The value of the Ramsey number $R(C_n, K_4)$ is $3(n - 1) + 1 \ (n \geq 4)$

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Abstract

The Ramsey number $R(C_n, K_m)$ is the smallest integer $p$ such that any graph $G$ on $p$ vertices either contains a cycle $C_n$ with length $n$ or contains an independent set with order $m$. In this paper we prove that $R(C_n, K_4) = 3(n - 1) + 1 \ (n \geq 4)$.

We shall only consider graphs without multiple edges or loops. The Ramsey number $R(C_n, K_m)$ is the smallest integer $p$ such that any graph $G$ on $p$ vertices either contains a cycle $C_n$ with length $n$ or contains an independent set with order $m$.

In 1973, J.A. Bondy and P. Erdős proved that following theorem.

Theorem 1.1 ([1]). $R(C_n, K_m) = (n - 1)(m - 1) + 1$ for $n \geq m^2 - 2$.

In 1976, R.H. Schelp and R.J. Faudree in [8] stated the following problem:

Problem 1.2 ([8]). Find the range of integers $n$ and $m$ such that $R(C_n, K_m) = (n - 1)(m - 1) + 1$. In particular, does the equality hold for $n \geq m$?

For this problem, the known results are $R(C_4, K_4) = 10$ (see [2]), $R(C_5, K_4) = 13$, $R(C_5, K_5) = 17$ (see [4], [5]) and $R(C_n, K_3) = 2n - 1 \ (n > 3)$ (see [3], [6]). However, so far, even for some fixed small $m$, the problem has not been solved.

In the following, we will prove that $R(C_n, K_4) = 3(n - 1) + 1$ for $n \geq 4$.

Lemma 1.3. Suppose $G$ is a graph that contains the cycle $(v_1, v_2, \ldots, v_{n-1})$ of length $n - 1$ but no cycle of length $n$. Let $X \subseteq V(G) \setminus \{v_1, v_2, \ldots, v_{n-1}\}$. Then

(a) No vertex $x \in X$ is adjacent to two consecutive vertices on the cycle.
(b) If $x \in X$ is adjacent to $v_i$ and $v_j$, then $v_{i+1}v_{j+1} \not\in E(G)$.
(c) If $x \in X$ is adjacent to $v_i$ and $v_j$, then no vertex $x' \in X$ is adjacent to both $v_{i+1}$ and $v_{j+2}$.

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Proof. (a) and (b) were used in [1]. If $x$ is adjacent to $v_i$ and $v_j$ and $x'$ is adjacent to $v_{i+1}$ and $v_{j+2}$, then $x \neq x'$ and $(v_i, x, v_j, v_{j-1}, \ldots, v_{i+1}, x', v_{j+2}, \ldots, v_{i-1})$ is a cycle of length $n$ in $G$; this proves (c). □

Theorem 1.4. For all $n \geq 4$, $R(C_n, K_4) = 3(n - 1) + 1$.

Proof. The example $G = 3K_{n-1}$ establishes the lower bound $R(C_n, K_4) \geq 3(n - 1) + 1$, so it suffices to prove that for $n \geq 4$ every graph $G$ of order $3(n - 1) + 1$ contains either $C_n$ or a 4-element independent set. Since the desired result is true for $n = 4$ and $n = 5$, we may take $n > 5$ and assume by induction that $R(C_{n-1}, K_4) = 3(n - 2) + 1$. Assume that $G(V, E)$ is a graph of order $3(n - 1) + 1$ that contains neither a $C_n$ nor a 4-element independent set. Using $R(C_n, K_3) = 2(n - 1) + 1$ and $R(C_{n-1}, K_4) = 3(n - 2) + 1$, we find that $G$ contains a 3-element independent set $X = \{x_1, x_2, x_3\}$, and, disjoint from $X$, a cycle $(v_1, v_2, \ldots, v_{n-1})$ of length $n - 1$. Let us refer to $(v_1, v_2, \ldots, v_{n-1})$ as simply the cycle. Since $G$ has no 4-element independent set, each vertex on the cycle is adjacent to at least one vertex in $X$. Since $n - 1 > 3$ at least one vertex in $X$ is adjacent to two or more vertices of the cycle. Thus we may assume that $x_1$ is adjacent to $v_i$ and $v_j$. By part (b) of Lemma 1.3 $v_{i+1}v_{j+1} \notin E$. Since $n > 5$ and $x_1$ cannot be adjacent to three or more vertices of the cycle by part (a) and (b) of Lemma 1.3, thus $x_1v_{j+2} \notin E$. By part (a) of Lemma 1.3, $x_1v_{i+1} \notin E$ and $x_1v_{j+1} \notin E$. Since $v_{j+2}$ is adjacent to some vertex in $X$, we may assume that $x_2v_{j+2} \in E$. By part (c) of Lemma 1.3 $x_2v_{i+1} \notin E$ and by part (a) of Lemma 1.3 $x_2v_{j+1} \notin E$. Thus $\{x_1, x_2, v_{i+1}, v_{j+1}\}$ is a 4-element independent set, a contradiction. □

REFERENCES

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