南京大学学报数学半年刊
第17卷 第1期 JOURNAL OF NANJING UNIVERSITY Vol. 17, No. 1
2000年5月 MATHEMATICAL BIQUARTERLY May, 2000

ON CRITICAL RAMSEY DECOMPOSITION*

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Abstract After inducing the definitions of decomposition, Ramsey decomposition and critical Ramsey decomposition, we deduce some propositions of these kinds of decompositions and a lower bound formula for Ramsey numbers.

Keywords decomposition, Ramsey decomposition

AMS(1991) subject classifications 05C55, 05D10, 05C65

Let $H_1,H_2,\cdots,H_k$ be graphs or hypergraphs, K_p^r be a complete r-uniform graph or hypergraph. The Ramsey number $R_r(H_1,H_2,\cdots,H_k)$ is the smallest integer p such that for any k-edge coloring $(E_1,E_2,\cdots,E_k)$ of K_p^r, it contains some i, H_i as a subgraph in $K_p^r[E_i]$. Let $R(G_1,G_2,\cdots,G_k):=R_2(G_1,G_2,\cdots,G_k)$. A k-edge coloring $(E_1,E_2,\cdots,E_k)$ of K_p^r is called a $(r;H_1,H_2,\cdots,H_k;p)((r;n_1,n_2,\cdots,n_k;p)$, resp.)-decomposition if for all i, $H_i(K_{n_i}^r$, resp.) is not contained in $K_p^r[E_i]$. It is clear that $R_r(H_1,H_2,\cdots,H_k)=p_0+1$ iff $p_0=\max\{p\mid$ there exists a $(r;H_1,H_2,\cdots,H_k,p)$-decomposition$\}$. The $(r;H_1,H_2,\cdots,H_k;p)((r;n_1,n_2,\cdots,n_k;p)$, resp.)-decomposition is called a $(r;H_1,H_2,\cdots,H_k;p)((r;n_1,n_2,\cdots,n_k;p)$, resp.)-Ramsey decomposition if $p=R_r(H_1,H_2,\cdots,H_k)-1(R_r(n_1,n_2,\cdots,n_k)-1$, resp.). Let $(G_1,G_2,\cdots,G_k;p)$-(Ramsey) decomposition $:=(2;G_1,G_2,\cdots,G_k;p)$-(Ramsey) decomposition. The $(G_1,G_2,\cdots,G_k;p)$-Ramsey decomposition is called a $(G_1,G_2,\cdots,G_k;p)$-critical Ramsey decomposition if for any edge e not in $K_p[E_1]$, $K_p[E_1]+e$ contains G_1. Note that this kind of decomposition must exist, for given a $(G_1,G_2,\cdots,G_k;p)$-Ramsey decomposition, if for any e in $K_p[E_i]$, $i\neq1$ and $K_p[E_1]+e$ does not contain G_1, then let $E_1:=E_1+e$, $E_i:=E_i-e$, along this way we finally obtain a $(G_1,G_2,\cdots,G_k;p)$-critical Ramsey decompositon. All terminology not defined here can be found in [1,3].

It is well-known that finding Ramsey numbers is very hard. However if we get the order

* The project supported by NSFC and NSFJS;
Received: Jun, 6 1996;
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of some Ramsey decomposition, then the corresponding Ramsey number follows. In fact, such as the critical connected graph and coloring etc., we only need to research a special kind of decomposition with rich properties i. e. Ramsey decompositions. So, to find the properties of critical Ramsey decomposition is interesting.

Lemma In each $(K_n, G_2, G_3, \cdots, G_k; p)$-critical Ramsey decomposition $(E_1, E_2, \cdots, E_k)$, there are at least $n-2$ vertices incident to u and v in $K_p[E_1]$, where $uv \notin E_1$.

Proof If follows by the definition of critical Ramsey decomposition.

Theroem 1 Let $(E_1, E_2, \cdots, E_k)$ be a $(r; n_1, n_2, \cdots, n_k; p)$-Ramsey decomposition. For any vertex v in K_p^r and $i \in \{1, 2, \cdots, k\}$, there exists a $K_{n_i-1}^r$ in $K_p^r[E_i]$ such that v is in $K_{n_i-1}^r$.

