ON CRITICAL RAMSEY DECOMPOSITION

Shi Lingsheng  Zhang Kemm
(Dept. of Math., Nanjing University, 210093, Nanjing, PRC)

Abstract  After inducing the definitions of decomposition, Ramsey decomposition and critical Ramsey decomposition, we deduce some propositions of these kinds of decompositions and a lower bound formula for Ramsey numbers.

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Let $H_1, H_2, \ldots, H_s$ be graphs or hypergraphs, $K_s$ be a complete $r$-uniform graph or hypergraph. The Ramsey number $R(H_1, H_2, \ldots, H_s)$ is the smallest integer $p$ such that for any $k$-edge coloring $(E_1, E_2, \ldots, E_s)$ of $K_p$, it contains some $i, H_i$ as a subgraph in $K_p[E_i]$. Let $R(G_1, G_2, \ldots, G_s) := R_s(G_1, G_2, \ldots, G_s)$. A $k$-edge coloring $(E_1, E_2, \ldots, E_s)$ of $K_p$ is called a $(r; H_1, H_2, \ldots, H_s; p)((r; n_1, n_2, \ldots, n_s; p)$)-decomposition if for all $i, H_i, (K_p, \text{ resp.})$ is not contained in $K_p[E_i]$. It is clear that $R(H_1, H_2, \ldots, H_s) = p_0 + 1$ iff $p_0 = \max_i p_i$, there exists a $(r; H_1, H_2, \ldots, H_s; p)$-decomposition. The $(r; H_1, H_2, \ldots, H_s; p)((r; n_1, n_2, \ldots, n_s; p)$-decomposition is called a $(r; H_1, H_2, \ldots, H_s; p)((r; n_1, n_2, \ldots, n_s; p)$-Ramsey decomposition if $p = R(H_1, H_2, \ldots, H_s) - 1(R(n_1, n_2, \ldots, n_s) - 1, \text{ resp.})$. Let $(G_1, G_2, \ldots, G_s; p)$-Ramsey decomposition := $(2; G_1, G_2, \ldots, G_s; p)$-Ramsey decomposition. The $(G_1, G_2, \ldots, G_s; p)$-Ramsey decomposition is called a $(G_1, G_2, \ldots, G_s; p)$-critical Ramsey decomposition if for any edge $e$ not in $K_p[E_i], K_p[E_i] + e$ contains $G_i$. Note that this kind of decomposition must exist, for given a $(G_1, G_2, \ldots, G_s; p)$-Ramsey decomposition, if for any $e$ in $K_p[E_i], i \neq 1$ and $K_p[E_i] + e$ does not contain $G_1$, then let $E_1 := E_1 \cup e, E_i := E_i - e$, along this way we finally obtain a $(G_1, G_2, \ldots, G_s; p)$-critical Ramsey decomposition. All terminology not defined here can be found in [1, 3].

It is well-known that finding Ramsey numbers is very hard. However if we get the order

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第一作者简介: 史灵生, 男, 1975 年 1 月生, 数学专业硕士生.
of some Ramsey decomposition, then the corresponding Ramsey number follows. In fact, such as the critical connected graph and coloring etc., we only need to research a special kind of decomposition with rich properties i.e. Ramsey decompositions. So, to find the properties of critical Ramsey decomposition is interesting.

**Lemma** In each \((K_*, G_2, G_3, \ldots, G_4; p)\)-critical Ramsey decomposition \((E_1, E_2, \ldots, E_4)\), there are at least \(n-2\) vertices incident to \(u\) and \(v\) in \(K_p[E_1]\), where \(uv \in E_1\).

**Proof** If follows by the definition of critical Ramsey decomposition.

**Theorem 1** Let \((E_1, E_2, \ldots, E_4)\) be a \((r; n_1, n_2, \ldots, n_4; p)\)-Ramsey decomposition. For any vertex \(v\) in \(K_p\) and \(i \in \{1, 2, \ldots, k\}\), there exists a \(K_{p-1}^*\) in \(K_p[E_1]\) such that \(v\) is in \(K_{p-1}^*\).

