On a conjecture involving cycle-complete graph Ramsey numbers

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Abstract

It has been conjectured that $r(C_n, K_m) = (m-1)(n-1)+1$ for all $(n, m) \neq (3,3)$ satisfying $n \geq m$. We prove this for the case $m = 5$.

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1 Introduction

The *independence number* $\alpha(G)$ of a graph G is the cardinality of its largest independent set. Given a graph H without isolated vertices, the *Ramsey number* $r(H, K_m)$ is the smallest integer N such that every graph G of order N either contains H as a subgraph or satisfies $\alpha(G) \geq m$. In one of the earliest contributions to graphical Ramsey theory [1], Bondy and Erdős proved the following result for the case where $H \cong C_n$, a cycle of length n.

Theorem (Bondy, Erdős). *For all* $n \geq m^2 - 2$,

$$r(C_n, K_m) = (m-1)(n-1) + 1.$$

The condition $n \geq m^2 - 2$ is required because of the proof technique, and it has been thought from the beginning that the conclusion is likely to hold under a rather less restrictive hypothesis. The problem of determining for each m the smallest n for which $r(C_n, K_m) = (m-1)(n-1) + 1$ is among those given in [3], and it is conjectured in [8] and elsewhere that $r(C_n, K_m) = (m-1)(n-1) + 1$ for all $(n, m) \neq (3, 3)$ satisfying $n \geq m$. This is trivial for $m = 2$. It was confirmed for $m = 3$ in early work on graphical Ramsey theory [4], and recently it was proved for $m = 4$ [9]. In this paper, we shall prove that the conjecture is true for $m = 5$.

Theorem 1. *For all* $n \geq 5$, $r(C_n, K_5) = 4n - 3$.

Note. The condition $n \geq 5$ is best possible. From early work of Clancy [2], it is known that $r(C_4, K_5) = 14$. There is a unique graph G of order 13 such that $C_4 \not\subset G$ and $\alpha(G) \leq 4$. This graph is exhibited in [6] and elsewhere.

To reach our goal, it is only necessary to prove that for $n \geq 5$ every C_n-free graph G of order $4n - 3$ satisfies $\alpha(G) \geq 5$. The fact that $r(C_n, K_5) \geq 4n - 3$ follows from the simple example of $G \cong 4K_{n-1}$, which contains no C_n and has independence number $\alpha(G) = 4$.

2 Proofs

The proof of Theorem 1 will be given through a sequence of lemmas. As usual, $\delta(G)$ denotes the minimum degree, that is $\delta(G) = \min_{v \in V(G)} \deg v$.

Lemma 1. *Suppose that for some* $n \geq 4$ *there exists a graph* G *of order* $4(n-1)+1$ *such that* $C_n \not\subset G$ *and* $\alpha(G) \leq 4$. *Then* $\delta(G) \geq n - 1$.

Proof. Suppose to the contrary that some vertex $v \in V(G)$ satisfies $\deg v \leq n - 2$. Deleting v and its neighborhood, we obtain a graph H of order at least $3(n-1) + 1$. By the result in [9] either $C_n \subset H$ or $\alpha(H) \geq 4$. Since $C_n \not\subset G$, we must assume that latter. But then v together with the appropriate four vertices from $V(H)$ yields a five-element independent set in G, a contradiction. □

The following lemma is proved in [7].

Lemma 2. *Suppose $\delta(G) \geq 4$ and $C_5 \not\subseteq G$. Then $\alpha(G) \geq \Delta(G)$.*

The following result is given in [5]. In the interest of completeness, it is included here with proof.

Lemma 3. $r(C_5, K_5) = 17$.

Proof. Suppose there exists a graph G of order 17 such that $C_5 \not\subseteq G$ and $\alpha(G) \leq 4$. By Lemma 1 we know that $\delta(G) \geq 4$. Let $u \in V(G)$ be a vertex of degree $\delta(G)$, let Γ denote the neighborhood of u, and let W denote the set of vertices that remain after u and its neighborhood have been deleted. There are two cases.

