EQUITABLE COLORINGS OF LINE GRAPHS AND COMPLETE r-PARTITE GRAPHS*

WANG Weifan

(*Department of Mathematics, Liaoning University, Shenyang* 110036, *China*)

ZHANG Kemin

(*Department of Mathematics, Nanjing University, Nanjing* 210093, *China*)

Abstract. It is shown in this paper that Meyer's conjecture on the equitable coloring holds for line graphs and complete r-partite graphs.

Key words. Equitable chromatic number, line graph, complete r-partite graph.

1. Introduction

Let G be a graph with vertex set $V(G)$, edge set $E(G)$, maximum degree $\Delta(G)$, minimum degree $\delta(G)$, vertex chromatic number $\chi(G)$, and edge chromatic number $\chi'(G)$. G is equitably k-colorable if $V(G)$ can be partitioned into k independent sets V_1, $V_2, \cdots, V_k$ such that $||V_i| - |V_j|| \leq 1$ for all i and j. The such smallest integer k as above is called the equitable chromatic number of G, denoted by $\chi_=(G)$. Similarly we can define the equitable edge chromatic number of a graph G and denote it by $\chi'_=(G)$. Hajnal and Szemerédi[1] proved that $\chi_=(G) \leq \Delta(G)+1$ for every graph G. Meyer[2] conjectured that $\chi_=(G) \leq \Delta(G)$ if a connected graph G is neither a complete graph nor an odd cycle. This conjecture has been confirmed for a few special cases such as trees[3], bipartite graphs[4], outerplanar graphs[5], and graphs G having either $\Delta(G) \geq \frac{1}{2}|V(G)|$ or $\Delta(G) \leq 3$[6]. Recently, Zhang and Yap[7] proved that every planar graph G with $\Delta(G) \geq 13$ is equitably $\Delta(G)$-colorable.

The purpose of this paper is to investigate the equitable colorings of line graphs and complete r-partite graphs. We shall use $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the largest integer $\leq x$ and the smallest integer $\geq x$, respectively.

2. Line Graphs

The line graph, denoted by $L(G)$, of a graph G is a graph in which $V(L(G)) = E(G)$ and two vertices are adjacent in $L(G)$ iff they are adjacent as edges of G. By the definition, the following result is straightforward.

Lemma 2.1 *If H is the line graph of a graph G, then $\chi(L(G)) = \chi'(G)$.*

Theorem 2.2 *Let G be a graph and k an integer with $k \geq \chi'(G)$. Then G is equitably k-edge colorable.*

Proof Let $(E_1, E_2, \cdots, E_k)$ be a k-edge coloring of G such that the sum $\sum_{i,j}^{k} ||E_i| - |E_j||$ as small as possible. We claim that $||E_i| - |E_j|| \leq 1$ for all i and j, equivalently, $(E_1, E_2, \cdots, E_k)$

Received September 30, 1997.

Revised January 9, 1999.

*This research is supported by the NSFC(No.19871040), NSFJ and SFECLN.

is an equitable k-edge coloring of G. In fact, if there are i_0 and j_0 such that $||E_{i_0}| - |E_{j_0}|| \geq 2$, we may suppose that $|E_{i_0}| \geq |E_{j_0}| + 2$. Let $H = G[E_{i_0} \cup E_{j_0}]$ denote the subgraph of G induced by the edge subset $E_{i_0} \cup E_{j_0}$. Thus every vertex of H is of degree one or two in H since it is incident to at most one edge of E_{i_0} and one edge of E_{j_0}. This implies that each component of H is either an even cycle or a path with edges alternately in E_{i_0} and E_{j_0}. It follows from $|E_{i_0}| \geq |E_{j_0}| + 2$ that there exists a path component P of H such that P starts and ends at the edges of E_{i_0}. Suppose that $P = u_0 e_1 u_1 e_2 u_2 \cdots e_{2m+1} u_{2m+1}$ be a such path, where $e_t \in E_{i_0}$, $t = 1, 3, \cdots, 2m+1$, $e_s \in E_{j_0}$, $s = 2, 4, \cdots, 2m$, $m \geq 0$. We form a new k-edge coloring of G as follows:

