On the Ramsey number $R(C_n \text{ or } K_{n-1}, K_m)$ ($m = 3, 4$)

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Abstract

The Ramsey number $R(C_n \text{ or } K_{n-1}, K_m)$ is the smallest integer $p$ such that every graph $G$ on $p$ vertices contains either a cycle $C_n$ with length $n$ or a $K_{n-1}$, or an independent set of order $m$. In this paper we prove that $R(C_n \text{ or } K_{n-1}, K_3) = 2(n - 2) + 1$ ($n \geq 5$), $R(C_n \text{ or } K_{n-1}, K_4) = 3(n - 2) + 1$ ($n \geq 7$). In particular, we prove that $R(C_4 \text{ or } K_3, K_3) = 6,$ $R(C_4 \text{ or } K_3, K_4) = 8,$ $R(C_5 \text{ or } K_4, K_4) = 11$ and $R(C_6 \text{ or } K_5, K_4) = 14.$

1. Introduction.

We shall consider only graphs without multiple edges or loops.

The Ramsey number $R(C_n \text{ or } K_{n-1}, K_m)$ is the smallest integer $p$ such that every graph $G$ on $p$ vertices contains either a cycle $C_n$ with length $n$ or a complete graph $K_{n-1}$ on $n - 1$ vertices, or an independent set of order $m$.

In 1976, R.H. Schelp and R.J. Faudree in [2] stated the following problem:

**Problem 1.1** ([2]). *Is it true that $R(C_n \text{ or } K_{n-1}, K_m) = (n - 2)(m - 1) + 1$ ($n \geq m$)?*

With this problem, the aim of Schelp and Faudree was to solve the following problem:

**Problem 1.2** ([2]). *Find the range of integers $n$ and $m$ such that $R(C_n, K_m) = (n - 1)(m - 1) + 1$. In particular, show that the equality holds for $n \geq m$.*

However, we think that Problem 1.1 is more difficult than Problem 1.2. And in fact, the statement is false for $m \leq n \leq 2(m - 1)$. (See Lemma 2.3 below.)

In [3], we proved that $R(C_n, K_4) = 3(n - 1) + 1$ ($n \geq 4$).

In this paper, we prove that $R(C_n \text{ or } K_{n-1}, K_3) = 2(n - 2) + 1$ ($n \geq 5$) and $R(C_n \text{ or } K_{n-1}, K_4) = 3(n - 2) + 1$ ($n \geq 7$). In particular, we prove that $R(C_4$

* The project supported by NSFC and NSFJS

$K_3, K_3) = 6, \ R(C_4 \ or \ K_3, K_4) = 8, \ R(C_5 \ or \ K_4, K_4) = 11 \ and \ R(C_6 \ or \ K_5, K_4) = 14.$

The following notation will be used in this paper. If $G$ is a graph, the vertex set (resp. edge set) of $G$ is denoted by $V(G)$ (resp. $E(G)$). For $x \in V(G)$, $N(x) = \{v \in V(G) \mid xv \in E(G)\}$ and $N[x] = N(x) \cup \{x\}$. If $X \subseteq V(G)$, then $\langle X \rangle$ is the subgraph induced by $X$. We denote by $\alpha(G)$ the independence number of $G$, and by $g(G)$ the girth of $G$.

2. Lemmas.

For convenience, in Lemma 1 to Lemma 3 below, we assume $G$ is a graph that contains the cycle $(v_1, v_2, \ldots, v_n)$ of length $n$ but no cycle of length $n+1$.

Lemma 2.1 ([3]). Let $X \subseteq V(G) \setminus \{v_1, v_2, \ldots, v_n\}$. Then
(a) No vertex $x \in X$ is adjacent to two consecutive vertices on the cycle.
(b) If $x \in X$ is adjacent to $v_i$ and $v_j$, then $v_{i+1}v_{j+1} \not\in E(G)$.
(c) If $x \in X$ is adjacent to $v_i$ and $v_j$, then no vertex $x' \in X$ is adjacent to both $v_i+1$ and $v_j+2$.

Lemma 2.2. Let $I_{m-1}$ be an independent set of order $m-1$ with $I_{m-1} \subseteq V(G) \setminus \{v_1, v_2, \ldots, v_n\}$. If $n \geq 2m-3$ and $|N(x) \cap \{v_1, v_2, \ldots, v_n\}| = k$, where $x \in I_{m-1}$, then $k \leq m-3$.

