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NEW RESULTS OF B(27) AND B(52)*

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Abstract In this paper, we give a broadcast graph of order 27 with 48 edges and prove $44 \leqslant B(27) \leqslant 48$, moreover we give a new broadcast graph of order 52 with 98 edges, that is to say $B(52) \leqslant 98$.

Keywords broadcast graph, communication network, minimum broadcast graph

AMS(1991) subject classifications 05C10, 68M10, 90C35

1 Introduction

Broadcasting is the process of distributing information from an originator to all other vertices of a communication network. The problem addressed in this paper is under the assumptions that only one piece of information is to be distributed, each communication involves exactly two adjacent vertices and takes one unit time, and no vertex is involved in two or more simultaneous communications.

Let G be a graph of order n, representing a communication network. It is easy to see that at least $\lceil \log_2 n \rceil$ time units are required to complete broadcasting under the above assumptions. For $u \in V(G)$, a broadcast from u determines a spanning tree of G, denoted by $T(u)$ called broadcast tree from u, and we define the broadcast time of u, denoted by $b(u)$, to be the minimum number of time units required to complete broadcasting from vertex u, the broadcast time of G, denoted by $b(G)$, to be the maximum broadcast time of any vertex in G, i.e., $b(G) = \max\{b(u) \mid u \in V(G)\}$. If $b(G) = \lceil \log_2 n \rceil$, then G is called a broadcast graph. The

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broadcast function $B(n)$ is the minimum number of edges in any broadcast graph with n vertices. A broadcast graph with n vertices and $B(n)$ edges is called a minimum broadcast graph or *mbg*.

From the point of view of application, minimum broadcast graphs represent the cheapest possible communication networks such that broadcasting can be accomplished from any vertex as fast as theoretically possible.

A survey of the history of these problems and a list of references can be found in [3], [4]. Minimum broadcast graphs are difficult to construct and there is no known method for constructing a *mbg* for arbitrary n. In fact, even determining the value of $b(u)$ in an arbitrary graph is NP-complete (see [5]). For $B(n)$, only the values of $B(n)$ for $n \leqslant 22$ or $n=31,63$ or $n=2^k, 2^k-2$ if $k \geqslant 5$ have been determined. So far $B(27) \leqslant 49$, $B(52) \leqslant 100$ (see [1]).

In this paper, we prove $B(27) \leqslant 48$ by constructing a broadcast graph of order 27 with 48 edges, and using a method developed in [1] we construct a broadcast graph of order 52 with 98 edges which proves that $B(52) \leqslant 98$, we also give a lower bound of $B(27)$. We refer to [2] for notations and terminology not defined here.

2 New upper and Lower Bound of $B(27)$

Let G be a minimum broadcast graph on 27 vertices, then we have

Proposition 2.1 $\delta(G)=3$.

Proof If there exists in G a vertex v of degree 1 or 2, then we can easily show that $b(v) > \lceil \log_2 27 \rceil = 5$; If $\delta > 3$ then $|E(G)| > 49$, a contradiction. Thus, complete the proof.

Proposition 2.2 Let $v \in G$ be a vertex of degree 3, then v must be adjacent to a vertex with degree at least 4.

Proof If not, then we can show that broadcasting from v can at most inform 25 vertices in five time units. □

Proposition 2.3 $B(27) \geqslant 44$.

Proof Assume that G contains exactly m vertices of degree at least 4, which are collected in $V' \subset V(G)$, then $\sum_{v \in V'} d(v) \geqslant 4m$. On the other hand, from Propositions 2.1 and 2.2 we have $\sum_{v \in V'} d(v) \geqslant 27 - m$.

1. If $4m \geqslant 27 - m$ then $m \geqslant 6$, $|E(G)| \geqslant \lceil (6 \cdot 4 + 21 \cdot 3)/2 \rceil = 44$.
2. If $27 - m > 4m$ then $m \leqslant 5$ and $\sum_{v \in V'} d(v) \geqslant 22$. Thus