Case (i): $\delta(G) = 4$. In this case $\langle W \rangle$ is a C_5-free graph of order 12 with no four-element independent set. All such graphs are found in [7], and they are listed in the Appendix (§3) of this paper for the reader's convenience. Inspection shows that each one contains a K_4 with at least two vertices of degree three. In particular, for each possibility there is a cycle $(w_1, w_2, w_3, w_4, w_1)$ in which w_1 and w_2 have degree three in $\langle W \rangle$. Since $\delta(G) = 4$, w_1 is adjacent to some vertex in Γ and so is w_2. If w_1 and w_2 are each adjacent to $v \in \Gamma$ then $(v, w_1, w_4, w_3, w_2, v)$ is a C_5 in G. If w_1 and w_2 are adjacent to v_1 and v_2, respectively, where $v_1 \neq v_2$, then $(u, v_1, w_1, w_2, v_2, u)$ is a C_5 in G. In either case, we have obtained the desired contradiction.

Case (ii): $\delta(G) \geq 5$. In this case $\alpha(G) \geq \Delta(G) \geq 5$ by Lemma 2, a contradiction. □

Lemma 4. $r(C_6, K_5) = 21$.

Proof. Suppose there exists a graph G of order 21 such that $C_6 \not\subseteq G$ and $\alpha(G) \leq 4$. Let $V(G) = \{v_1, v_2, \dots, v_{21}\}$. By Lemma 1, $\delta(G) \geq 5$. Also, $r(K_1 + P_4, K_5) = 19$ [5] and $r(C_6, K_4) = 16$, so we may assume that v_1 is adjacent to each vertex of the path (v_2, v_3, v_4, v_5), and $I \stackrel{\text{def}}{=} \{v_6, v_7, v_8, v_9\}$ is an independent set. It is easy to check that since $C_6 \not\subseteq G$, no vertex in $\{v_6, v_7, \dots, v_{21}\}$ is adjacent to two or more vertices of $\{v_2, v_3, v_4, v_5\}$. [If w is adjacent to v_2 and v_3 then $(w, v_2, v_1, v_5, v_4, v_3, w)$ is a C_6 in G, if w is adjacent to v_2 and v_4 then $(w, v_2, v_3, v_1, v_5, v_4, w)$ is a C_6 in G, and so on.] Since $\alpha(G) \leq 4$ each vertex of $V(G) \setminus I$ is adjacent to at least one vertex of I. In view of these two facts, we may assume $\{v_2v_6, v_3v_7, v_4v_8, v_5v_9\} \subset E(G)$. No vertex in $\{v_{10}, \dots, v_{21}\}$ is adjacent to two or more vertices of I; otherwise, G contains a C_6. Consider v_6. Note that $v_1v_6 \notin E(G)$; otherwise $(v_1, v_5, v_4, v_3, v_2, v_6, v_1)$ is a C_6 in G. Since $\delta(G) \geq 5$ we may assume that $v_6v_j \in E(G)$ for $10 \leq j \leq 13$. Note that $\{v_6, v_{10}, v_{11}, v_{12}, v_{13}\}$ spans a complete subgraph; if $v_iv_j \notin E(G)$ for some $\{i, j\} \subset \{10, 11, 12, 13\}$, then $\{v_7, v_8, v_9, v_i, v_j\}$ is a five-element independent set in G. Now the argument can be repeated, except instead of simply containing $K_1 + P_4$, we may assume that the subgraph induced by $\{v_1, v_2, \dots, v_5\}$ is complete. Then either some $i \leq 5$ makes $\{v_i, v_6, v_7, v_8, v_9\}$ a five-element independent set in G or else some $v_j \in I$ is adjacent to two or more vertices of $\{v_1, v_2, \dots, v_5\}$ yielding a C_6 in G, a contradiction. □

The following lemma provides tools which will be used repeatedly in the remaining proofs. Parts (a) and (b) were used in [1] and parts (c) and (d) appear in [9].

