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A note on hypertournaments

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Abstract It is proved that for given integer $k \geqslant 2$, almost all k-hypertournaments are strong and in almost all k-hypertournaments, every pair of vertices lies on a 3-cycle.

Keywords: tournament, hypertournament, cycle, strongly connected digraph.

GIVEN two integers n and k, $n \geqslant k > 1$, a k-hypertournament H on n vertices is a pair (V, A), where V is a set of vertices, $(|V| = n)$ and A is a set of k-tuple of vertices, called arcs, so that for any k-subset S of V, A contains exactly one of the $k!$ k-tuples whose entries belong to S. That is, H may be thought of as arising from an orientation of the hyperedges of the complete k-uniform hypergraph[1]. Clearly a 2-hypertournament is merely a tournament.

Let $H = (V, A)$ denote a k-hypertournament H on n vertices. A path in H is a sequence $v_1 a_1 v_2 a_2 v_3 \cdots v_{t-1} a_{t-1} v_t$ of distinct vertices $v_1, v_2, \cdots, v_t$, $t \geqslant 1$, and distinct arcs $a_1, a_2, \cdots, a_{t-1}$ such that v_i precedes v_{i+1} in a_i, $1 \leqslant i \leqslant t-1$. A cycle in H is a sequence $v_1 a_1 v_2 a_2 v_3 \cdots v_{t-1} a_{t-1} v_t a_t v_1$ of distinct vertices $v_1, v_2, \cdots, v_t$ and distinct arcs $a_1, \cdots, a_t$, $t \geqslant 1$, such that v_i precedes v_{i+1} in a_i, $1 \leqslant i \leqslant t$ $(v_{t+1} = v_1)$. A path or cycle Q in H is Hamiltonian if $V(Q) = V(H)$. H is Hamiltonian if it has a Hamiltonian cycle. A path from x to y is called an (x, y)-path. H is called strong if H has an (x, y)-path for every (ordered) pair x, y of distinct vertices in H. A strong component H' of a k-hypertournament H is a maximal subhypertournament which is strong.

Reference [2] proved that every strong k-hypertournament with n vertices, where $2 \leqslant k \leqslant n-2$, contains a Hamiltonian cycle. In this note, we prove that for given integer $k \geqslant 2$, almost all k-hypertournaments are Hamiltonian, and we obtain extensions of the theorems on tournaments: almost all tournaments are strong[3], and in almost all tournaments, every pair of vertices lies on a 3-cycle[4]. We prove that for given integer $k \geqslant 2$, almost all k-hypertournaments are strong and in almost all k-hypertournaments, every pair of vertices lies on a 3-cycle.

Theorem 1. For given integer $k \geqslant 2$, almost all k-hypertournaments are strong.

To prove this theorem, we need the following three lemmas.

Lemma 1. If $1 \leqslant i < k$, then $i!^a (k-i)!^b \leqslant (k-1)!^{\max\{a, b\}}$.

The inequality can be easily verifed by induction on k.

Lemma 2. $\binom{j}{k} + \binom{n-j}{k} \leqslant \binom{n-1}{k}$ for $1 \leqslant j \leqslant n-1, k \geqslant 2$, where $\binom{n}{k}$ denotes binomial coefficient, when $n < k$, $\binom{n}{k} = 0$.

Proof. Without loss of generality, we assume $j \leqslant \frac{n}{2}$. If $j < k$, the lemma is clearly valid. Hence, suppose that $k \leqslant j \leqslant n/2$, we have $\binom{j}{k} - \binom{j-1}{k} = \binom{j-1}{k-1} \leqslant \binom{n-j}{k-1} = \binom{n-j+1}{k} - \binom{n-j}{k}$, i.e. $\binom{j}{k} + \binom{n-j}{k} \leqslant \binom{j-1}{k} + \binom{n-j+1}{k} \leqslant \binom{j-2}{k} + \binom{n-j+2}{k} \leqslant \cdots \leqslant \binom{k-1}{k} +$

$\binom{n-k+1}{k}$. So $\binom{j}{k} + (n - j = k) \leqslant \binom{n-k+1}{k} \leqslant \binom{n-1}{k}$. Q.E.D.

Lemma 3. Let H be a not strong k-hypertournament with $n(>k)$ vertices. Then the strong components of H, $C_1, C_2, \cdots, C_t$ can be labeled such that there is no arc in which the vertices from C_j precede the vertices from C_i, for $1 \leqslant i < j \leqslant t$.

