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ON CONTAINER LENGTH AND WIDE-DIAMETER IN UNIDIRECTIONAL HYPERCUBES

Lu Changhong and Zhang Kemin

Abstract. In this paper, two unidirectional binary n-cubes, namely, $Q_1(n)$ and $Q_2(n)$, proposed as high-speed networking schemes by Chou and Du, are studied. We show that the smallest possible length for any maximum fault-tolerant container from a to b is at most $n+2$ whether a and b are in $Q_1(n)$ or in $Q_2(n)$. Furthermore,we prove that the wide-diameters of $Q_1(n)$ and $Q_2(n)$ are equal to $n+2$. At last, we show that a conjecture proposed by Jwo and Tuan is true.

1. Introduction

The hypercube is one of the best candidates for *high-speed computing* [12, 13], and using *optical fibers* as point-to-point transmission links, *Metropolitan Area Networks* (MANs) with hypercube topology can support *high-speed*, *high-bandwith*, *short-delay*, and *parallel communications* [2, 3, 6, 15, 16]. As pointed in [10] by Jwo and Tuan, due to the lack of a bidirectional electrical/optical converter and the high cost of a *full-duplex* tansmission, a unidirectional topology is desirable for MANs [3, 4]. In particular, Chou and Du [3] proposed two different schemes, namely, $Q_1(n)$ and $Q_2(n)$, to define the orientations of the edges in the binary n-cube as follows: $\eta(x)$ is the number of 1's in the binary representation of x. Consider the two vertices $a = a_{n-1}a_{n-2}\cdots a_{i+1}a_ia_{i-1}\cdots a_1a_0$ and $b = a_{n-1}a_{n-2}\cdots a_{i+1}\overline{a}_ia_{i-1}\cdots a_1a_0$.

$Q_1(n)$: Let $P(a, i)$ be the *polarity* of the ith communication port of a which is defined as

$$P(a, i) = (-1)^{\eta(a)+i}.$$

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If $P(a;i)$ is positive, then there is a directed edge from a to b; otherwise, there is a directed edge from b to a. The unidirectional hypercube defined by the above polarity function is called a *positive* $Q_1(n)$. A *negative* $Q_1(n)$ is defined in the same way but with a different polarity function:

$$P(a;i) = (-1)^{\eta(a)+i+1}.$$

Clearly, $Q_1(n)$ and its negative counterpart are isomorphic. Unless otherwise stated, we shall consider the positive $Q_1(n)$ only.

Observe that $Q_1(n)$ can be constructed by one $Q_1(n-1)$, one negative $Q_1(n-1)$, and 2^{n-1} edges between them.

$Q_2(n)$: Like $Q_1(n)$, the orientations of the edges in $Q_2(n)$ are defined by the polarities of the corresponding communication ports. If n is odd, $a_{n-1} = 1$ and $0 \le i \le n-2$, then the corresponding polarity function is

$$P(a;i) = (-1)^{\eta(a)-1+i};$$

otherwise, the polarity $P(a;i)$ is the same as that for $Q_1(n)$. In fact, when n is odd, $Q_2(n)$ can be constructed by two $Q_1(n-1)$'s and 2^{n-1} edges between them. Since $Q_2(n)$ is identical to $Q_1(n)$ when n is even, we shall only consider $Q_2(n)$ when n is odd.

General results and more details on $Q_1(n)$ and $Q_2(n)$ can be found in [3, 10].

Any set of vertex-disjoint paths from vertex x to vertex y, denoted by $C(x;y)$, is called an $(x;y)$-container [6]. The width of $C(x;y)$, written as $w(C(x;y))$, is its cardinality. The length of $C(x;y)$, written as $l(C(x;y))$, is the longest path length in $C(x;y)$. Define $D_w(x;y)$ to be the minimum possible length of any $(x;y)$-container with width w. Let $\xi(x;y)$ denote the maximum number of vertex-disjoint paths from x to y. The wide-diameter of a graph G [5, 7], denoted by $WD(G)$, is the maximum of $D_{\xi(x;y)}(x;y)$ for all pairs of vertices x and y. Obviously, the wide-diameter of a graph is no less than its diameter. The wide-diameter, proposed by Hsu [7], and Flandrin and Li [5] independently, is a good index to characterize the reliability of transmission delay in a network, and has received much attention recently [5-9, 11, 14]. We refer to [1] for notations and terminology not defined here.

Recently, Jwo and Tuan [10] have shown that $\xi(x;y) = \min(\mathrm{out}(x);\mathrm{in}(y))$ for all pairs of vertices x and y in $Q_1(n)$ or $Q_2(n)$, i.e., both $Q_1(n)$ and $Q_2(n)$ are maximum fault-tolerant. Furthermore, they have also shown that $D_{\xi(x;y)}(x;y)$ is at most (1) $l+4$, where l is the shortest path length in $Q_1(n)$ from x to y, and (2) $l+5$, where l is the shortest path length in $Q_2(n)$ (n is odd) (i) from x to y when x and y have the same leading-bit values and (ii) from x to y'(y' and y only differ at leading-bit position) when otherwise. They also suggest that the constructed container in [9] has the smallest possible length among all maximum fault-tolerant containers from x to y.