$\overline{E}_{i_0} = (E_{i_0} - \{e_1, e_3, \cdots, e_{2m+1}\}) \cup \{e_2, e_4, \cdots, e_{2m}\}$,

$\overline{E}_{j_0} = (E_{j_0} - \{e_2, e_4, \cdots, e_{2m}\}) \cup \{e_1, e_3, \cdots, e_{2m+1}\}$,

$\overline{E}_j = E_j$, $j = 1, 2, \cdots, k$ and $j \neq i_0, j_0$.

Obviously we have

$$\sum_{i,j}^{k} ||\overline{E}_i| - |\overline{E}_j|| < \sum_{i,j}^{k} ||E_i| - |E_j||.$$

This contradicts the choice of $(E_1, E_2, \cdots, E_k)$. ∎

Theorem 2.3 *If G is a line graph, then $\chi_=(G) = \chi(G)$.*

Proof Let G be the line graph of a graph H. Then, by Lemma 2.1, $\chi(G) = \chi'(H)$. We write $q = \chi'(H)$. By Theorem 2.2, H has an equitable q-edge coloring $(E_1, E_2, \cdots, E_q)$. In view of the definition of line graph, $V(G)$ is partitioned into q subsets $V_1, V_2, \cdots, V_q$, where V_i corresponds to E_i, and $|V_i| = |E_i|$, $i = 1, 2, \cdots, q$. Since E_i is an independent set of edges in H, V_i also is an independent set of vertices in G. So $(V_1, V_2, \cdots, V_q)$ forms a q-vertex coloring of G. Furthermore, since $||V_i| - |V_j|| = ||E_i| - |E_j|| \leq 1$, $(V_1, V_2, \cdots, V_q)$ is an equitable q-coloring of G. This implies that

$$\chi_=(G) \leq q = \chi'(H) = \chi(G).$$

Conversely, $\chi_=(G) \geq \chi(G)$ is trivial. Therefore

$$\chi_=(G) = \chi(G).$$ ∎

Corollary 2.4 *Meyer's conjecture is true for all line graphs.*

Proof First it follows from Theorem 2.3 that $\chi_=(G) = \chi(G)$ for every line graph G. Next, by Brooks theorem on vertex coloring, we have $\chi(G) \leq \Delta(G)$ when G is neither a complete graph nor an odd cycle. Consequently, $\chi_=(G) \leq \Delta(G)$. ∎

Lemma 2.5[8] *Let G be a graph. Then the following statements are equivalent:*

(1) *G is a line graph;*

(2) *The edges of G can be partitioned into a union of complete subgraphs such that each vertex of G is in at most two such complete subgraphs;*

(3) *G is $K_{1,3}$-free and any induced subgraph isomorphic to $K_4 - e$ has at least one of its triangles even. (We call a triangle even if any vertex in G is adjacent to an even number of its vertices.)*

By Lemma 2.5 and Theorem 2.3, we immediately have

Corollary 2.6 *If G is a graph satisfying* (2) *or* (3) *in Lemma* 2.5, *then $\chi_=(G) = \chi(G)$.*

Corollary 2.7 *If a graph G satisfies one of the following conditions, then $\chi_=(G) = \chi(G)$.*

(a) *G contains no induced subgraph which is isomorphic to $K_{1,3}$ or $K_4 - e$.*

(b) *G is connected, every block of G is complete, and each cut-vertex lies in exactly two blocks.*

Proof It suffices to note by [8] that G is the line graph of a K_3-free graph (or a tree) if G satisfies (a) (or (b)) in the corollary. ∎