Lemma 5. *Suppose G contains the cycle $(u_1, u_2, \dots, u_{n-1}, u_1)$ of length $n-1$ but no cycle of length n. Let $X \subseteq V(G) \setminus \{u_1, u_2, \dots, u_{n-1}\}$. Then*

- *(a) No vertex $x \in X$ is adjacent to two consecutive vertices on the cycle.*
- *(b) If $x \in X$ is adjacent to u_i and u_j then $u_{i+1}u_{j+1} \notin E(G)$.*
- *(c) If $x \in X$ is adjacent to u_i and u_j then no vertex $x' \in X$ is adjacent to both u_{i+1} and u_{j+2}.*
- *(d) Suppose $\alpha(G) = m-1$ where $m \leq (n+3)/2$, and $\{x_1, x_2, \dots, x_{m-1}\} \subset X$ is an $(m-1)$-element independent set. Then no member of this set is adjacent to $m-2$ or more vertices on the cycle.*

Proof. (a) Obvious.

(b) If $x \in X$ is adjacent to u_i and u_j where $u_{i+1}u_{j+1} \in E(G)$ then

$$(u_i, x, u_j, u_{j-1}, \dots, u_{i+1}, u_{j+1}, \dots, u_{i-1}, u_i)$$

is a C_n in G, a contradiction.

(c) If x is adjacent to u_i and u_j and x' is adjacent to u_{i+1} and u_{j+2} then

$$(u_i, x, u_j, u_{j-1}, \dots, u_{i+1}, x', u_{j+2}, \dots, u_{i-1}, u_i)$$

is a C_n in G, a contradiction.

(d) First notice as did Bondy and Erdős that no $x \in X$ can be adjacent to $m-1$ or more vertices of the cycle. For, if $1 \leq j_1 < j_2 < \dots < j_{m-1} \leq n-2$ and $x \in X$ satisfies $xu_j \in E(G)$ for all $j \in J = \{j_1, j_2, \dots, j_{m-1}\}$, then in view of (a) and (b) we see that $\{x\} \cup \{u_{j+1} | \ j \in J\}$ is an m-element independent set. Now suppose that $1 \leq k_1 < k_2 < \dots < k_{m-2} \leq n-3$ and $x \in \{x_1, x_2, \dots, x_{m-1}\}$ satisfies $xu_k \in E(G)$ for all $k \in K = \{k_1, k_2, \dots, k_{m-2}\}$. [The condition $n \geq 2m-3$ ensures that there is such an indexing of the vertices on the cycle.] By what was just proved, x is not adjacent to any more vertices on the cycle, in particular x is not adjacent to v_s where $s = k_{m-2}+2$. But v_s is adjacent to some $x' \in \{x_1, x_2, \dots, x_{m-1}\}$ since otherwise there would be an m-element independent set. By (b) we know that $\{u_{k+1} | \ k \in K\}$ is an independent set, and by (c) no member of this set is adjacent to x'. It follows that $\{x, x'\} \cup \{u_{k+1} | \ k \in K\}$ is an m-element independent set, a contradiction. □

The Standard Configuration. To prove that $r(C_n, K_5) = 4(n-1)+1$ for $n \geq 7$, we shall in each case assume to the contrary that there exists a graph G of order $4(n-1)+1$ such that $C_n \not\subseteq G$ and $\alpha(G) \leq 4$. By Lemma 1, $\delta(G) \geq n-1$. By induction, $r(C_{n-1}, K_5) = 4(n-2)+1$. Hence we may assume that $(u_1, u_2, \dots, u_{n-1}, u_1)$ is a cycle of length $n-1$ in G and, disjoint from this cycle, there is a four-element independent set $I = \{v_1, v_2, v_3, v_4\}$. Let $C = V(C_{n-1}) = \{u_1, u_2, \dots, u_{n-1}\}$ denote the set of vertices on the cycle, and let $W = V(G) \setminus (C \cup I) = \{w_1, w_2, \dots, w_{3n-6}\}$ denote the set of vertices disjoint from $C \cup I$. Since $\alpha(G) \leq 4$ each vertex in C is adjacent to at least one vertex in I. In view of part (d) of Lemma 5 (with $m = 5$), no member of I is adjacent to 3 or more vertices on the cycle. Thus the set of edges $E(C, I) = \{uv|\ u \in C,\ v \in I\}$ satisfies $|C| \le |E(C, I)| \le 8$. If $v \in I$ is adjacent to u_i and u_j and these two vertices have no other neighbors in I then $u_iu_j \in E(G)$; otherwise, u_i, u_j and the three members of $I \setminus \{v\}$ yield a five-element independent set. Note that each vertex in I is adjacent to at least $n-3$ vertices in W. Since $4(n-3) > 3n-6$, we may assume (if needed) that there are two vertices in I that are commonly adjacent to some vertex $w \in W$. The structure just described will be called the *standard configuration.*

Lemma 6. $r(C_7, K_5) = 25$.