$$|E(G)| \geqslant \lceil (22 + 22 \cdot 3)/2 \rceil = 44.$$

Therefore $B(27)\geqslant 44$. □

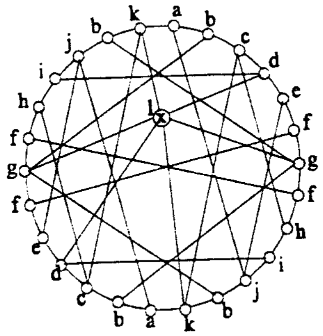

Fig. 1　A broadcast graph on 27 vertices

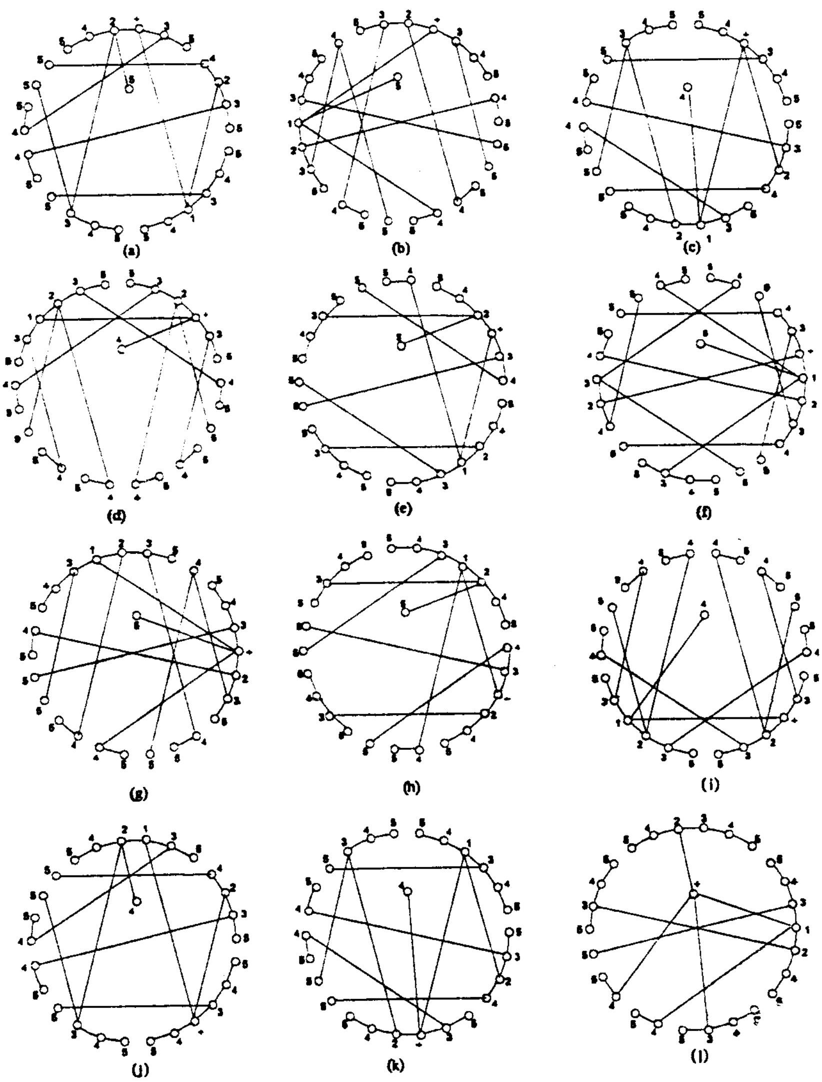

Fig. 2　Broadcasting schemes

Theorem 2. 1 $44 \leqslant B(27) \leqslant 48$.

Proof The left side of the inequality is true by proposition 2. 3. To prove the right side of the inequality, we construct a graph G described in Fig. 1 as a broadcast graph on 27 vertices with 48 edges, the vertex inside the cycle is denoted by x. Broadcasting schemes are showed in Fig. 2. In fact, by the symmetry of $G-x$ we need only show twelve broadcasting schemes for the broadcast graph. In Fig. 1, the letter beside a vertex indicates which scheme in Fig. 2 can be used. In a scheme, '+' indicates originator and a label beside a vertex indicates the time unit of the vertex receiving the message. Thus, we have prove the theorem. □

3 New Upper Bound of B(52)

In [1], Bermond et al. developed a method called solid 2-cover to construct new broadcast graphs from smaller broadcast graphs. In this paper, we apply this method on a *mbg* of order 26 to obtain a new broadcast graph of order 52 with 98 edges which improves the old upperd bound of 100. First of all, we introduce some terms defined in [1].

Definition 3. 1 Given a broadcast tree in a broadcast graph on n vertices, a vertex u is idle at time unit $t \leqslant \lceil \log_2 n \rceil$ if and only if u is aware of the message at (the beginning of) the time unit t and u does not communicate with any of its neighbors during the time unit.

Definition 3. 2 Given a broadcast graph G on n vertices. a subset of vertices M is called a solid 2-cover if and only if M convers all the paths of length 2 in G, and for each $u \notin M$, there is a broadcast tree from u such that at least one of the following three conditions holds:

1. All neighbors of u belong to M and at least one neighbor of u is idle at some time during the broadcast.

2. Exactly one neighbor v of u does not belong to M, at least one of the neighbors of u distinct from v is idle at some time during the broadcast, and at least one of the neighbors of v distinct from u is idle at some time during the broadcast.