Proof. We take $v_i \in V(C_i), i = 1, 2, \cdots, t$ and construct a tournament H' with vertices $v_1, v_2, \cdots, v_t$. An arc v_iv_j is in H' if and only if the vertices from C_i precede the vertices from C_j in all arcs which contain vertices from C_i and vertices from C_j in H. Clearly, H' is a transitive tournament. So the vertices of H' can be labeled such that v_jv_i is not in H' for $1 \leqslant i < j \leqslant t$. Therefore the components of H can be labeled such that there is no arc in which the vertices from C_j precede the vertices from C_i, for $1 \leqslant i < j \leqslant t$.

Proof of Thoerem 1. When $k = 2$, ref. [3] proved that almost all tournaments are strong. Hence, suppose that $k \geqslant 3$. Let $p(n)$ be the probability that a k-hypertournament on $n(\geqslant k)$ vertices, chosen at random from the set of $k!^{\binom{n}{k}}$ possible ones, will be strong. $p(1) = 1$, by definition.

If a given k-hypertournament on $n(\geqslant k)$ vertices is not strong, then by Lemma 3 there exists a maximal proper subset V' of the vertices such that V' together with arcs whose entries belong to V' form a strong subhypertournament, and in all arcs which contain vertices not in V' and vertices which are in V', the vertices not in V' precede the vertices which are in V'.

The probability that this subset consists of j vertices, $1 \leqslant j \leqslant n-1$, denoted by $q(j)$, is

$$q(j) = P(j)\binom{n}{j} k!^{\binom{j}{k}} k!^{\binom{n-j}{k}} \Big(1!^{\binom{n-j}{1}} (k-1)!^{\binom{j}{k-1}} + 2!^{\binom{n-j}{2}} (k-2)!^{\binom{j}{k-2}} + \cdots + (k-1)!^{\binom{n-j}{k-1}} 1!^{\binom{j}{1}}\Big) / k!^{\binom{n}{k}}.$$

As these cases are mutually exclusive and exhaustive, summing over j we have $p(n) = 1 - \sum_{j=1}^{n-1} q(j)$.

In order to estimate $p(n)$, we need bounds of $q(j)$.

By Lemmas 1, 2 and $0 \leqslant p(j) \leqslant 1$, when $n \geqslant 2k$, we have

$$\begin{aligned} q(j) &\leqslant \binom{n}{j} k!^{\binom{j}{k}+\binom{n-j}{k}} \Big((k-1)(k-1)!^{\binom{n-1}{k-1}}\Big) / k!^{\binom{n}{k}} \\ &= \frac{\binom{n}{j} k!^{\binom{j}{k}+\binom{n-j}{k}} (k-1) k!^{\binom{n-1}{k-1}}}{k!^{\binom{n}{k}} k^{\binom{n-1}{k-1}}} \\ &= \frac{\binom{n}{j}(k-1)}{k!^{\binom{n}{k}-\binom{n-1}{k-1}-\binom{j}{k}-\binom{n-j}{k}} k^{\binom{n-1}{k-1}}} \\ &= \frac{\binom{n}{j}(k-1)}{k!^{\binom{n-1}{k}-\binom{j}{k}-\binom{n-j}{k}} k^{\binom{n-1}{k-1}}} \\ &\leqslant \frac{\binom{n}{j}(k-1)}{k^{\binom{n-1}{k-1}}}. \end{aligned}$$

Hence

$$0 \leqslant \sum_{j=1}^{n-1} q(j) \leqslant \frac{(k-1)\sum_{j=1}^{n-1}\binom{n}{j}}{k^{\binom{n-1}{k-1}}} < \frac{(k-1)2^n}{k^{(n-1)(n-2)}}.$$

So when $n \to \infty$, $\sum_{j=1}^{n-1} q(j) \to 0$ since $k \geqslant 3$. Therefore we have $p(n) \to 1$ as $n \to \infty$.

This completes the proof of the theorem.

Reference [2] proved that every strong k-hypertournament with n vertices, where $3 \leqslant k \leqslant n-2$, contains a Hamiltonian cycle. By this result and Theorem 1 we have the following

Corollary. For given integer $k \geqslant 2$, almost all k-hypertournaments are Hamiltonian.

Theorem 2. For given integer $k \geqslant 2$, in almost all k-hypertournaments, every pair of vertices lies on a 3-cycle.

Proof. Let u, v and w be vertices in a random k-hypertournament. Let a be any arc which contains u, v. Without loss of generality we assume u precedes v in a. The probability that there is an arc b which is distinct from a and in which v precedes w and there is another arc which is distinct from a and b and in which w precedes u is $\frac{1}{4}$. If z is a fourth vertex, then the probability that neither w nor z with u, v yields a 3-cycle is $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$. In general, in a random k-hypertournament of order n, the probability that u and v do not lie on a 3-cycle is $\left(\frac{3}{4}\right)^{n-2}$. So the probability that u and v lie on a 3-cycle approaches 1 as $n \to \infty$.

This completes the proof of the theorem.

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