In this paper, we shall prove that $D_{\kappa(x,y)}(x,y)$ is no more than $n+2$ for any pairs of vertices x and y in $Q_1(n)$ or $Q_2(n)$. Furthermore, we prove that the wide-diameters of $Q_1(n)$ and $Q_2(n)$ are equal to $n+2$ and the conjecture in [9] is true. Since the diameters of $Q_1(n)$ and $Q_2(n)$ are $n+1$ when n is even and the diameters of $Q_1(n)$ and $Q_2(n)$ re $n+2$ when n is odd, we have that $|WD(Q_i(n)) - \mathrm{Diam}(Q_i(n))| \le 1, \ i = 1, 2.$

2. Preliminaries

Suppose that $a = a_{n-1}a_{n-2}\cdots a_0$ and $b = b_{n-1}b_{n-2}\cdots b_0$ are two vertices in $Q_1(n)$ (resp. $Q_2(n)$). Define $DP_i(a,b) = a_i \oplus b_i$, where $0 \le i \le n-1$ and $\oplus$ is Boolean addition. $DP(a,b)$ is defined as the n-bit sequence: $DP_{n-1}(a,b)\cdots DP_1(a,b)DP_0(a,b)$. The polarity of $DP(a,b)$ is the same as that of a. $\hat{p}$ and $\hat{n}$ denote the number of $1'$s in $DP(a,b)$ with positive and negative polarity, respectively. For instance, if $a = 1111$ and $b = 0001$, then $DP(a,b) = 1110$ and $\hat{p} = 1, \hat{n} = 2$. For notational simplicity, we will use z_i to represent $DP_i(a,b)$.

Fact 1 [3]. *Given two vertices* a *and* b *in* $Q_1(n)$ (*resp.* $Q_2(n)$), *the shortest path length from* a *to* b *can be computed as follows*:

$$\begin{cases} 2\hat{p} - (\hat{p} - \hat{n}) \mod 2, & \text{if } \hat{p} > \hat{n}; \\ 2\hat{n} + (\hat{n} - \hat{p}) \mod 2, & \text{if } \hat{p} \le \hat{n}. \end{cases}$$

Fact 2 [3]. *Given two vertices* a *and* b *in* $Q_1(n)$, *let* l (*resp.* l') *be the shortest path length from* a (*resp.* b) *to* b (*resp.* a). *Then*,

$$\begin{cases} l - l' = 0, & \text{if } (\hat{p} - \hat{n}) \mod 2 = 0; \\ l - l, = 2, & \text{if } (\hat{p} - \hat{n}) \mod 2 = 1. \end{cases}$$

Fact 3 [3]. *The diameter of* $Q_i(n)$ *is (a)* $n+1$, *if* n *is even*, *(b)* $n+2$, *if* n *is odd*, $i = 1, 2$.

Fact 4 [9]. *Let* a *and* b *be two vertices of* $Q_1(n)$ (*resp.* $Q_2(n)$). *Then* $\kappa(a,b) = \min(\mathrm{out}(a), \mathrm{in}(b))$. *In other words*, *both* $Q_1(n)$ *and* $Q_2(n)$ *are maximum fault-tolerant.*

Lemma 1. $WD(Q_i(n)) \ge n+1$, *if* n *is even*, $WD(Q_i(n) \ge n+2$, *otherwise*, $i = 1, 2$.

Proof. By Fact 3 and the definition of wide-diameter, it is obvious. ∎

Lemma 2. *Let* a *and* b *be two vertices of* $Q_1(n)$ *where* n *is odd*; *and* l *be the shortest path length from* a *to* b. *Then,* $l = n + 2$ *if and only if*

$$\begin{cases} (\hat{n} - \hat{p}) \mod 2 = 1; \\ \hat{n} = \frac{n+1}{2}. \end{cases}$$

Proof. By Fact 1, we have $l = n + 2$ if and only if

$$n + 2 = 2\hat{p} - (\hat{p} - \hat{n}) \mod 2; \quad \text{if } \hat{p} > \hat{n};$$

or

$$n + 2 = 2\hat{n} + (\hat{n} - \hat{p}) \mod 2; \quad \text{if } \hat{p} \leq \hat{n}.$$

Since $n + 2$ is odd and $\hat{p} \leq (n + 2)/2$, we easily find

$$l = n + 2 \iff \begin{cases} (\hat{n} - \hat{p}) \mod 2 = 1; \\ \hat{n} = \frac{n+1}{2}. \end{cases}$$

∎

Lemma 3. *Let* a *and* b *be two vertices of* $Q_1(n)$ *where* n *is odd*; *and* l *be the shortest path length from* a *to* b. *Then* $l = n + 1$ *if and only if*

$$\begin{cases} (\hat{n} - \hat{p}) \mod 2 = 0; \\ \hat{n} = \frac{n+1}{2}; \end{cases} \text{ or } \begin{cases} (\hat{n} - \hat{p}) \mod 2 = 0; \\ \hat{p} = \frac{n+1}{2}. \end{cases}$$