Proof. For $x \in I_{m-1}$ suppose the neighbors of $x$ on the cycle are $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$. By parts (a) and (b) of Lemma 2.1 we know that that $\{x, v_{i_1+1}, \ldots, v_{i_k+1}\}$ is an independent set; hence $k+1 \leq m-1$. To prove that $k \leq m-3$, suppose to the contrary that $k = m-2$. Then $2k = 2m-4$ so since $n \geq 2m-3$ we may put $z = v_{i_k+2}$, where $i_k + 2 \not\equiv i_1 \pmod{n}$. Then $xz \not\in E(G)$. If $x'z \in E(G)$ for some $x' \in I_{m-1}$ then by part (c) of Lemma 2.1 $\{x, x', v_{i_1}, \ldots, v_{i_k}\}$ is an $m$-element independent set; otherwise $I_{m-1} \cup \{z\}$ is an $m$-element independent set. Hence $k \leq m-3$. \hfill $\Box$

Corollary. If $n > (m-1)(m-3) + 1$ and $G$ contains a $C_{n-1}$ and a vertex disjoint independent set $I_{m-1}$ with size $m-1$, then $G$ either contains a $C_n$ or an independent set of $m$ vertices.

Proof. If there is no independent set of $m$ vertices then each vertex on the $C_{n-1}$ is adjacent to at least one vertex in $I_{m-1}$. But then some vertex in $I_{m-1}$ is adjacent to at least $\lceil (n-1)/(m-1) \rceil \geq m-2$ vertices on the cycle, contradicting Lemma 2.2. \hfill $\Box$

Lemma 2.3.

(1) $R(C_n \ or \ K_{n-1}, K_m) \geq (n-2)(m-1) + 1 \ (n \geq m)$.
(2) $R(C_n \ or \ K_{n-1}, K_m) \geq (n-2)(m-1) + 2 \ (m \leq n \leq 2(m-1))$.

Proof.
(1) This is trivial.
(2) Starting with the cycle $(x_1, y_2, \ldots, x_{m-1}, y_{m-1}, x_m)$, let $G$ be the graph obtained by replacing each $y_i$ by a $K_{n-3}$. (Thus each vertex in the $K_{n-3}$ that
replaces $y_i$ is adjacent to $x_i$ and $x_{i+1}$.) It is easy to see that $G$ contains no $K_{n-1}$ and $\alpha(G) \leq m - 1$. If the edge $x_m x_1$ is removed, then each block of the resulting graph has $n - 1$ vertices; hence there is no $C_n$. Any other cycle in $G$ must use the edge $x_m x_1$, and then it must have length at least $2(m - 1) + 1 \geq n + 1$. □

Lemma 2.4 [1]. Let $G$ be a graph on $n \geq 3$ vertices. If $\delta(G) \geq n/2$, then $G$ either is pancyclic or else $G = K_{n/2,n/2}$.

3. The Ramsey number $R(C_n$ or $K_{n-1}, K_m)$ for $m = 3, 4$.

Theorem 3.1. $R(C_n$ or $K_{n-1}, K_3) = 2(n - 2) + 1$ ($n \geq 5$).

Proof. Let $G$ be a graph with order $2(n - 2) + 1$. Suppose $\alpha(G) \leq 2$ and suppose $G$ contains neither a $C_n$ nor a $K_{n-1}$.

Let $x \in V(G)$ and $V_x = V(G) \setminus N(x)$. Then $\langle V_x \rangle$ is a clique of $G$. Since $G$ does not contain a $K_{n-1}$ then $|V_x| \leq n - 2$. Thus $d(x) \geq n - 2$.

If $d(x) \geq n - 1$ for every $x \in V(G)$ then by Lemma 2.4 $G$ is pancyclic, a contradiction.

Thus there is a vertex $x \in V(G)$ such that $d(x) \leq n - 2$. (Note $n \geq 5$). Hence we have $d(x) = n - 2$ and $\langle V_x \rangle \cong K_{n-2}$. It is clear that there are two nonadjacent vertices in $N(x)$, say $y_1, y_2$. Since $\alpha(G) \leq n - 2$, there is a vertex $z_1$ in $V_x$ such that $z_1 \notin N(y_1)$. Thus $z_1 \in N(y_2)$ since $\alpha(G) \leq 2$. Similarly, there is a vertex in $V_x$, say $z_2$, such that $z_2 \notin N(y_2)$ and $z_2 \in N(y_1)$.

Thus $(x, y_1, v_1, v_2, \cdots, v_{n-2}, v_{n-3}, y_2)$ is a cycle of $G$, where $v_1 = z_1$, $v_{n-3} = z_2$ and $\{v_2, v_3, \cdots, v_{n-4}\} \subseteq V_x \setminus \{z_1, z_2\}$, a contradiction.

Therefore $R(C_n$ or $K_{n-1}, K_3) = 2(n - 2) + 1$ ($n \geq 5$). □

Theorem 3.2.

(1) $R(C_4$ or $K_3, K_3) = 6$.

(2) $R(C_4$ or $K_3, K_4) = 8$.

(3) $R(C_5$ or $K_4, K_4) = 11$.