3. Complete r-Partite Graphs

A graph is called a complete r-partite graph, denoted by $K_{n_1,n_2,\cdots,n_r}$, if its vertex set can be partitioned into r independent sets V_1, V_2, $\cdots$, V_r so that every vertex in V_i is joined to every vertex in V_j, $j \neq i$, where $|V_i| = n_i \geq 1$, $i = 1,2,\cdots,r$, $r \geq 2$. Without loss of generality, we may suppose that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$, and $r \geq 3$ because the case $r = 2$ was studied by Lih and Wu[4]. It is easy to see that $\Delta(K_{n_1,n_2,\cdots,n_r}) = \sum_{i=2}^{r} n_i$ and $\delta(K_{n_1,n_2,\cdots,n_r}) = \sum_{i=1}^{r-1} n_i$. In particular, $K_{n_1,n_2,\cdots,n_r} = K_r$ if $n_1 = n_2 = \cdots = n_r = 1$, where K_r denotes the complete graph of order r. Thus, in this case, we obtain $\chi_=(K_{n_1,n_2,\cdots,n_r}) = \chi_=(K_r) = \chi(K_r) = r = \Delta(K_r) + 1$.

Theorem 3.1 *Let $K_{n_1,n_2,\cdots,n_r}$ be a complete r-partite graph with $r > 1$, then*

$$\chi_=(K_{n_1,n_2,\cdots,n_r}) \leq \Delta(K_{n_1,n_2,\cdots,n_r}).$$

Proof It is not difficult to see that the problem of finding the equitable chromatic number of $K_{n_1,n_2,\cdots,n_r}$ can be reduced to solve the following Integer Linear Programming:

$$(ILP_k)\quad \begin{cases} f_k = \min\sum\limits_{i=1}^{r} x_i \\ kx_i + y_i = n_i, \ i = 1,2,\cdots,r; \\ 0 \leq y_i \leq x_i, \quad i = 1,2,\cdots,r; \\ x_i, y_i \ \text{ integer.} \end{cases}$$

Let $I = \{1,2,\cdots,n_1\}$ and $J = \{k \in I \,|\, (ILP_k) \text{ has a feasible solution }\}$. We denote by $(x_1^k, x_2^k, \cdots, x_r^k, y_1^k, y_2^k, \cdots, y_r^k)$ and f_k^* an optimal solution and the optimal value of (ILP_k) for $k \in J$, respectively. Then

$$\chi_=(K_{n_1,n_2,\cdots,n_r}) = \min_{k\in J} f_k^*.$$

In order to obtain the required result, we now prove that $f_1^* = \sum_{i=1}^{r} x_i^1 \leq \sum_{i=2}^{r} n_i$. Obviously, $1 \in J$, and $x_i^1 = \lceil \frac{n_i}{2} \rceil, i = 1,2,\cdots,r$. Let us consider two cases as follows.

Case 1 $1 \leq n_1 \leq 2$. In this case, $n_r \geq 2, x_1^1 = 1$ and $x_r^1 = \lceil \frac{n_r}{2} \rceil \geq 1$. It follows that

$$\begin{aligned} \sum_{i=1}^{r} x_i^1 &= x_1^1 + x_r^1 + \sum_{i=2}^{r-1} x_i^1 = 1 + \left\lceil \frac{n_r}{2} \right\rceil + \sum_{i=2}^{r-1} \left\lceil \frac{n_i}{2} \right\rceil \\ &\leq n_r + \sum_{i=2}^{r-1} n_i = \sum_{i=2}^{r} n_i. \end{aligned}$$

Case 2 $n_1 \geq 3$. If $n_1 = n_2 = \cdots = n_r = 3$, then by $r \geq 3$, we have

$$\sum_{i=1}^{r} x_i^1 = \left\lceil \frac{3}{2} \right\rceil r = 2r \leq 3(r-1) = \sum_{i=2}^{r} n_i.$$

If $n_r \geq 4$ and $\lceil \frac{n_1}{2} \rceil \leq r$, then

$$\begin{aligned} \sum_{i=1}^{r} x_i^1 &= \sum_{i=1}^{r} \left\lceil \frac{n_i}{2} \right\rceil \leq r + \sum_{i=2}^{r} \left\lceil \frac{n_i}{2} \right\rceil \\ &= \sum_{i=2}^{r-1} \left(\left\lceil \frac{n_i}{2} \right\rceil + 1 \right) + \left\lceil \frac{n_r}{2} \right\rceil + 2 \leq \sum_{i=2}^{r} n_i. \end{aligned}$$