3. Exactly one neighbor v of u does not belong to M and at least one of the neighbors of u distinct from v is idle at some time $t < \lceil \log n \rceil$ during the broadcast or at least one of the neighbors of v distinct from u is idle at some time $t < \lceil \log_2 n \rceil$ during the broadcast.

In the following, we show a solid 2-cover of a *mbg* G on 26 vertices

The set of vertices represented by a stuffed cycle (see Fig. 3) are a solid 2-conver of the *mbg* on 26 vertices, denoted by M, the number beside a vertex shows which of the three conditions are satisfied, we refer to [5] for relative broadcast schemes. Then join the two copies of G as in Fig. 4, for $u \in V(G)$, u and its copy v is joined if and only if $u \in M$, then we can get a broadcast graph on 52 vertices with 98 edges. Thus we have the following theorem

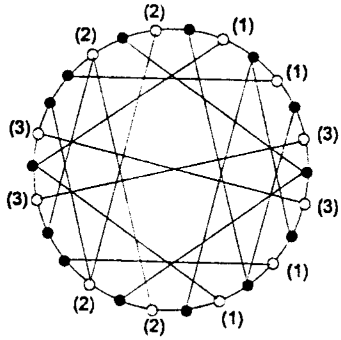

Fig. 3　A solid 2-conver of a mbg on 26 vertices

Theorme 3. 1　$B(52)\leqslant 98$.

Proof　Let G' be the graph as Fig. 4, we show that $b(G')=\lceil\log_2 26\rceil+1=6$. Let $u\in V(G_1)$ be the originator of the broadcast. If $u\in M$ then u may sends the message to its copy v, and then u and v broadcast in G_1 and G_2 respectively, this takes $\lceil\log_2 26\rceil+1$ time units.

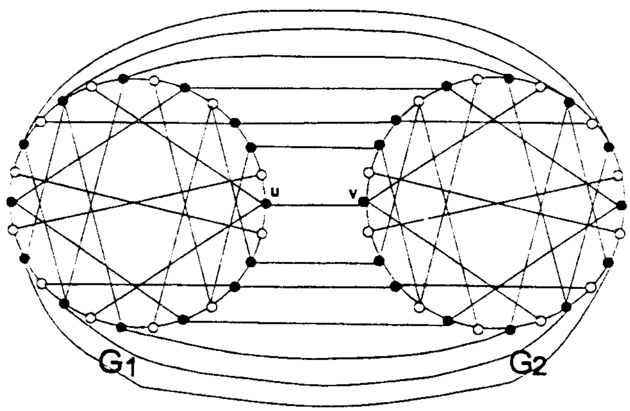

Fig. 4　A Broadcast graph on 52 vertices obtained from solid 2-cover

Suppose that u does not belong to M, since M is a solid 2-cover of G, one of the three conditions in Definition 3. 2 holds, so we have,

Case 1　If Condition 1 holds, then all neighbors of u belong to M. Let T be the relative broadcast tree for u in G_1, and let $u(i)$, $i=1,2,\cdots$, be the neighbors of u in G_1 that are informed by u at time units $i=1,2,\cdots$. Then we can expand T to inform vertices of G_2 as follows. When a $u(i)$ receives the message, it first sends it to its copy $v(i)\in G_2$, then $u(i)$ and $v(i)$ both complete broadcast according to T in G_1 and G_2, respectively. Thus all vertices of G_1 and all vertices of G_2, except v will be informed after $\lceil\log_2 26\rceil+1$ time units. Since Condition 1 is satisfied, we can ensure that some neighbor of v in G_2 is idle at some time unit during the broadcast and this vertex can inform v in his idle time unit.

Case 2 If Condition 2 or 3 holds, similar to the proof of Case 1, we can use the choosed broadcast tree T to ensure that all vertices except the copy of u and the copy of one of its neighbors $u(i) \in M$ are aware of the message at time $\lceil \log_2 26 \rceil + 1$. Since M is a solid 2-cover, there are enough idle vertices to inform the copies of u and $u(i)$ by the end of time unit $\lceil \log_2 26 \rceil + 1$.

Thus the proof is completed. □

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广播函数 B(27)和 B(52)的新结果

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摘要 本文给出了新的 27 和 52 个顶点的广播图，进一步给出了广播函数 B(27)，B(52)的界：$44 \leqslant B(27) \leqslant 48$，$B(52) \leqslant 98$.

关键词 广播图，最小广播图，通信网络

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