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A note on Ramsey numbers with two parameters☆

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Abstract

The Ramsey number $R(G_1, G_2)$ is the smallest integer p such that for any graph G on p vertices either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ denotes the complement of G. In this paper, some new bounds with two parameters for the Ramsey number $R(G_1, G_2)$, under some assumptions, are obtained. Especially, we prove that $R(K_6 - e, K_6) \leq 116$ and $R(K_6 - e, K_7) \leq 202$, these improve the two upper bounds for the classical Ramsey number in [S.P. Radziszowski, Small Ramsey number, Electron. J. Combin. DS1 (2002) 1–36].

MSC: 05C55; 05D10

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The Ramsey number $R(G_1, G_2)$ is the smallest integer p such that for any graph G on p vertices either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ denotes the complement of G. A graph G with order p is called a $(G_1, G_2; p)$-graph, if G does not contain G_1 and $\overline{G}$ does not contain G_2. It is easy to see that $R(G_1, G_2) = p_0 + 1$ iff $p_0 = \max\{p \mid$ there exists a $(G_1, G_2 : p)$-graph$\}$. The $(G_1, G_2; p)$-graph is called a $(G_1, G_2; p)$-Ramsey graph if $p = R(G_1, G_2) - 1$. Let G be a graph with order p, d_i be the degree of vertex i in G, and $\overline{d_i} = p - 1 - d_i$, where $1 \leq i \leq p$. Assume that G^{p-1} is the subgraph of

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G with one vertex being deleted from G and G^{p-2} is the subgraph of G with one vertex being deleted from G^{p-1}.

Theorem 1. *Let G_1 (resp. G_2) be a graph with order m (resp. n) and $3 \le m \le n$. Assume that $R(G_1^{m-2}, G_2) \le \alpha+1$, $R(G_1, G_2^{n-2}) \le \beta+1$, $R(G_1^{m-1}, G_2) \le \gamma+1$, and $0 < x < 3$, $y \ge 2(3-x)\gamma$. Then the following inequalities must hold:*

(a) *If $\frac{9}{4} < x < 3$, then*

$$R(G_1, G_2) \le 2 + \frac{1}{4x-9}\left\{C + \sqrt{C^2 + (4x-9)D}\right\}. \tag{a}$$

(b) *If $0 < x < \frac{9}{4}$, then*

$$R(G_1, G_2) \ge 2 + \frac{1}{9-4x}\left\{-C + \sqrt{C^2 + (4x-9)D}\right\}$$

or

$$R(G_1, G_2) \le 2 + \frac{1}{9-4x}\left\{-C - \sqrt{C^2 + (4x-9)D}\right\}. \tag{b}$$

(c) *If $x = \frac{9}{4}$, then*

$$R(G_1, G_2) \le F(y) = 2 + \frac{1}{6y - 6\alpha - 3\beta - 9} \times \left\{(y+\beta-\alpha)^2 + 9\gamma y - \frac{27}{4}\gamma^2\right\} \tag{c}$$

where $C = 2x + 2x\beta - 3\beta + 3\alpha - 3y$ and $D = (y+\beta-\alpha)^2 + 4xy\gamma - 4x(3-x)\gamma^2$.

Proof. For any $(G_1, G_2; p)$-Ramsey graph, denote the number of subgraphs K_3 in G (resp. in $\overline{G}$) by $|K_3|$ (resp. $|\overline{K}_3|$). One has

$$3|K_3| = \sum_{ij \text{ is an edge of } G} \left|N(i) \bigcap N(j)\right|.$$

Since any subgraph of G does not contains the subgraph G_1, $\left|N(i) \bigcap N(j)\right| \le \alpha$ for ij being an edge of G, thus $3|K_3| \le \frac{1}{2}\alpha \sum_{i=1}^{p} d_i$. Similarly, $3|\overline{K}_3| \le \frac{1}{2}\beta \sum_{i=1}^{p} \overline{d_i}$. Now by [1], we have:

$$3\binom{p}{3} - \frac{3}{2}\sum_{i=1}^{p} d_i\overline{d_i} = 3|K_3| + 3|\overline{K}_3| \le \frac{1}{2}\left\{\beta(p-1)p - (\beta-\alpha)\sum_{i=1}^{p} d_i\right\}.$$

It is clear that $d_i \le \gamma$, and

$$\begin{aligned} p(p-1)(p-2-\beta) &\le \sum_{i=1}^{p}\{-3d_i^2 + (3p-3-\beta+\alpha)d_i\} \\ &= \sum_{i=1}^{p}\{-xd_i^2 + (3p-3-\beta+\alpha-y)d_i - (3-x)d_i^2 + yd_i\} \\ &\le p\left\{\frac{1}{4x}(3p-3-\beta+\alpha-y)^2 - (3-x)\gamma^2 + y\gamma\right\}. \end{aligned}$$

Furthermore, we obtain:

$$(4x-9)(p-1)^2 - 2C(p-1) - D \leq 0.$$

Now, (a)–(c) follow from the above inequality.

Remark 1. If $y_0 = \frac{1}{2}\{2\alpha+\beta+3\sqrt{\gamma(4\alpha+2\beta+6-3\gamma)+(\beta+1)^2}\}$, then $\frac{\mathrm{d}F}{\mathrm{d}y}(y_0) = 0$ and $\frac{\mathrm{d}^2F}{\mathrm{d}y^2}F(y_0) > 0$. This implies that $R(G_1, G_2) \leq F(y_0)$, and this is the formula of [3].

Corollary 1. *$R(K_6-e, K_6) \leq 116$ and $R(K_6-e, K_7) \leq 202$.*

Proof. Let $G_1 = K_6 - e$, $G_2 = K_6$ in the Theorem 1, by [2] we may assume that $(\alpha, \beta, \gamma) = (20, 35, 54)$. Let $x = \frac{10}{4}$, $y = 60$ (resp. $x = 1.5$, $y = 162$), by the formula (a) (resp. (b)), we obtain $R(K_6-e, K_6) \leq 116$ (from the formula (b) we obtain $R(K_6-e, K_6) \leq 116$ or $R(K_6-e, K_6) \geq 170$, however from [2] we know $R(K_6-e, K_6) \leq 119$, thus $R(K_6-e, K_6) \leq 116$).

Let $G_1 = K_6 - e$, $G_2 = K_7$, and $(\alpha, \beta, \gamma) = (33, 66, 87)$. Assume $x = 1.5$, $y = 300$, by the formula (b), $R(K_6-e, K_7) \leq 202$ or $R(K_6-e, K_7) \geq 334$, however we know $R(K_6-e, K_7) \leq 204$, thus $R(K_6-e, K_7) \leq 202$.

Remark 2. (a) Let $G_1 = K_6 - e$, $G_2 = K_6$, from [2], we may assume $(\alpha, \beta, \gamma) = (20, 35, 54)$. Let $x = \frac{1}{4}$, $y = 299.095$, now from the formula (b), $R(K_6-e, K_6) \geq 118$ or $R(K_6-e, K_6) \leq 117$, certainly this can not tell us anything, however, from the Corollary above, the formula (b) of the Theorem 1 is useful for some cases.

(b) The right hand of the formula (a) is a function of degression on the variable γ, thus if we know the smaller value of γ, we can obtain better the upper bound of $R(G_1, G_2)$. For example, if we assume that $\gamma = 50$ in (2) of the Corollary above, we obtain $R(K_6-e, K_6) \leq 115$.

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