(4) $R(C_6$ or $K_5, K_4) = 14$.

Proof. (1) It is clear that $R(C_4$ or $K_3, K_3) = 6$.

(2) Suppose $G$ is of order eight and girth at least five. We shall prove that $\alpha(G) \geq 4$. If $G$ is bipartite, this conclusion is immediate, so we assume that $G$ contains an odd cycle. If $\langle X \rangle \cong C_7$ is the shortest odd cycle in $G$, then the remaining vertex $u$ is adjacent to at most one vertex in $X$. But any five-element subset of $X$ contains a three-element independent set; hence $\{x_i, x_j, x_k, u\}$ is an independent set for appropriate $i, j, k$. If $\langle X \rangle \cong C_5$ is the shortest odd cycle in $G$, then since $\{u, v, w\} = V(G) \setminus X$ does not span $K_3$ we may assume that $u$ and $v$ are nonadjacent. Since $g(G) \geq 5$ neither $u$ nor $v$ is adjacent to more than one vertex in $X$. Hence there are three vertices in $X$, none of which is adjacent to either $u$ or $v$. Since $G$ contains no $K_3$, we thus find that $\{x_i, x_j, u, v\}$ is an independent set for appropriate $i, j$.
(3) Suppose $G$ is a graph of order eleven that contains neither $C_5$ nor $K_4$, and $\alpha(G) \leq 3$. In view of the result $R(C_5, K_4, K_3) = 7$ obtained earlier, we have $\delta(G) \geq 4$. Using $R(C_4, K_4) = 10$ as well, we may assume that

$$V(G) = X \cup Y \cup Z = \{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3\} \cup \{z_1, z_2, z_3, z_4\},$$

where $(x_1, x_2, x_3, x_4)$ is a $C_4$ in $G$ and $Y$ is an independent set. Since each vertex in $X$ is adjacent to at least one vertex in $Y$ and $G$ contains no $C_5$, there is no loss of generality in assuming $x_1y_1, x_3y_1, x_2y_2 \in E(G)$. Then $x_2x_4 \notin E(G)$; otherwise, $(x_1, y_1, x_3, x_2, x_4)$ is a $C_5$ in $G$. Since there is no $C_5$ in $G$, it is apparent that $x_2y_1 \notin E(G)$ and $x_4y_1 \notin E(G)$. In the same way $x_1y_2 \notin E(G)$ and $x_3y_2 \notin E(G)$. Since $\delta(G) \geq 4$, we have $y_1z \in E(G)$ for some $z \in Z$. Note that $x_1z \notin E(G)$, $y_2z \notin E(G)$, $zx_3 \notin E(G)$; otherwise $G$ contains $(z, x_1, x_2, x_3, y_1)$, $(z, y_2, x_2, x_3, y_1)$, $(z, x_3, x_2, x_1, y_1)$, respectively. Now $x_1x_3 \in E(G)$; otherwise $(x_1, x_3, y_2, z)$ is an independent set. Then $x_4y_2 \notin E(G)$; otherwise $G$ contains the cycle $(x_4, y_2, x_2, x_1, x_3)$. Since $x_4y_1 \notin E(G)$ and $x_4y_2 \notin E(G)$, we have $x_4y_3 \in E(G)$ and thus $x_3y_3 \notin E(G)$. Finally, if $y_3 \in E(G)$ then $G$ contains the cycle $(z, y_3, x_4, x_1, y_1)$ and if $y_3 \notin E(G)$ then $\{x_2, y_2, y_3, z\}$ is an independent set.

(4) Suppose $G$ is a graph of order fourteen that contains neither $C_6$ nor $K_5$, and $\alpha(G) \leq 3$. In view of the results $R(C_5, K_4) = 12$ and $R(C_6, K_5, K_3) = 9$, we may assume that $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3\}$ are disjoint subsets of $V(G)$ such that $(x_1, \ldots, x_5)$ is a $C_5$ in $G$ and $Y$ is an independent set. Since $6 > (4 - 1)(4 - 3) + 1$, the desired result follows from the corollary to Lemma 2.2. □

Lemma. If $G$ is a graph of order $2m$ having independence number $\alpha(G) < 3$ and containing neither $K_{m+1}$ nor $C_{m+2}$ then $G \supseteq 2K_m$.

Proof. In view of Bondy’s theorem, we may assume that $\delta(G) \leq m - 1$. Let $x \in V(G)$ be a vertex of degree $\delta(G)$, and set $A = N[x]$ and $B = V(G) \setminus A$. Then $|B| \geq m$ and $\langle B \rangle$ is complete since $\alpha(G) < 3$. Since $K_{m+1} \not\subseteq G$, we have $\delta(G) = m - 1$. If $\langle A \rangle$ is complete then $G \supseteq 2K_m$, so let us assume $u, v \in A$ and $uv \notin E(G)$. Since $K_m \not\subseteq G$ and $\alpha(G) < 3$ there are distinct vertices $w, z \in B$ such that $uw \notin E(G)$ and $vz \notin E(G)$. Then the path $w, v, x, u, z$ together with the appropriate path of length $m - 2$ joining $w$ and $z$ in $\langle B \rangle$ yields a $C_{m+2} \subseteq G$ and thus a contradiction. □

Theorem 3.3. $R(C_n \text{ or } K_{n-1}, K_4) = 3(n - 2) + 1$ ($n \geq 7$).