Proof. By Fact 1, we have $l = n + 1$ if and only if

$$n + 1 = 2\hat{p} - (\hat{p} - \hat{n}) \mod 2; \quad \text{if } \hat{p} > \hat{n};$$

or

$$n + 1 = 2\hat{n} + (\hat{n} - \hat{p}) \mod 2; \quad \text{if } \hat{p} \leq \hat{n}.$$

Since $n + 1$ is even, we easily find

$$l = n + 1 \iff \begin{cases} (\hat{n} - \hat{p}) \mod 2 = 0; \\ \hat{n} = \frac{n+1}{2}; \end{cases} \text{ or } \begin{cases} (\hat{p} - \hat{n}) \mod 2 = 0; \\ \hat{p} = \frac{n+1}{2}. \end{cases}$$

∎

Lemma 4 [10]. *Let* a *and* b *be two vertices of* $Q_1(n)$ *with* $z_i = 1$ *for every even integer* i *in* $[0; n - 1]$. *Then* $D_{\pi(a;b)}(a; b)$ *equals the shortest path length from* a *to* b.

For $a = a_{n-1} \cdots a_1 a_0$ and $b = b_{n-1} \cdots b_1 b_0$ in $Q_1(n)$, if $z_{n-1} = 1$ and $z_i = 0$ for some even integer i, then each vertex $x = x_{n-1} \cdots x_1 x_0$ can be relabeled by the mapping defied as follows:

1. If n is odd, then choose an even integer i with $z_i = 0$ and define

$$\alpha_i : x \to x_i x_{n-2} x_{n-3} \cdots x_{i+1} x_{n-1} x_{i-1} \cdots x_0.$$

2. If n is even, then arbitrarily choose an i with $z_i = 0$ and define

$$\alpha_i \to \begin{cases} x_i x_{n-2} x_{n-3} \cdots x_{i+1} x_{n-1} x_{i-1} \cdots x_0; & \text{if i is odd;} \\ \overline{x}_i x_{n-1} x_{n-2} \cdots x_{i+1} x_0 x_{i-1} \cdots x_1; & \text{if i is even.} \end{cases}$$

The following result is due to Jwo and Tuan [10], which is also easy to deduce.

Lemma 5 [10]. *Let* a *and* b *be two vertices of* $Q_1(n)$ *with* $z_{n-1} = 1$ *and* $z_i = 0$ *for some even integer* i*. The relabeling mapping* α_i *described above is an automorphism of* $Q_1(n)$.

3. The Container Length and Wide-diameter of $Q_1(n)$

In this section, we shall first prove the following theorem:

Theorem 1. *Let* a *and* b *be two vertices of* $Q_1(n)$*. Then* $D_{\xi(a;b)}(a; b) \le n + 2$.

Proof. We proceed by induction on n. When $n = 2$, it is trivial. Assume that Theorem 1 is true for $n \le k - 1$ and $k \ge 3$.

Let $n = k$. If $z_i = 1$ for every even integer i with $0 \le i \le k - 1$, Lemma 4 and Fact 3 guarantee that Theorem 1 is true. Without loss of generality, we may assume that there exists an even integer i such that $z_i = 0$. By Lemma 5, we can assume that $z_{k-1} = 0$, i.e., a and b are in the same subcube $Q_1(k - 1)$. Let $Q_1^1(k - 1)$ represent the subcube containing a and b, and $Q_1^2(k - 1)$ respresent the other subcube. Given an n-bit binary number $v = v_{n-1} \cdots v_1 v_0$, let v' denote the n-bit binary number $\overline{v}_{n-1} v_{n-2} \cdots v_0$ and v'' denote the $(n - 1)$-bit binary number $v_{n-2} v_{n-3} \cdots v_0$. Clearly, a'' and b'' are two vertices in a $Q_1(k - 1)$. By Fact 4, $\xi(a''; b'') = \min(\text{out}(a''); \text{in}(b''))$ and $\xi(b''; a'') = \min(\text{out}(b''); \text{in}(a''))$.

Suppose that $a_{k-1} = b_{k-1} = 0$ (resp. 1). Let $P_1; P_2; \cdots; P_r$ be a collection of the maximum number of vertex-disjoint paths from a to b in $Q_1^1(k - 1)$, where $r = \xi(a''; b'')$ (resp. $r = \xi(b''; a'')$). Obviously, we can regard $P_1; P_2; \cdots; P_r$ as a maximum amount of vertex-disjoint paths from a'' to b'' (resp. from b'' to a'') in $Q_1(k - 1)$. By induction hypothesis, we can assume that each of the r paths has length at most $k + 1$.