Proof. Assume the standard configuration. Then $6 \le |E(C, I)| \le 8$. The proof is divided into two parts. The first part deals with the possibility $7 \le |E(C, I)| \le 8$ and the second part with $|E(C, I)| = 6$.

Part I: $7 \le |E(C, I)| \le 8$. Note that each vertex in I is adjacent to at least one vertex in C. If not, then some other vertex in I is adjacent to at least $\lceil 7/3 \rceil = 3$ vertices in C, contradicting part (d) of Lemma 5 (with $m = 5$). In case (i) below, we use the prerogative of assuming that v_1 and v_2 are commonly adjacent to some $w \in W$. We may assume that v_1 is adjacent to two vertices in C. There are two cases.

Case (i): v_1 is adjacent to u_1 and u_3. Note that $u_2u_4 \notin E(G)$ and $u_2u_6 \notin E(G)$, both by part (b) of Lemma 5. Also $u_4v_2 \notin E(G)$; otherwise $(w, v_1, u_1, u_2, u_3, u_4, v_2, w)$ is a C_7 in G. In the same way, $u_6v_2 \notin E(G)$. We now make two claims.

Claim 1: $u_5v_2 \notin E(G)$. Suppose $u_5v_2 \in E(G)$. Then $u_2v_2 \notin E(G)$ by part (c) of Lemma 5 and $u_4u_6 \notin E(G)$ as well; otherwise $(w, v_1, u_1, u_6, u_4, u_5, v_2, w)$ is a C_7 in G. In this case, $\{u_2, u_4, u_6, v_1, v_2\}$ is a five-element independent set in G, a contradiction.

Claim 2: $u_2v_2 \in E(G)$. Suppose $u_2v_2 \notin E(G)$. Then $u_4u_6 \in E(G)$ since otherwise $\{u_2, u_4, u_6, v_1, v_2\}$ is a five-element independent set in G. Then $u_1v_2 \notin E(G)$; otherwise $(w, v_1, u_3, u_4, u_6, u_1, v_2, w)$ is a C_7 in G. In the same way $u_3v_2 \notin E(G)$. Then $uv_2 \notin E(G)$ for all $u \in C$, a contradiction.

In view of part (a) of Lemma 5 and previously established facts, this means that v_2 is adjacent to precisely one vertex in C. Hence if $|E(C, I)| = 8$, we have already reached a contradiction. Now we may assume that u_4 and u_6 are both adjacent to v_3 and u_5 is adjacent to v_4. Then $u_4u_6 \in E(G)$; otherwise $\{u_2, u_4, u_6, v_1, v_4\}$ is a five-element independent set in G. Note that $u_2v_4 \notin E(G)$ by part (c) of Lemma 5. Also, $u_1v_4 \notin E(G)$ and $u_3v_4 \notin E(G)$, both by part (b) of Lemma 5. It follows that v_4 is adjacent to precisely one vertex on the cycle, so $|E(C, I)| = 2 + 1 + 2 + 1 = 6$, a contradiction. This completes the proof in case (i).

Case (ii): v_1 is adjacent to u_1 and u_4. In this case, we do not make use of the assumption that v_1 and v_2 are commonly adjacent to $w \in E(G)$. This means that the three vertices v_2, v_3, v_4 are on an equal footing. A simple argument, sketched below, shows that a second vertex, which we may take to be v_2, is adjacent to u_2 and u_5 or to u_3 and u_6. [If we deny this conclusion and use part (c) of Lemma 5, we find that if $v \in \{v_2, v_3, v_4\}$ is adjacent to two vertices in C, then one of those vertices must be u_1 or u_4. For each such v there is an extra edge in $E(C,I)$ over the six that are required by the fact that each vertex in C is adjacent to at least one vertex in I. Suppose there are k such vertices. By the observation just made, $|E(C,I)| \geq 6+k$. On the other hand, the appropriate degree sum for vertices in I yields $|E(C,I)| = 2(k+1)+(3-k) = 5+k$.] Hence there are two subcases.

