The value of the Ramsey number $R(C_n, K_4)$ is $3(n-1)+1$ $(n \geq 4)$

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Abstract

The Ramsey number $R(C_n, K_m)$ is the smallest integer p such that any graph G on p vertices either contains a cycle C_n with length n or contains an independent set with order m. In this paper we prove that $R(C_n, K_4) = 3(n-1)+1$ $(n \geq 4)$.

We shall only consider graphs without multiple edges or loops.

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In 1973, J.A. Bondy and P. Erdös proved that following theorem.

Theorem 1.1 ([1]). *$R(C_n, K_m) = (n-1)(m-1)+1$ for $n \geq m^2 - 2$.*

In 1976, R.H. Schelp and R.J. Faudree in [8] stated the following problem:

Problem 1.2 ([8]). *Find the range of integers n and m such that $R(C_n, K_m) = (n-1)(m-1)+1$. In particular, does the equality hold for $n \geq m$?*

For this problem, the known results are $R(C_4, K_4) = 10$ (see [2]), $R(C_5, K_4) = 13$, $R(C_5, K_5) = 17$ (see [4], [5]) and $R(C_n, K_3) = 2n-1$ $(n > 3)$ (see [3], [6]). However, so far, even for some fixed small m, the problem has not been solved.

In the following, we will prove that $R(C_n, K_4) = 3(n-1)+1$ for $n \geq 4$.

Lemma 1.3. *Suppose G is a graph that contains the cycle $(v_1, v_2, \cdots, v_{n-1})$ of length $n-1$ but no cycle of length n. Let $X \subseteq V(G) \setminus \{v_1, v_2, \cdots, v_{n-1}\}$. Then*

(a) *No vertex $x \in X$ is adjacent to two consecutive vertices on the cycle.*
(b) *If $x \in X$ is adjacent to v_i and v_j, then $v_{i+1}v_{j+1} \notin E(G)$.*
(c) *If $x \in X$ is adjacent to v_i and v_j, then no vertex $x' \in X$ is adjacent to both v_{i+1} and v_{j+2}.*

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Proof. (a) and (b) were used in [1]. If x is adjacent to v_i and v_j and x' is adjacent to v_{i+1} and v_{j+2}, then $x \neq x'$ and $(v_i, x, v_j, v_{j-1}, \cdots, v_{i+1}, x', v_{j+2}, \cdots, v_{i-1})$ is a cycle of length n in G; this proves (c). $\square$

Theorem 1.4. *For all $n \geq 4$, $R(C_n, K_4) = 3(n-1)+1$.*

Proof. The example $G = 3K_{n-1}$ establishes the lower bound $R(C_n, K_4) \geq 3(n-1)+1$, so it suffices to prove that for $n \geq 4$ every graph G of order $3(n-1)+1$ contains either C_n or a 4-element independent set. Since the desired result is true for $n = 4$ and $n = 5$, we may take $n > 5$ and assume by induction that $R(C_{n-1}, K_4) = 3(n-2)+1$. Assume that $G(V, E)$ is a graph of order $3(n-1)+1$ that contains neither a C_n nor a 4-element independent set. Using $R(C_n, K_3) = 2(n-1)+1$ and $R(C_{n-1}, K_4) = 3(n-2)+1$, we find that G contains a 3-element independent set $X = \{x_1, x_2, x_3\}$, and, disjoint from X, a cycle $(v_1, v_2, \cdots, v_{n-1})$ of length $n-1$. Let us refer to $(v_1, v_2, \cdots, v_{n-1})$ as simply **the cycle**. Since G has no 4-element independent set, each vertex on the cycle is adjacent to at least one vertex in X. Since $n-1 > 3$ at least one vertex in X is adjacent to two or more vertices of the cycle. Thus we may assume that x_1 is adjacent to v_i and v_j. By part (b) of Lemma 1.3 $v_{i+1}v_{j+1} \notin E$. Since $n > 5$ and x_1 cannot be adjacent to three or more vertices of the cycle by part (a) and (b) of Lemma 1.3, thus $x_1v_{j+2} \notin E$. By part (a) of Lemma 1.3, $x_1v_{i+1} \notin E$ and $x_1v_{j+1} \notin E$. Since v_{j+2} is adjacent to some vertex in X, we may assume that $x_2v_{j+2} \in E$. By part (c) of Lemma 1.3 $x_2v_{i+1} \notin E$ and by part (a) of Lemma 1.3 $x_2v_{j+1} \notin E$. Thus $\{x_1, x_2, v_{i+1}, v_{j+1}\}$ is a 4-element independent set, a contradiction. $\square$

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