Case 1. k *is odd.*

Subcase 1.1. $\eta(a)$ *is odd or* $\eta(b)$ *is even.*

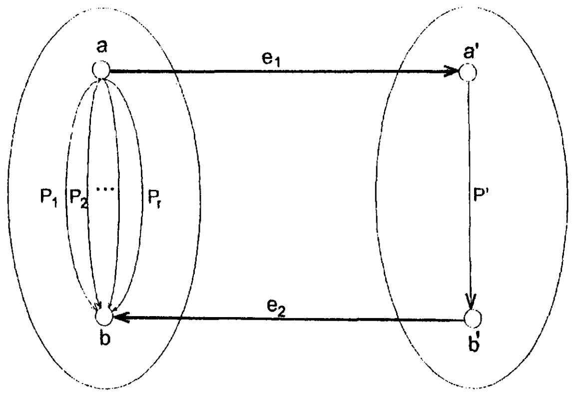

FIG. 1. k is odd, $\omega(a)$ is even, and $\omega(b)$ is odd, or k is even, $\omega(a)$ is odd, and $\omega(b)$ is even: the $r+1$ vertex-disjoint paths from a to b in $Q_1(k)$.

In this situation, we have $\min(\mathrm{out}(a), \mathrm{in}(b)) = \xi(a'';b'')$ (resp. $\min(\mathrm{out}(a), \mathrm{in}(b)) = \xi(b'';a'')$). By Fact 3, $\xi(a;b) = \xi(a'';b'')$ (resp. $\xi(a;b) = \xi(b'';a'')$). Thus, $P_1;P_2;\cdots;P_r$ is also a collectoin of the maximum number of vertex-disjoint paths from a to b in $Q_1(k)$, where $r = \xi(a;b)$. So, $D_{\xi(a;b)}(a;b) \le k+1$.

Subcase 1.2. $\omega(a)$ *is even and* $\omega(b)$ *is odd.*

We have $\min(\mathrm{out}(a), \mathrm{in}(b)) = \xi(a'';b'')+1$ (resp. $\min(\mathrm{out}(a), \mathrm{in}(b)) = \xi(b'';a'')+1$). By Fact 3, $\xi(a;b) = \xi(a'';b'')+1$ (resp. $\xi(a;b) = \xi(b'';a'')+1$). See Figure 1. Since a_{k-1} has positive polarity and b_{k-1} has negative polarity, there exist an edge e_1 from a to a' and an edge e_2 from b' to b. Let P' be a shortest path from a' to b' in $Q_1^2(k-1)$. It is easy to see that there exists a new path $P = e_1 + P' + e_2$, which certainly is vertex-disjoint with all the paths $P_1;P_2;\cdots;P_r$ from a to b in $Q_1^1(k-1)$. Since P' is a shortest path in $Q_1^2(k-1)$, the length of P' is no more than k by Fact 3, and the length of P is no more than $k+2$. So, the length of the maximum fault-tolerant $(a;b)$-container $P_1;P_2;\cdots;P_r;P$ is no more than $k+2$.

Case 2. k *is even.*

Subcase 2.1. $\omega(a)$ *is odd and* $\omega(b)$ *is even.*

We have $\min(\mathrm{out}(a), \mathrm{in}(b)) = \xi(a'';b'')+1$ (resp. $\min(\mathrm{out}(a), \mathrm{in}(b)) = \xi(b'';a'')+1$). By Fact 3, $\xi(a;b) = \xi(a'';b'')+1$ (resp. $\xi(a;b) = \xi(b'';a'')+1$)). See Fig. 1. Proceed similarly to that in Subcase 1.2 and obtain that $P_1;P_2;\cdots;P_r;P$ are a maximum fault-tolerant $(a;b)$-container. We calculate the length of P. When $a_{n-1} = 0$, the length of P' is equal to the length of the shortest path from b'' to a'' in $Q_1(k-1)$. Obviously, this is at most $k+1$ by Fact 3. Since $\omega(b'') = \omega(b)$ is

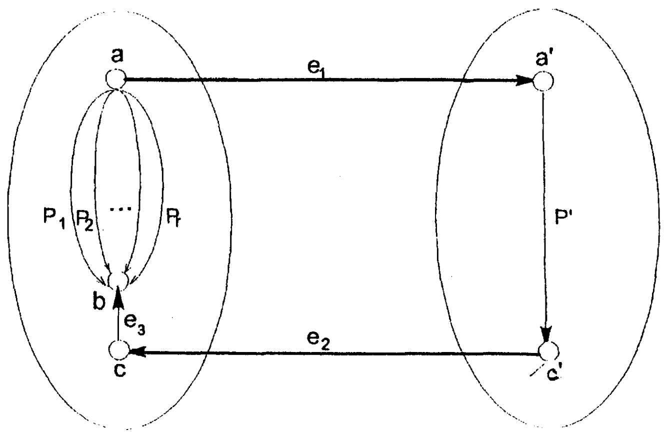

FIG. 2. k is even, $\phi(a)$ is odd and $\phi(b)$ is odd: the $r+1$ disjoint-paths from a to b in $Q_1(k)$.

even, we know that ω of $DP(b'';a'')$ is less than $k/2$ and the shortest path from b'' to a'' has length at most k by Lemma 2. When $a_{n-1}=1$, the length of P' is equal to the length of the shortest path from a'' to b'' in $Q_1(k-1)$. Obviously, this is also no more than $k+1$ by Fact 3. Since $\phi(a'') = \phi(a) - 1$ is even, we know that ω of $DP(a'';b'')$ is less than $k/2$ and the shortest path from b'' to a'' also has length at most k. In a word, P' has length at most k. So, P has length at most $k+2$. By the induction hypothesis, we easily see that the constructed maximum fault-tolerant $(a;b)$-container $P_1;P_2;\cdots;P_r;P$ has length at most $k+2$.