Subcase (a): v_2 is adjacent to u_2 and u_5. Then $u_3u_6 \notin E(G)$ by part (b) of Lemma 5. For $v \in \{v_3, v_4\}$, either $u_3v \in E(G)$ or $u_6v \in E(G)$; otherwise $\{u_3, u_6, v_1, v_2, v\}$ is a five-element independent set in G. If $u_3v \in E(G)$ then $u_1v \notin E(G)$ and $u_4v \notin E(G)$, by parts (c) and (a), respectively, of Lemma 5. If $u_6v \in E(G)$ then $u_1v \notin E(G)$ and $u_4v \notin E(G)$ by parts (a) and (c), respectively, of Lemma 5. In view of this, $\{u_1, u_4, v_2, v_3, v_4\}$ is a five-element independent set in G, a contradiction.

Subcase (b): v_2 is adjacent to u_3 and u_6. The proof is similar to that of subcase (a). First $u_2u_5 \notin E(G)$ by part (b) of Lemma 5. Then for $v \in \{v_3, v_4\}$ either $u_2v \in E(G)$ or $u_5v \in E(G)$. Finally, for $u_1v \notin E(G)$ and $u_4v \notin E(G)$ for $v \in \{v_3, v_4\}$, so $\{u_1, u_4, v_1, v_3, v_4\}$ is a five-element independent set in G, a contradiction. This completes the proof in Part I.

Part II: $|E(C,I)| = 6$. In this part, each vertex in C is adjacent to precisely one vertex in I, so if $v \in I$ is adjacent to u_i and u_j then $u_iu_j \in E(G)$. Do not assume that v_1 and v_2 are both adjacent to $w \in W$, only that some pair $v_i, v_j \in I$ have this property. Without loss of generality, v_1 is adjacent to two vertices in C. There are two cases.

Case (i): v_1 is adjacent to u_1 and u_4. Then we may assume that u_2 is adjacent to v_2. In view of parts (a), (b), and (c) of Lemma 5 and the fact that each vertex in C is adjacent to precisely one vertex in I, it is clear that $u_iv_2 \notin E(G)$ for $i \neq 2$. In the same way, we may assume that u_3 is adjacent to v_3 and then find that $u_iv_3 \notin E$ for $i \neq 3$. Then we may assume that u_5 is adjacent to v_4. Finally, however, v_6 cannot be adjacent to any vertex in I, a contradiction.

Case (ii): v_1 is adjacent to u_1 and u_3. Then $u_1u_3 \in E(G)$. We may assume that u_2 is adjacent to v_2. As before, we then find that $u_iv_2 \notin E(G)$ for $i \neq 2$. Then, in the only acceptable configuration, u_4 and u_6 are both adjacent to v_3, $u_4u_6 \in E(G)$, $u_5v_4 \in E(G)$ and $u_iv_4 \notin E(G)$ for $i \neq 5$. Now we use the fact that there are two vertices $v_i, v_j \in I$ that are both adjacent to $w \in W$. If v_1 and v_3 are both adjacent to w then $(w, v_1, u_1, u_2, u_3, u_4, v_3, w)$ is a C_7 in G. If v_1 and v_4 are both adjacent to w then $(w, v_1, u_1, u_3, u_4, u_5, v_4, w)$ is a C_7 in G. If v_2 and v_4 are adjacent to w then $(w, v_2, u_2, u_3, u_4, u_5, v_4, w)$ is a C_7 in G. Hence, by symmetry, we may assume that v_1 and v_2 are both adjacent to $w \in W$. Let $Z = \{u_1, \ldots, u_6, v_1, \ldots, v_4, w\}$ and $Z' = V(G) \setminus Z$.