If $n_r \geq 4$ and $\lceil \frac{n_1}{2} \rceil \geq r+1$, then $n_i \geq n_1 \geq 2r \geq 6$, thus

$$\begin{aligned}\sum_{i=1}^{r} x_i^1 &= \left\lceil \frac{n_1}{2} \right\rceil + \sum_{i=2}^{r} \left\lceil \frac{n_i}{2} \right\rceil \\ &= \sum_{i=2}^{r} \left(\left\lceil \frac{n_i}{2} \right\rceil + \frac{1}{r-1} \left\lceil \frac{n_1}{2} \right\rceil \right) \\ &\leq \sum_{i=2}^{r} \left(\left\lceil \frac{n_i}{2} \right\rceil + \left\lceil \frac{n_1+1}{2(r-1)} \right\rceil \right) \leq \sum_{i=2}^{r} \left(\left\lceil \frac{n_i}{2} \right\rceil + \left\lceil \frac{n_i+1}{4} \right\rceil \right) \leq \sum_{i=2}^{r} n_i.\end{aligned}$$

If we can further prove that $f_k^* \leq f_m^*$ for any $k, m \in J$ and $k > m$, then the proof will be complete by the following inequality

$$\chi_=(K_{n_1,n_2,\cdots,n_r}) = \min_{k \in J} f_k^* \leq f_1^* \leq \sum_{i=2}^{r} n_i = \Delta(K_{n_1,n_2,\cdots,n_r}).$$

Indeed, for $i = 1, 2, \cdots, r$, we have $kx_i^k + y_i^k = mx_i^m + y_i^m = n_i$. Thus

$$\begin{aligned} m(x_i^m - x_i^k) &= (k-m)x_i^k + y_i^k - y_i^m \\ &\geq (k-m)x_i^k + y_i^k - x_i^m \\ &\geq (k-m)(x_i^k - x_i^m) + y_i^k. \end{aligned}$$

Equivalently

$$k(x_i^m - x_i^k) \geq y_i^k \geq 0.$$

From $k \geq 1$, it follows that $x_i^m - x_i^k \geq 0$. Therefore

$$f_m^* - f_k^* = \sum_{i=1}^{r} x_i^m - \sum_{i=1}^{r} x_i^k = \sum_{i=1}^{r} (x_i^m - x_i^k) \geq 0.$$

∎

Corollary 3.2 *$\chi_=(K_{n_1,n_2,\cdots,n_r}) = r$ if and only if $|n_i - n_j| \leq 1$ for all i and j.*

Proof First note that $\chi(K_{n_1,n_2,\cdots,n_r}) = r$. Let $(V_1, V_2, \cdots, V_r)$ be an r-partition of the vertices of $K_{n_1,n_2,\cdots,n_r}$ with $|V_i| = n_i$, $i = 1, 2, \cdots, r$. If $\chi_=(K_{n_1,n_2,\cdots,n_r}) = r$, then, for any equitable r-coloring of $K_{n_1,n_2,\cdots,n_r}$, each color occurs in at most one V_i and at least one color is required to color each V_i. Thus it follows that each color is exactly assigned to one V_i. This implies that $|n_i - n_j| = ||V_i| - |V_j|| \leq 1$ for all i and j. Conversely, if $|n_i - n_j| \leq 1$ for all i and j, we can form an equitable r-coloring of $K_{n_1,n_2,\cdots,n_r}$ by coloring every vertex in V_i with the color i, $i = 1, 2, \cdots, r$. Hence $\chi_=(K_{n_1,n_2,\cdots,n_r}) \leq r$. On the other hand, the bound