Subcase 2.2. $\phi(a)$ *is odd and* $\phi(b)$ *is odd.*

We similarly have $\kappa(a;b) = \kappa(a'';b'') + 1$ (resp. $\kappa(a;b) = \kappa(a'';b'') + 1$). See Figure 2. Since a_{k-1} has positive polarity, there exists an edge e_1 from a to a'. Althouhg b has $k/2$ incoming ports available within $Q_1^1(k-1)$, only $(k/2)-1$ incoming ports are used by the collection of vertex-disjoint paths $P_1;P_2;\cdots;P_r$, where $r = \kappa(a'';b'')$ (resp. $r = \kappa(b'';a'')$). Thus, there is an unused incoming port, say port j, of b which results in the edge e_3 from the vertex $c = b_{k-1}\cdots b_{j+1}\overline{b_j}b_{j-1}\cdots b_0$ to b. Note that c is not in any of $P_1;P_2;\cdots;P_r$, and $c' = \overline{b_{k-1}}\cdots b_{j+1}\overline{b_j}b_{j-1}\cdots b_0$. Since c and c' differ in the $(k-1)$th bit and the polarity of that bit in c' is positive, there is an edge e_2 from c' to c. Let P' be a shortest path from a' to c' in $Q_1^2(k-1)$. Then, the new path $P = e_1 + P' + e_2 + e_3$ does not intersect any internal vertex in $P_1;P_2;\cdots;P_r$. So, $P_1;P_2;\cdots;P_r$ and P is a maximum fault-tolerant $(a;b)$-container.

Since $P_1;P_2;\cdots;P_r$ is identical to a maximum amount of vertex-disjoint paths from a'' (resp. b'') to b'' (resp. a'') in $Q_1(k-1)$, and since we assume that each of

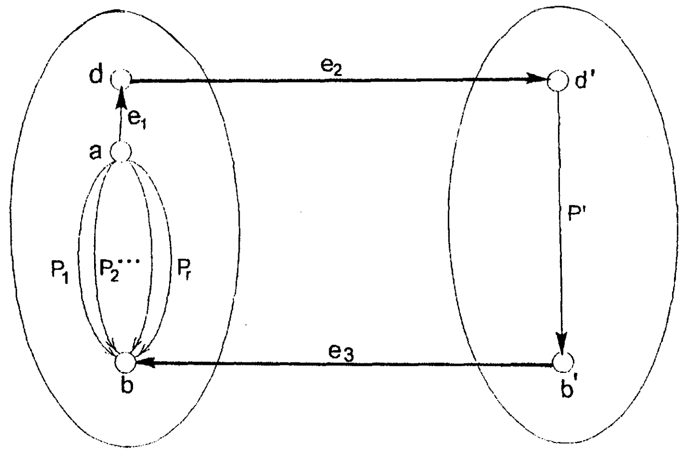

FIG. 3. k is even, $\omega(a)$ is even and $\omega(b)$ is even: the $r+1$ vertex-disjoint paths from a to b in $Q_1(k)$.

the r paths has length at most $k+1$, it is sufficient to prove that the new path P has length at most $k+2$. When $a_{k-1}=0$, the length of P' is equal to that of the shortest path from c'' to a'' in $Q_1(k-1)$. Since $\omega(c'')$ is even and $\omega(a'')$ is odd, we have $\hat{n} \neq (k/2)+1$ and $(\hat{p}-\hat{n}) \mod 2 \neq 0$. By Lemmas 2 and 3, we have the length of the shortest path from c'' to a'' in $Q_1(k-1)$ is at most $k-1$. When $a_{k-1}=1$, the length of P' is equal to that of the shortest path from a'' to c'' in $Q_1(k-1)$. Note that $\omega(a'')$ is even and $\omega(c'')$ is odd. Similarly, we obtain that P' has length at most $k-1$. So, we know that P always has length at most $k+2$. Thus, $D_{\xi(a,b)}(a,b) \leq k+2$.