**Proof** If for some \(i, v\) is not in any \(K_{p-1}^*\) contained in \(K_p[E_1]\), then we can give a \(k\)-edge coloring of \(K_{p-1}^*\) as follows,

\[
E' = E \cup \{uvw_{i-1}v_{i-1} \mid v_{i-1} \in E_1, \forall v_1, v_2, \ldots, v_{i-1} \in V(K_p)\}
\]

\[
E' = E \cup \{uvw_{i-1}v_{i-1} \mid v_{i-1} \in E_1, \forall v_1, v_2, \ldots, v_{i-1} \in V(K_p)\}
\]

It is clear that \(K_4^*\) is not contained in \(K_{p-1}[E_1]\) as a subgraph for any \(i\). This contradicts that \((E_1, E_2, \ldots, E_4)\) is a \((r; n_1, n_2, \ldots, n_4; p)\)-Ramsey decomposition. Thus the theorem holds.

**Theorem 2**

\[
R_r(n_1, \ldots, n_{i-1}, n_i + n_i - 1, n_{i+1}, \ldots, n_k) \geq R_r(n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_k)
\]

\[
+ R_r(n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_k) - 1 \quad \text{for } i = 1, 2, \ldots, k.
\]  

**Proof** Let \((E_1, E_2, \ldots, E_4)\) and \((F_1, F_2, \ldots, F_4)\) be \((r; n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_k; p_1)\) and \((r; n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_k; p_2)\)-Ramsey decompositions respectively. Then we give a \(k\)-edge coloring of \(K_{p-1}^*\) as follows,

\[
E' = E \cup F, \quad \text{for } j \neq i.
\]

It is clear that there are no \(K_{p-1}^*\) in \(K_{p-1}^*[E_1]\) and no \(K_{p-1}^*\) in \(K_{p-1}^*[E_1]\) for \(j \neq i\). Hence,

\[
R_r(n_1, \ldots, n_{i-1}, n_i + n_i - 1, n_{i+1}, \ldots, n_k) - 1 \geq p_1 + p_2.
\]

Since \(p_1 = R_r(n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_k) - 1\) and \(p_2 = R_r(n_1, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_k) - 1\), (1) holds.

**Theorem 3** Let \((E_1, E_2, \ldots, E_4)\) be a \((3, n_2, n_3, \ldots, n_4; p)\)-critical Ramsey decomposition and \(p\) be odd. Then for any \(v \in V(K_p)\), there exists a 5-cycle \(C_v\) containing \(v\) in \(K_p[E_1]\). Furthermore, it is also true for each edge in \(K_p[E_1]\).

**Proof** For each vertex \(v\) in \(V(K_p)\), there exists an edge \(uv \in E_1\) by Lemma. If \(N_{K_p}(v) \cup N_{K_p}(v) = V(K_p)\) then \(p = |N_{K_p}(v) \cup N_{K_p}(v)| = |N_{K_p}(v)| + |N_{K_p}(v)| \leq 2R(n_2, n_3, \ldots, n_k) - 2\). Noting that \(p = R(3, n_2, n_3, \ldots, n_k) - 1\) and \(p\) is an odd integer, we have \(R(3, n_2, n_3, \ldots, n_k) < 2R(n_2, n_3, \ldots, n_k) - 1\) which contradicts Theorem 2. So there exists a vertex \(z\) in \(v(k_p) - N_{K_p}(v) \cup N_{K_p}(v)\), a vertex \(x\) in \(N_{K_p}(v)\), and a vertex \(y\) in \(N_{K_p}(v)\).
(v) such that \(zx\) and \(yz\) in \(K_p[E_i]\). Thus we obtain a \(C_{13}\).

**Theorem 4** In each \((K_3, G_2, G_3, \ldots, G_i; p)\)-critical Ramsey decomposition \((E_1, E_2, \ldots, E_i), \delta(K_p[E_i]) \geq [R(K_3, G_2, G_3, \ldots, G_i) - 2] / [R(G_2, G_3, \ldots, G_i) - 1].\)

**Proof** Assume \(d_{K_p[E_i]}(v) = \delta(K_p[E_i]).\) If vertex \(u\) is not in \(N_{K_p[E_i]}(v)\) then, by Lemma, there exists a vertex \(w\) in \(N_{K_p[E_i]}(v), uw \in E_i.\) Thus \(p - 1 - d_{K_p[E_i]}(v) \leq d_{K_p[E_i]}(v) [R(G_2, G_3, \ldots, G_i) - 2].\) Since \(p = R(K_3, G_2, G_3, \ldots, G_i) - 1,\) the theorem holds.

