$ or $n\geqslant 4$. So $\exp_D(v_1)=n$. Thus $\exp_D(1)=n-1$, $\exp_D(2)=\exp_D(3)=n$ and $\exp_D(k)=n+k-3$ for $4\leqslant k\leqslant n-1$. Hence $\exp_D(k)=n-3+k$ for $3\leqslant k\leqslant n-1$.

Lemma 6 $n-1\in\hat{E}_n(2)$ for $n\geqslant 3$.

Proof When $n=3$, let D be a digraph which consists of $C_3=(v_1,v_2,v_3,v_1)$ and $C'_3=(v_3,v_2,v_1,v_3)$. Then it is easy to check that $\exp_D(2)=2$. In the following we assume that $n\geqslant 4$. Let D be a digraph which consists of $C_n=(v_1,v_2,\cdots,v_{n-1},v_1)$ and further arcs (v_1,v_n), (v_n,v_1) and (v_1,v_{n-2}). Then $D\in PD_n(0)$ and $L(D)=\{2,3,n\}$. Thus $\exp_D(v_1,v_y)\leqslant d(v_1,v_y)+\varphi(2,3,n)\leqslant n-4+2=n-2$. hence $\exp_D(v_1)\leqslant n-2$. If $v_1\to v_{n-3}[n-3]$, then there exist nonnegative integers k_1,k_2 and k_3 such that $n-3=n-4+2k_1+3k_2+nk_3$, i. e. $\varphi(2,3,n)=1$, a contradiction. So $\exp_D(v_1)=n-2$. Since $R_1(v_n)=\{v_1\}$ and $R_i(v_{n-i})=\{v_1\}$ for $2\leqslant i\leqslant n-1$, $\exp_D(v_1)<\exp_D(v_n)$ and $\exp_D(v_1)<\exp_D(v_{n-1})<\exp_D(v_{n-2})<\cdots<\exp_D(v_3)<\exp_D(v_2)$. Thus $\exp_D(1)=n-2$, $\exp_D(2)=n-1$ for $n\geqslant 4$.

To sum up, we get

Main Theorem $\hat{E}_n(k)=E_n(k)\setminus\{1\}$ for any integers $n(\geqslant 3)$, k with $1\leqslant k\leqslant n-1$.

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无环本原有向图的局部指数集

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摘要　设 $D=(V,E)$ 是一个本原有向图，我们定义 D 的在顶点 $u\in V$ 处的局部指数，记作 $\exp_D(u)$，是使得对每个 $v\in V$ 从 u 到 v 均有长为 k 的有向通道的最小整数 k. 令 $V=\{1,2,\cdots,n\}$，我们对顶点排序使得 $\exp_D(1)\leqslant\exp_D(2)\leqslant\cdots\leqslant\exp_D(n)$. 令

$$\hat{E}_n(k):=\{\exp_D(k)\mid D\in PD_n(0)\},$$

这里 $PD_n(0)$ 表示所有 n 阶无环本原有向图的集合．1990 年，$\hat{E}_n(n)$ 已经解决，本文完全解决了 $\hat{E}_n(k)(n\geqslant 3,1\leqslant k\leqslant n-1)$.

关键词　本原有向图，局部指数，缺数

中图法分类号　O157.5