Subcase 2.3. $\omega(a)$ *is even and* $\omega(b)$ *is even.*

In this situation, $\xi(a,b)=\xi(a'',b'')+1$ (resp. $\xi(a,b)=\xi(b'',a'')$). As shown in Figure 3, $P_1,P_2,\ldots,P_r$ and the new path $P=e_1+e_2+P'+e_3$ is a maximum fault-tolerant (a,b)-container with width $\xi(a,b)$, where $r=\xi(a'',b'')$ (resp. $r=\xi(b'',a'')$) and P' is a shortest path from d' to b' in $Q_1^2(k-1)$. Similarly, we can prove that P' has length at most $k-1$ and the length of P is no more than $k+2$. Thus, $D_{\xi(a,b)}(a,b) \leq k+2$. The detail is left to readers.

Subcase 2.4. $\omega(a)$ *is even and* $\omega(b)$ *is odd.*

In this situation, $\xi(a,b)=\xi(a'',b'')$ (resp. $\xi(a,b)=\xi(b'',a'')$). We easily know $D_{\xi(a,b)}(a,b) \leq k+2$. The proof is similar to that of Subcase 1.1.

By induction, we get that $D_{\xi(a,b)}(a,b) \leq n+2$.

The proof of Thereom 1 is completed. ■

Due to Thereom 1, we know that the wide-diameter of $Q_1(n)$ is no more than $n+2$ and, when n is odd, $WD(Q_1(n)) = n+2$ by Lemma 1. On the other hand, if there exists some even number k (≥ 4) such that $WD(Q_1(k)) = k+1$, then consider two vertices $a = 00\cdots0$ and $b = 0011\cdots1$ in $Q_1(k)$. Since $\omega(a)$ is even and $\omega(b)$ is even, we know $\kappa(a;b) = \kappa(a'';b'')+1$, so any $(a;b)$-container with width $\kappa(a;b)$ must have a path, say P, which passes the vertex $b' = 1011\cdots1$, and $(b';b)$ is the last edge in P. Let P' be a shortest path from a to b' in $Q_1(k)$. By Fact 1, we calculate that the length of P is equal to $k+1$ since $DP(a;b') = 1011\cdots1$ and $\hat{p} = (k-2)/2$, $\hat{n} = k/2$. Then P has length at least $k+2$. So the length of any $(a;b)$-container with width $\kappa(a;b)$ is at least $k+2$, a contradiction. Thus, we have the follow thoerem:

Theorem 2. *The wide-diameter of* $Q_1(n)$ *(*$n \geq 3$*) is equal to* $n+2$.

Remark 1. In [10], Jwo and Tuan have shown that the smallest possible length for any maximum fault-tolerant container from a to b is at most $l+4$, where l is the shortest path in $Q_1(n)$ from a to b. Now, we show that this upper bound is best. When $n \geq 4$ is even, consider the two vertices $a = 00\cdots0$ and $b = 0011\cdots1$ in $Q_1(n)$ ($n \geq 4$ is even). Since $DP(a;b) = 0011\cdots1$ and $\hat{p} = \hat{n} = (n-2)/2$, we have $l = n-2$ by Fact 1. As above, we know that the length for any maximum fault-tolerant container from a to b is at least $n+2$. By Theorem 1, we see that the smallest possible length for any maximum fault-tolerant container from a to b is equal to $n+2$, i.e., it equals $l+4$. Thus the upper bound given by Jwo and Tuan in [10] is in a sense best possible.

4. The Container Length and Wide-diameter of $Q_2(n)$

By the definition, it is enough to consider for odd n. Let a and b be two vertices in $Q_2(n)$. We know $Q_2(n)$ is constructed from two $Q_1(n-1)$'s in [3], say, $Q_1^1(n-1)$ and $Q_1^2(n-1)$. And we assume $a \in Q_1^1(n)$. Note that if there exists a path $a = v_0 \to v_1 \to \cdots \to v_k$ in $Q_1^1(n-1)$, then there is a corresponding path $a' = v_0' \to v_1' \to \cdots \to v_k'$ in $Q_1^2(n-1)$. Suppose that $P_1; P_2; \cdots; P_r$ are a collection of maximum number of vertex-disjoint paths from a to $a_{n-1}b''$ in $Q_1^1(n-1)$, where $r = \kappa(a'';b'')$, and $P_1'; P_2'; \cdots; P_r'$ are their counterparts in $Q_1^2(n-1)$. Obviously, $P_1; P_2; \cdots; P_r$ is identical to a maximum fault-torelant $(a'';b'')$-container in $Q_1(n-1)$. By Theorem 1, we assume each of paths $P_1; P_2; \cdots; P_r$ has length at most $n+1$.