**Theorem 5** If there exists a 5-cycle \(C_5\) in \(K_p[E_i]\) for a \((K_3, G_2, G_3, \ldots, G_i; p)\)-critical Ramsey decomposition \((E_1, E_2, \ldots, E_i),\) then there are at least \([R(K_3, G_2, G_3, \ldots, G_i) - 2R(G_2, G_3, \ldots, G_i)]/5 + 1\) separate \(C_5\) in \(K_p[E_i].\)

**Proof** Assume \(K_p[E_i]\) contains \(n \leq [R(K_3, G_2, G_3, \ldots, G_i) - 2R(G_2, G_3, \ldots, G_i)]/5\) separate \(C_5\). Let \(V = V(G) - V(nC_5).\) If for each vertex \(v\) in \(V, d_{K_p[E_i]}(v) < 2\) or \(d_{K_p[E_i]}(v) > 5\) and for each vertex \(u\) in \(N_{K_p[E_i]}(v), d_{K_p[E_i]}(v) < 2,\) then \(K_p[E_i][V]\) is a bipartite graph without cycles. Noting that \(E_1 \cap E(K_p[V])\) is also in a \((K_3, G_2, G_3, \ldots, G_i; |V|)\)-decomposition, we have \(|V| \leq 2R(G_2, G_3, \ldots, G_i) - 1.\) Thus \(R(K_3, G_2, G_3, \ldots, G_i) \leq 5 + 2R(G_2, G_3, \ldots, G_i) - 2R(G_2, G_3, \ldots, G_i) / 5 + 2R(G_2, G_3, \ldots, G_i) \leq R(K_3, G_2, G_3, \ldots, G_i)\) which is absurd. So \(U = \{v \in V \mid d_{K_p[E_i]}(v) > 1\} \cap \exists u \in N_{K_p[E_i]}(v)\) s. t. \(d_{K_p[E_i]}(v) > 1\) is nonempty.

If for each vertex \(v\) in \(U,\) each vertex \(u\) in \(N_{K_p[E_i]}(v)\) and each vertex \(w\) in \(N_{K_p[E_i]}(u)\) \(N_{K_p[E_i]}(v) \subset N_{K_p[E_i]}(w),\) then \(K_p[E_i][V]\) only contains even cycles, i.e. \(K_p[E_i][V]\) is a bipartite graph which is impossible. So \(P = \{wx \in E_1 \cap E(K_p[V]) \mid \exists v \in V\ s. t. u, x \in N_{K_p[E_i]}(v)\) and \(w \in N_{K_p[E_i]}(u)\} \neq \emptyset.\)

By Lemma, for each \(wx\) in \(P,\) there exists a vertex \(y\) in \(V(K_p)\) such that \(wy\) and \(xy\) are in \(E_i.\) If all these kinds of \(y\) are not in \(V\) then \(E_1 \cap E(K_p[V]) \cup P\) is also in a \((K_3, G_2, G_3, \ldots, G_i; |V|)\)-decomposition and only contains even cycles which is impossible. So there exists some vertex \(y\) in \(V\) such that \(wy\) and \(xy\) in \(E_1 \cap E(K_p[V])\) and \(uwyxw\) is a 5-cycle. The proof is complete.

**Corollary** There are at least \([R(3, n_2, n_3, \ldots, n_4) - 2R(n_2, n_3, \ldots, n_4)]/5 + 1\) separate 5-cycles in \(K_p[E_i]\) for a \((3, n_2, n_3, \ldots, n_4; p)\)-critical Ramsey decomposition \((E_1, E_2, \ldots, E_4, p)\) as \(p\) is odd.

**Proof** It follows by Theorem 3 and 5.

All lemma and theorems (partly) generalize the corresponding results in [2].

**References**

关于临界 Ramsey 分解

史灵生 张克民
（南京大学数学系，南京 210093）

摘要 本文在给出分解、Ramsey 分解和临界 Ramsey 分解定义后，导出有关上述分解的某些性质和 Ramsey 数的下界公式。

关键词 分解、Ramsey 分解

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