Proof If for some i, v is not in any $K_{n_i-1}^r$ contained in $K_p^r[E_i]$, then we can give a k-edge coloring of K_{p+1}^r as follows,

$$E'_i = E_i \cup \{uv_1v_2\cdots v_{r-1} \mid vv_1v_2\cdots v_{r-1} \in E_i, \forall\, v_1, v_2, \cdots, v_{r-1} \in V(K_p^r)\}$$
$$\cup \{uvv_1v_2\cdots v_{r-2} \mid \forall\, v_1, v_2, \cdots, v_{r-2} \in V(K_p^r) - v\},$$
$$E'_j = E_j \cup \{uv_1v_2\cdots v_{r-1} \mid vv_1v_2\cdots v_{r-1} \in E_j, \forall\, v_1, v_2, \cdots, v_{r-1} \in V(K_p^r) - v\}, \quad \text{for } j \neq i.$$

It is clear that $K_{n_i}^r$ is not contained in $K_{p+1}^r[E'_i]$ as a subgraph for any i. This contradicts that $(E_1, E_2, \cdots, E_k)$ is a $(r; n_1, n_2 \cdots, n_k; p)$-Ramsey decomposition. Thus the theorem holds.

Theorem 2

$$R_r(n_1, \cdots, n_{i-1}, n_{i_1} + n_{i_2} - 1, n_{i+1}, \cdots, n_k) \geqslant R_r(n_1, \cdots, n_{i-1}, n_{i_1}, n_{i+1}, \cdots, n_k)$$
$$+ R_r(n_1, \cdots, n_{i-1}, n_{i_2}, n_{i+1}, \cdots, n_k) - 1 \quad \text{for } i = 1, 2, \cdots, k. \tag{1}$$

Proof Let $(E_1, E_2, \cdots, E_k)$ and $(F_1, F_2, \cdots, F_k)$ be $(r; n_1, \cdots, n_{i-1}, n_{i_1}, n_{i+1}, \cdots, n_k; p_1)$ and $(r; n_1, \cdots, n_{i-1}, n_{i_2}, n_{i+1}, \cdots, n_k; p_2)$-Ramsey decomposition respectively. Then we give a k-edge coloring of $K_{p_1+p_2}^r$ as follows,

$$E'_i = E_i \cup F_i \cup \{E \in K_{p_2+p_2}^r \mid E \cap V(K_{p_1}^r) \neq \varnothing \text{ and } E \cap V(K_{p_2}^r) \neq \varnothing\},$$
$$E'_j = E_j \cup F_j, \quad \text{for } j \neq i.$$

It is clear that there are no $K_{n_{i_1}+n_{i_2}-1}^r$ in $K_{p_1+p_2}^r[E'_i]$ and no K_{n_j} in $K_{p_1+p_2}^r[E'_j]$ for $j \neq i$. Hence,

$$R_r(n_1, \cdots, n_{i-1}, n_{i_1} + n_{i_2} - 1, n_{i+1}, \cdots, n_k) - 1 \geqslant p_1 + p_2.$$

Since $p_1 = R_r(n_1, \cdots, n_{i-1}, n_{i_1}, n_{i+1}, \cdots, n_k) - 1$ and $p_2 = R_r(n_1, \cdots, n_{i-1}, n_{i_2}, n_{i+1}, \cdots, n_k) - 1$, (1) holds.

Theorem 3 Let $(E_1, E_2, \cdots, E_k)$ be a $(3, n_2, n_3, \cdots, n_k; p)$-critical Ramsey decomposition and p be odd. Then for any $v \in V(K_p)$, there exists a 5-cycle C_5 containing v in $K_p[E_1]$. Furthermore, it is also true for each edge in $K_p[E_1]$.

Proof For each vertex v in $V(K_p)$, there exists an edge uv in E_1 by Lemma. If $N_{K_p[E_1]}(u) \cup N_{K_p[E_1]}(v) = V(K_p)$ then $p = |N_{K_p[E_1]}(u) \cup N_{K_p[E_1]}(v)| = |N_{K_p[E_1]}(u)| + |N_{k_p[E_1]}(v)| \leqslant 2R(n_2, n_3, \cdots, n_k) - 2$. Noting that $p = R(3, n_2, n_3, \cdots, n_k) - 1$ and p is an odd integer, we have $R(3, n_2, n_3, \cdots, n_k) < 2R(n_2, n_3, \cdots, n_k) - 1$ which contradicts Theorem 2. So there exists a vertex z in $v(k_p) - N_{k_p[E_1]}(u) \cup N_{k_p[E_1]}(v)$, a vertex x in $N_{K_p[E_1]}(u)$ and a vertex y in $N_{K_p[E_1]}$

(v) such that xz and yz in $K_p[E_1]$. Thus we obtain a C_5: $uxzyv$.

Theorem 4 In each $(K_3,G_2,G_3,\cdots,G_k;p)$-critical Ramsey decomposition $(E_1,E_2,\cdots,E_k)$, $\delta(K_p[E_1])\geqslant[R(K_3,G_2,G_3,\cdots,G_k)-2]/[R(G_2,G_3,\cdots,G_k)-1]$.