Case 1. $\omega(a)$ *is odd or* $\omega(b)$ *is even.*

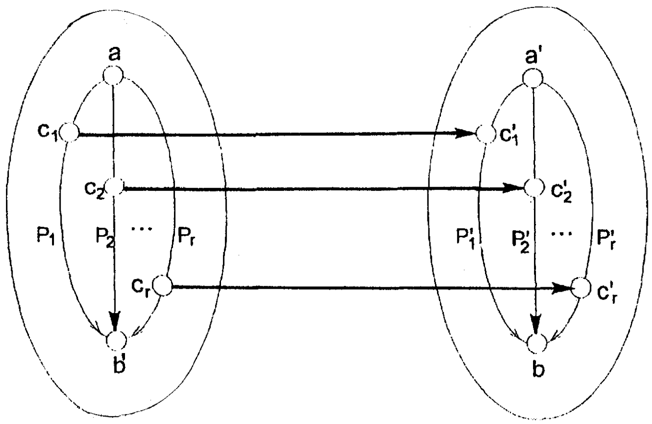

FIG. 4. $\eta(a)$ is odd or $\eta(b)$ is even, and $a_{n-1} \neq b_{n-1}$: the r vertex-disjoint paths from a to b in $Q_2(n)$.

Subcase 1.1. $a_{n-1} = b_{n-1}$:

In this situation, we know $\xi(a,b) = \xi(a'', b'')$. Therefore, $P_1, P_2, \cdots, P_r$ is a maximum fault-torelant (a,b)-container in $Q_2(n)$, and $D_{\xi(a,b)}(a,b) \leq n+1$.

Subcase 1.2. $a_{n-1} \neq b_{n-1}$

Similarly, $\xi(a,b) = \xi(a'', b'') = r$. a and b are not in the same subcube. Then a and b' are in $Q_1^1(n-1)$ and b and a' are in $Q_1^2(n-1)$. See Figure 4. Observe that among $P_1, P_2, \cdots, P_r$, (1) at most one path has length less than 3, and (2) each of the remaining paths has length more than 2 and thus contains at least two internal vertices. For $1 \leq i \leq r$, let u and v be two consecutive vertices in P_i and let u' and v' be their counterparts in P_i', respectively. Note that it is easy to check that there always exists u with an outgoing edge to u' or v to v'. Then we can select a vertex c_i in P_i with an outgoing edge to c_i' in P_i', $i = 1, 2, \cdots, r$. Evidently, the $2r$ vertices $c_1, c_2, \cdots, c_r$, $c_1', c_2', \cdots, c_r'$, are all distinct. For each i in $[1, r]$, a path from a to b in $Q_2(n)$ can be formed by first going through the subpath of P_i from a to c_i, then through the edge from c_i to c_i', and, finally, through the subpath of P_i' from c_i' to b. These newly formed r paths are vertex-disjoint and each of them has length at most $n+2$ since each of the paths $P_1, P_2, \cdots, P_r$ has length at most $n+1$ by Theorem 1. Then $D_{\xi(a,b)}(a,b) \leq n+2$.

Case 2. *$\eta(a)$ is even and $\eta(b)$ is odd.*

Subcase 2.1. $a_{n-1} = b_{n-1}$.

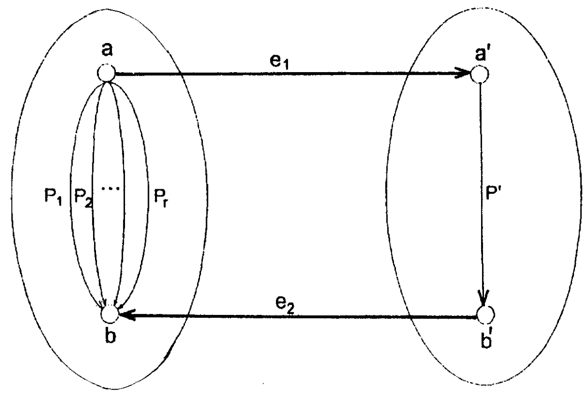

FIG. 5. $\eta(a)$ is even, $\eta(b)$ is odd, and $a_{n-1} = b_{n-1}$: the $r+1$ vertex-disjoint paths from a to b in $Q_2(n)$.

We know $\xi(a;b) = \xi(a'';b'') + 1$, and a and b are in the same subcube. See Figure 5, where P' is a shortest path from a' to b' in $Q_1^2(n)$. Since the $(n-1)$th port of a has positive polarity and that of b has negative polarity, there exist e_1 from a to a' and e_2 from b' to b. We easily get a new path $P = e_1 + P' + e_2$. Due to Fact 3 and the fact that $n-1$ is even, we know that the length of P' is at most n. Then P has length at most $n+2$. Now, it is easy to see $D_{\xi(a;b)}(a;b) \le n+2$.

Subcase 2.2. $a_{n-1} \ne b_{n-1}$.

We have $\xi(a;b) = \xi(a'';b'') + 1$ by Fact 4. See Figure 6, where P_t is a shortest path in $\{P_i | i \in [1;r]\}$. Since the $(n-1)$th port of a has positive polarity and that of b has negative polarity, e_1 is from a to a' and e_2 is from b' to b. For each pair P_i and P_i', $i \ne t$, there exists a vertex c_i in P_i and c_i' in P_i' such that a new path from a to b in $Q_2(n)$ is formed by taking the subpath from a to c_i in P_i, then through the edge from c_i to c_i', and finally from c_i' to b in P_i'. For the pair P_t and P_t', two new paths are formed: One is $e_1 + P_t'$ and the other is $P_t + e_2$. Since each of the paths $P_1; P_2; \cdots; P_r$ has length at most $n+1$ by Theorem 1, we easily see that each of the paths in the new container has length at most $n+2$. Thus $D_{\xi(a;b)}(a;b) \le n+2$.