Proof. Suppose $n \geq 7$ and $G$ is a graph of order $3(n-2)+1$ that contains neither $C_n$ nor $K_{n-1}$ and satisfies $\alpha(G) \leq 3$. Since $R(C_{n-1}, K_4) = 3(n - 2) + 1$ for $n \geq 5$, we may assume that $(x_1, x_2, \ldots, x_{n-1})$ is a cycle in $G$. With $X = \{x_1, x_2, \ldots, x_{n-1}\}$ consider the subgraph of $G$ spanned by $V(G) \setminus X$. If this graph has independence number 3 then we have $C_n \subseteq G$ or $\alpha(G) \geq 4$ by the corollary to Lemma 2.2. Hence the subgraph of $G$ spanned by $V(G) \setminus X$ has $2(n-2)$ vertices and its independence number is 2. By the preceding Lemma, we thus find a partition $V(G) \setminus X = (Y, Z)$.
such that \( \langle Y \rangle \cong \langle Z \rangle \cong K_{n-2} \). Since \( \langle X \rangle \) is not complete, we may assume that \( x_1x_k \notin E(G) \) where \( k \leq \lfloor (n + 1)/2 \rfloor \). If \( x_1v \notin E(G) \) and \( x_kv \notin E(G) \) for every \( v \in Y \cup Z \) then \( \{x_1, x_k, y, z\} \) is a 4-element independent set for arbitrary \( y \in Y \) and \( z \in Z \) such that \( yz \notin E(G) \). (There must be such a \( z \) since \( G \) contains no \( K_{n-1} \).) Hence by symmetry we may assume that \( x_1y_1 \in E(G) \) and (since \( G \) contains no \( K_{n-1} \)) \( x_1y_2 \notin E(G) \). Note that \( x_ky_i \notin E(G) \) for all \( i \neq 1 \); otherwise (since \( (n + 1)/2 + 1 \leq n \)) there is a cycle \( (x_1, \ldots, x_k, y_i, \ldots, y_1) \) in \( G \) of length \( n \). In particular, \( x_ky_2 \notin E(G) \). We now consider two cases.

Case (i). \( x_kz \notin E(G) \) for all \( z \in Z \). If \( x_1z_i \in E(G) \) for some \( z_i \in Z \) then \( y_2z \notin E(G) \) for all \( z \in Z \); otherwise there is a cycle \( (x_1, y_1, \ldots, y_2, z, z_i) \) of length \( n \in G \). Then since there is some \( z_j \in Z \) such that \( x_1z_j \notin E(G) \) we find that \( \{x_1, x_k, y_2, z_j\} \) is an independent set. If \( x_1z \notin E(G) \) for all \( z \in Z \) then we can pick a vertex \( z_j \in Z \) such that \( y_2z_j \notin E(G) \) and then \( \{x_1, x_k, y_2, z_j\} \) is an independent set.

Case (ii). \( x_kz_1 \in E(G) \) and \( x_ky_2 \notin E(G) \). By repeating an earlier argument, we have \( x_1z_2 \notin E(G) \). If \( y_2z_2 \notin E(G) \) then \( \{x_1, x_k, y_2, z_2\} \) is an independent set. Otherwise, \( (x_1, \ldots, x_k, z_1, \ldots, z_2, y_2, \ldots, y_1) \) is a cycle in \( G \) of length \( \geq k + 4 \) and \( G \) contains a \( C_n \) provided \( n \geq \lfloor (n + 1)/2 \rfloor + 4 \). This completes the proof in case \( n \geq 8 \). In case \( n = 7 \), we are left to consider the case \( k = 4 \). In particular, we may assume \( x_1x_3 \in E(G) \) and then the argument proceeds as before except that \( (x_1, x_3, x_4, z_1, z_2, y_2, y_1) \) provides the \( C_7 \). \( \square \)

Acknowledgement: The authors would like to thank the referee for many helpful suggestions.

References

[3] Yang Jian Sheng, Huang Yi Ru and Zhang Ke Min, The value of the Ramsey number \( R(C_n, K_4) \) is \( 3(n - 1) + 1 \) \( (n \geq 4) \), Australasian Journal of Combinatorics, 20 (1999), 205–206.

(Received 19/1/2000; revised 16/5/2000)