As one may readily verify, for each vertex $z \in Z \setminus \{v_1, v_2, w\}$ there is a path of length six from w to z. Also for each $z \in Z \setminus \{u_4, u_6, v_4\}$ there is a path of length six from u_5 to z. Since $C_7 \not\subset G$, the degrees of u_5, v_2, v_3, w in $\langle Z \rangle_G$ are $3, 2, 2, 2$, respectively. Since $\delta(G) \geq 6$ there are at least $3+4+4+4 = 15$ edges joining $S \overset{\text{def}}{=} \{u_5, v_2, v_3, w\}$ and Z'. Since $|Z'| = 14$, there must be two vertices in S that are adjacent to the same $w' \in Z'$. Finally, the following path system shows that any two vertices in S are joined by a path of length five in $\langle Z\rangle_G$:

$$(u_5, u_4, u_3, u_1, u_2, v_2), \quad (u_5, u_4, u_3, u_1, u_6, v_3),$$
$$(u_5, u_4, u_3, u_2, v_2, w), \quad (v_2, u_2, u_1, u_3, u_4, v_3),$$
$$(v_2, u_2, u_1, u_3, v_1, w), \quad (v_3, u_4, u_3, u_2, v_2, w).$$

Since there are two vertices in S that are both adjacent to $w' \in Z'$, this gives a C_7 in G, a contradiction. □

Lemma 7. $r(C_8, K_5) = 29$.

Proof. Assume the standard configuration. The edge count $7 = |C| \leq |E(C, I)| \leq 8$ gives two cases for consideration.

Case (i): $|E(C, I)| = 7$. In this case, each vertex in C is adjacent to exactly one vertex in I, one (exceptional) vertex in I is adjacent to only one vertex in C, and the other three are each adjacent to two vertices on the cycle. We may assume that v_1 is not the exceptional vertex. Let N denote the neighbors of v_1 in C. By symmetry, there are two subcases.

Subcase (a): $N = \{u_1, u_3\}$. Then $u_1u_3 \in E(G)$, and we may assume that u_2 is adjacent to v_2. It is easily checked that there is a path of order eight joining v_2 and u_i for $i = 4, 5, 6, 7$. Since there would be a C_8 otherwise, we may assume that $u_iv_2 \notin E(G)$ for $i = 1, 3, 4, 5, 6, 7$, so v_2 must be the exceptional vertex. Then we may assume that v_3 is adjacent to u_4 and u_6, and that v_4 is adjacent to u_5 and u_7, so $u_5u_7 \in E(G)$. But this violates part (b) of Lemma 5.

Subcase (b): $N = \{u_1, u_4\}$. Then $u_1u_4 \in E(G)$, and we may assume that that u_2 is adjacent to v_2 and u_3 is adjacent to v_3. Note that there is a path of order eight joining v_i and u_j for $i = 2, 3$ and $j = 5, 6, 7$. But v_2 and v_3 are not both exceptional, so we have a contradiction.

Case (ii): $|E(C, I)| = 8$. In this case, one (exceptional) vertex in C is adjacent to two vertices in I, and each vertex in I is adjacent to two vertices in C. As noted earlier, we may assume that there is a vertex $w \in W$ that is adjacent to both v_1 and v_2. Again let N denote the neighbors of v_1 in C.

Subcase (a): $N = \{u_1, u_3\}$. Note that there is a path of order eight joining v_2 and u_i for $i = 4, 5, 6, 7$, so v_2 cannot be adjacent to u_4, u_5, u_6 or u_7. Also v_2 cannot be adjacent to u_1 and u_2 or to u_2 and u_3 by part (a) of Lemma 5. Finally, v_2 cannot be adjacent to both u_1 and u_3 since there is only one exceptional vertex in C. Hence there do not exist two vertices on the cycle that can serve as neighbors of v_2, a contradiction.

Subcase (b): $N = \{u_1, u_4\}$. Note that there is a path of order eight joining v_2 and u_i for $i = 1, 4, 5, 7$. Hence we may assume that v_2 is adjacent to u_2 and u_6. However, this violates part (c) of Lemma 5.