From the above discussion, we have the following theorem:

Theorem 3. *Let* a *and* b *be two vertices of* $Q_2(n)$. *Then* $D_{\xi(a;b)}(a;b) \le n+2$:

From Lemma 1, we have:

Theorem 4. *The wide-diameter of* $Q_2(n)$ (n is odd) *is equal to* $n+2$.

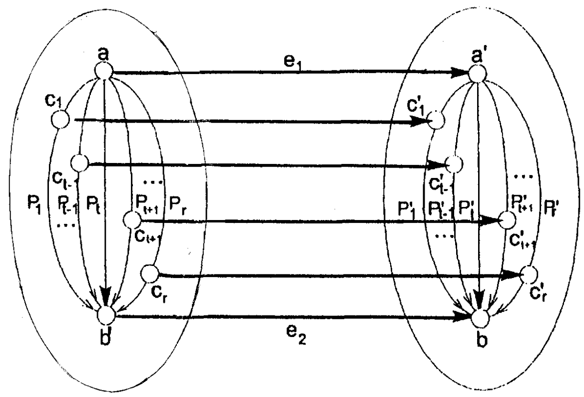

FIG. 6. $\eta(a)$ is even, $\eta(b)$ is odd, and $a_{n-1} \neq b_{n-1}$

Remark 2. For the two vertices $a = 00\cdots0$ and $b = 1001\cdots1$ in $Q_2(n)$ ($n \geq 3$ is odd), since $DP(a; b') = 00011\cdots1$ and $p = h = (n-3)/2$, we have $l = n - 3$ by Fact 1, where l is the shortest path length from a to b'. As Subcase 2.2 of Theorem 2, we know that for any maximum fault-tolerant container from a to b, there is a path through the edge $(a; c)$, where $c = 0010\cdots0$. We easily know that the shortest path from c to b in Q_2 has length $n + 1$. So we see that the smallest possible length for any maximum fault-tolerant container from a to b is equal to $n + 2$, i.e., it equals $l + 5$. Thus the upper bound given by Jwo and Tuan [10] is in a sense best possible.

5. Conclusion

In this paper, we give the wide-diameters of the two unidirectional binary n-cubes proposed by Chou and Du [3]. Since the constructed container in this paper is the same as that in [10], Remarks 1 and 2 show that the conjecture in [10] is true.

References

1. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan Press Ltd, London, 1976.
2. F. Boragonove and E. Cadorin, HR^4-net: A hierachical random-routing reliable and reconfigurable network for metropolitan area, in *Pro. IEEE INFOCOM*, March, 1987, pp. 320-326.
3. C. H. Chou and David H. C. Du, Unidirectional hypercubes, in: *Proc. Supercomputing '90*, 1990, pp. 254-263.

4. K. Day and A. Tripathi, Unidirectional star graphs, *Inform. Process. Lett.* **45** (1993), 123-129.

5. E. Flandrin and H. Li, Mengerian properties, Hamiltonicity and claw-free graphs, *Networks* **24** (1994), 660-678.

6. D. F. Hsu, On container width and length in graphs, groups, and networks, *IEICE Trans. Fundam.* **E(77A)** (1994), 668-680.

7. D. F. Hsu and Y. D. Lyuu, A graph-theoratical study of transmission delay and fault-tolerance, in: *Proc. of 4th ISMM International Conference on Parallel and Distributed Computing and Systems* 1991, pp. 20-24.

8. D. F. Hsu and T. Luszak, Note on the k-diameter of k-regular k-connected graphs, *Discrete Math.* **132** (1994), 291-296.

9. Y. Ishigami, The wide-diameter of the n-dimensionml toroidal mesh, *Networks* **27** (1996), 257-266.

10. J. S. Jwo and T. C. Tuan, On container length and connectivity in unidirectional hypercubes, *Networks* **32** (1998), 307-317.

11. Q. Li, D. Sotteau and J. M. Xu, 2-diameter of de Bruijn graphs, *Networks* **28** (1996), 7-14.

12. S. Lakshmivarahan and S. K. Dhall, *Analysis and Design of Parallel Algorithms*, McGraw-Hill, New York, 1990.

13. F. T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes*, Morgan Kaufmann, Son Mates, CA, 1992.

14. S. C. Liaw and G. J. Chang, Wide diameters of Butterfly networks, *Taiwanese J. Math.* **3** (1999), 83-88.

15. N. Maxemchuk, The Manhattan street network, in: *Proc. GLOBECOM'85, Del.*, 1985, pp. 255-261.

16. N. Maxemchuk, Routing in the Manhattan street network, *IEEE Trans. Commum.* **35** (1987), 503-512.

Lu Changhong and Zhang Kemin
Department of Mathematics, Nanjing University,
Nanjing, 210093, China.