$$\chi_=(K_{n_1,n_2,\cdots,n_r}) \geq \chi(K_{n_1,n_2,\cdots,n_r}) = r$$

is trivial. Therefore $\chi_=(K_{n_1,n_2,\cdots,n_r}) = r$. ∎

4. Chromatic Difference

For a graph G, the bound $\chi_=(G) \geq \chi(G)$ is trivial since each equitably k-coloring of G is always a k-coloring of G. By Theorem 3.1, there do exist some graphs G with $\chi_=(G) > \chi(G)$. Thus it is very natural to pose such a problem: Which graphs G have $\chi_=(G) = \chi(G)$? To study this problem, it is helpful to introduce a new parameter $\rho(G) = \chi_=(G) - \chi(G)$. We call $\rho(G)$ the chromatic difference of G. If $E(G) = \emptyset$, then $\rho(G) = 0$. Hence we always assume that $E(G)$ is non-empty and thus $\chi(G) \geq 2$. Applying the result of [1], we have $0 \leq \rho(G) \leq \Delta(G) + 1 - 2 = \Delta(G) - 1$.

A graph G is said to be a k-type graph if $\rho(G) = k$, $k = 0, 1, \cdots$. Theorem 2.3 shows that all line graphs are 0-type.

Conjecture 4.1 For any graph G, $0 \leq \rho(G) \leq \lfloor \frac{\Delta(G)}{2} \rfloor$.

Considering a star $K_{1,p}$ with $p \geq 3$, we have

$$\begin{aligned}\rho(K_{1,p}) &= \chi_=(K_{1,p}) - \chi(K_{1,p}) \\ &= \left\lceil \frac{\Delta(K_{1,p})}{2} \right\rceil + 1 - 2 = \left\lceil \frac{\Delta(K_{1,p})}{2} \right\rceil - 1 \leq \left\lfloor \frac{\Delta(K_{1,p})}{2} \right\rfloor.\end{aligned}$$

When p is odd, the above equality holds. This implies that the upper bound of Conjecture 4.1 is sharp. If Conjecture 4.1 were true, then we would deduce that Meyer's conjecture holds for all the graphs G having

$$\chi(G) \leq \frac{\Delta(G)}{2}$$

since

$$\chi_=(G) = \chi(G) + \rho(G) \leq \frac{\Delta(G)}{2} + \left\lfloor \frac{\Delta(G)}{2} \right\rfloor \leq \Delta(G).$$

In fact, Conjecture 4.1 holds trivially for $\chi_=(G) \leq \lfloor \frac{\Delta(G)}{2} \rfloor + 2$ or $\chi(G) \geq \lfloor \frac{\Delta(G)}{2} \rfloor + 2$. Moreover, by the results in [6], Conjecture 4.1 also holds for the graphs with $\Delta(G) \leq 4$.

References

[1] Hajnal, A. and Szemerédi, E., Proof of a conjecture of Eröds. In: Erdös, P., Rényi, A. and Sós, V. T. ed. Combinatorial Theory and Its Applications, II, *Colloq. Math. Soc. János Bolyai,* Vol.**4**, Amsterdam: North-Holland Press, 1970, 601–623.

[2] Meyer, W., Equitable coloring, *Amer. Math. Monthly*, 1973, **80**: 920–922.

[3] Chen Borliang and Lih Kowei, Equitable coloring of trees, *J. Combin. Theory, Ser.* B, 1994, **61**(1): 83–87.

[4] Lih Kowei and Wu Polin, On equitable coloring of bipartite graphs, *Discrete Math.*, 1996, **151**: 155–160.

[5] Yap, H. P., and Zhang Yi, The equitable Δ-coloring conjecture holds for outerplanar graphs, *Bulletin of the Institute of Mathematics, Academia Sinica*, 1997, **25**: 143–149.

[6] Chen Borliang, Lih Kowei and Wu Poulin, Equitable coloring and the maximum degree, *Europ. J. Combinatorics*, 1994, **15**: 443–447.

[7] Zhang Yi and Yap, H. P., Equitable colorings of planar graphs, *J. Combin. Math. Combin. Comput.*, 1998, **27**: 97–105.

[8] Harary, F., Graph Theory, Addison-Wesley, 1969.