Since a contradiction arises in each subcase, the lemma is proved. □

Lemma 8. $r(C_9, K_5) = 33$.

Proof. Assume the standard configuration. The edge count $8 = |C| \leq |E(C, I)| \leq 8$ requires each vertex in C to be adjacent to exactly one vertex of I and each vertex in

I to be adjacent to exactly two vertices in C. We may assume that there is a vertex $w \in W$ that is adjacent to both v_1 and v_2. Let $N = \{u_i, u_j\}$ denote the neighbors of v_1 on the cycle. Since there is no five-element independent set, $u_i u_j \in E(G)$. By symmetry, there are three cases.

Case (i): $N = \{u_1, u_3\}$. It is easily checked that for $4 \leq i \leq 8$ there is a path of order seven joining v_1 and u_i. The paths $(v_1, u_1, u_8, u_7, u_6, u_5, u_4)$ and $(v_1, u_3, u_1, u_8, u_7, u_6, u_5)$ serve for $i = 4$ and $i = 5$, respectively, and their counterparts by symmetry take care of $i = 8$ and $i = 7$. The required path for $i = 6$ may be taken to be $(v_1, u_1, u_2, u_3, u_4, u_5, u_6)$. Hence there do not exist two vertices on the cycle that can serve as neighbors of v_2.

Case (ii): $N = \{u_1, u_4\}$. In this case for $5 \leq i \leq 8$ there is a path of order seven joining v_1 and u_i. The paths $(v_1, u_4, u_1, u_8, u_7, u_6, u_5)$ and $(v_1, u_1, u_2, u_3, u_4, u_5, u_6)$ serve for $i = 5$ and $i = 6$, respectively, and symmetric counterparts take care of $i = 8$ and $i = 7$. Therefore v_2 cannot be adjacent to any of the vertices u_5, u_6, u_7, u_8. By part (a) of Lemma 5, v_2 cannot be adjacent to u_2 and u_3. Hence there do not exist two vertices on the cycle that can serve as the neighbors of v_2.

Case (iii): $N = \{u_1, u_5\}$. In this case, there is a path of order seven joining v_1 to u_i for $i = 2, 4, 6, 8$, so the neighbors of v_2 on the cycle must be u_3 and u_7. Without loss of generality, u_2 is adjacent to v_3, and by symmetry the neighbors of v_3 on the cycle are either u_2 and u_4 or u_2 and u_6. In the first instance, $u_2 u_4 \in E(G)$ and $(v_1, u_1, u_8, u_7, v_2, u_3, u_2, u_4, u_5, v_1)$ is a C_9 in G. In the second, $u_2 u_6 \in E(G)$ and $(v_1, u_1, u_8, u_7, u_6, u_2, u_3, u_4, u_5, v_1)$ is a C_9 in G.

Since a contradiction arises in each case, the proof is complete. □

Completion of the proof of Theorem 1. For $n \geq 10$, the edge count $n - 1 = |C| \leq |E(C, I)| \leq 8$ gives an immediate contradiction. □

3 Appendix - Possible Induced Subgraphs $\langle W \rangle$ for Case (i) in Lemma 3

Here we give the promised collection of graphs of order 12 that contain no C_5 and have independence number 3.

Proposition. *If G is a graph of order twelve such that $C_5 \not\subset G$ and $\alpha(G) = 3$ then G is isomorphic to $3K_4$ or to one of the five graphs shown below, obtained by adding appropriate edges to $3K_4$.*

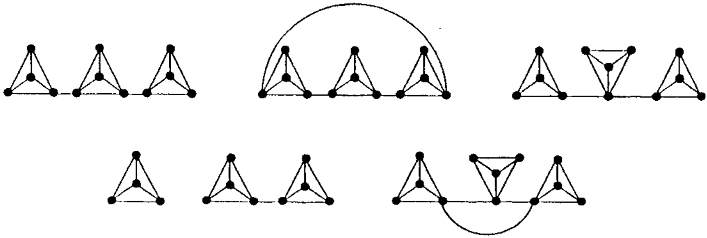

FIGURE 1. Graphs of order twelve with $C_5 \not\subseteq G$ and $\alpha(G) = 3$.

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