Proof Assume $d_{K_p[E_1]}(v)=\delta(K_p[E_1])$. If vertex u is not in $N_{K_p[E_1]}(v)$ then, by Lemma, there exists a vertex w in $N_{K_p[E_1]}(v)$, $uw\in E_1$. Thus $p-1-d_{K_p[E_1]}(v)\leqslant d_{K_p[E_1]}(v)[R(G_2,G_3,\cdots,G_k)-2]$. Since $p=R(K_3,G_2,G_3,\cdots,G_k)-1$, the theorem holds.

Theorem 5 If there exists a 5-cycle C_5 in $K_p[E_1]$ for a $(K_3,G_2,G_3,\cdots,G_k;p)$-critical Ramsey decomposition $(E_1,E_2,\cdots,E_k)$, then there are at least $\lfloor[R(K_3,G_2,G_3,\cdots,G_k)-2R(G_2,G_3,\cdots,G_k)]/5\rfloor+1$ separate C_5 in $K_p[E_1]$.

Proof Assume $K_p[E_1]$ contains $n(\leqslant\lceil[R(K_3,G_2,G_3,\cdots,G_k)-2R(G_2,G_3,\cdots,G_k)]/5\rfloor)$ separate C_5. Let $V=V(G)-V(nC_5)$. If for each vertex v in V, $d_{K_p[E_1][v]}(v)<2$ or $d_{K_p[E_1][V]}(v)>1$ and for each vertex u in $N_{K_p[E_1][V]}(v)$, $d_{K_p[E_1][V]}(u)<2$, then $K_p[E_1][V]$ is a bipartite graph without cycles. Noting that $E_1\cap E(K_p[V])$ is also in a $(K_3,G_2,G_3,\cdots,G_k;|V|)$-decomposition, we have $|V|<2R(G_2,G_3,\cdots,G_k)-1$. Thus $R(K_3,G_2,G_3,\cdots,G_k)<5n+2R(G_2,G_3,\cdots,G_k)\leqslant5\lfloor[R(K_3,G_2,G_3,\cdots,G_k)-2R(G_2,G_3,\cdots,G_k)]/5\rfloor+2R(G_2,G_3,\cdots,G_k)\leqslant R(K_3,G_2,G_3,\cdots,G_k)$ which is absurd. So $U=\{v\in V\mid d_{K_p[E_1][V]}(v)>1$ and $\exists\ u\in N_{K_p[E_1][V]}(v)$ s. t. $d_{K_p[E_1][V]}(u)>1\}\neq\varnothing$.

If for each vertex v in U, each vertex u in $N_{K_p[E_1][V]}(v)$ and each vertex w in $N_{K_p[E_1][V]}(u)$, $N_{K_p[E_1][V]}(v)\subset N_{K_p[E_1][V]}(w)$, then $K_p[E_1][V]$ only, contains even cycles, i. e. $K_p[E_1][V]$ is a bipartite graph which is impossible. So $P=\{wx\notin E_1\cap E(K_p[V])\mid\exists\ v\in V$ s. t. $u,x\in N_{K_p[E_1][V]}(v)$ and $w\in N_{K_p[E_1][V]}(u)\}\neq\varnothing$.

By Lemma, for each wx in P, there exists a vertex y in $V(K_p)$ such that wy and xy are in E_1. If all these kinds of y are not in V then $E_1\cap E(K_p[V])\cup P$ is also in a $(K_3,G_2,G_3,\cdots,G_k;|V|)$-decomposition and only contains even cycles which is impossible. So there exists some vertex y in V such that wy and xy in $E_1\cap E(K_p[V])$ and $uvxyw$ is a 5-cycle. The proof is complete.

Corollary There are at least $\lfloor[R(3,n_2,n_3,\cdots,n_k)-2R(n_2,n_3,\cdots,n_k)]/5\rfloor+1$ separate 5-cycles in $K_p[E_1]$ for a $(3,n_2,n_3,\cdots,n_k;p)$-critical Ramsey decomposition $(E_1,E_2,\cdots,E_k)$ as p is odd.

Proof It follows by Theorem 3 and 5.

All lemma and theorems (partly) generalize the corresponding results in [2].

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关于临界 Ramsey 分解

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摘要　本文在给出分解、Ramsey 分解和临界 Ramsey 分解定义后，导出有关上述分解的某些性质和 Ramsey 数的下界公式.

关键词　分解，Ramsey 分解
中图法分类号　O157.5