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A NEW SUFFICIENT CONDITION FOR D−CIRCUITS*

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Abstract A Benhocine et al[1] proved that G contains a D-circuit if G is a connected, almost bridgeless graph of order $n\geqslant 3$ and $\deg(u)+\deg(v)\geqslant\frac{2n+1}{3}$ for every pair of nonadjacent vertices u and v. We generalize this result, and obtain that, if G is a connected, almost bridgeless graph of order $n>3$ and $\deg(x)+\deg(y)+\deg(z)>n$ for every triple independent set $\{x,y,z\}$, then G contains a D-circuit.

Key words: bridgeless, circuit, D-circuit.

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1 Introduction

We use [2] for basic terminology and notation. If H is a subgraph of G, the neighborhood of u on H is $N_H(u)=\{v\in V(H)\mid uv\in E(G)\}$, and $N_H(G')=\bigcup_{u\in V(G')}N_H(u)$. Sometimes, we simplify $N(u)=:N_H(u)$, $N(G')=:N_H(G')$. $d_H(u,v)$ denotes the distance of u and v in H. In [4] a circuit was defined as a nontrivial closed trail. C is a circuit if and only if C is a nontrivial connected subgraph such that every vertex of C has even degree in C. A dominating circuit, short say D-circuit, of G is a circuit such that every edge of G is incident with at least one vertex of the circuit. We say G is almost bridgeless, if every bridge of G is incident with a vertex of degree one.

In [1] A. Benhocine, L. Clark, N. Köhler, H. J. Veleman proved the following theroem.

Theorem 1[1] Let G be a connected, almost bridgeless graph of order $n\geqslant 3$. If $\deg(u)+\deg(v)\geqslant(2n+1)/3$ for every pair of nonadjacent vertices u and v, then G contains a D-circuit.

In this note, we get that:

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Main Theorem Let G be a connected, almost bridgeless graph of order $n>3$. If $\deg(u)+\deg(v)+\deg(w)>n$ for every triple independent set $\{u,v,w\}$ of G, then G contains a D-circuit.

It is obviously that the condition of Theorem 1 satifies the condition of Main Theorem and G (see Fig. 1) is satisfied the later but is not satisfied the former.

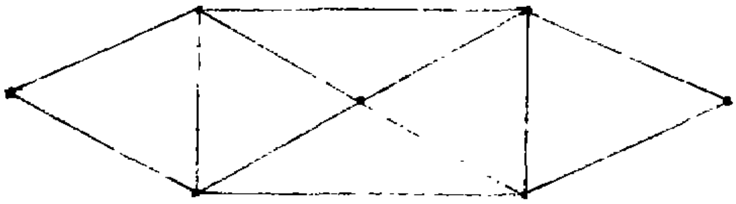

Fig. 1

2 Preliminary Lemmas

In order to prove the main theorem, we must establish some lemmas.

Lemma 1[1] Let G be a connected graph and C a circuit of G with maximum number vertices. Then G contains no circuit C' satisfyimg

$$V(C')\cap V(C)\neq\varnothing\neq V(C')\cap(V(G)-V(C)) \quad \text{and } |E(C')\cap E(C)|\leqslant 1.$$

Lemma 2 Let G be a connected graph and C a circuit with maximum number of vertices, K a nontrivial compoinent of $G-V(C)$. If there are $u_1,u_2\in N_c(K)$ such that for any $u\neq v\in N_c(K)$, $2=d_{G-V(K)}(u_1,u_2)\leqslant d_{G-V(K)}(u,v)$, then

2.1 For any $w\in N_c(u_1)\cap N_c(u_2)$, we have $\{u_1w,wu_2\}\subset E(C)$;

2.2 There is no cycle $C_1\subset C$ such that C_1 contains exactly one of the edges u_1w *and* u_2w. And $P_0=u_1wu_2$ is contained in *a* shortest cycle $C_2\subset C$;

2.3 In 2.2, let $C_2=wx_0x_1\cdots x_sx_{s+1}w\,(x_0=u_1,x_{s+1}=u_2)$. then

2.3.1

$$\{x_1w,x_sw\}\cap E(G)=\varnothing$$
$$N(x_1)\cap N(w)\subset\{u_1,x_2\}$$
$$N(x_s)\cap N(w)\subset\{u_2,x_{s-1}\};$$

2.3.2 For any $v\in V(K)$, we have

$$\{vw,x_1v\}\cap E(G)=\varnothing$$
$$N(x_1)\cap N(v)\subset\{u_1,x_2\}$$
$$N(x_s)\cap N(v)\subset\{u_2,x_{s-1}\}.$$

Proof Let P be an u_1-u_2 path with $\Phi\neq V(P)-\{u_1,u_2\}\subset V(K)$ such that $|V(P)|$ is minimum.

2.1 If $\{u_1w,wu_2\}\not\subset E(C)$, let $C'=\{u_1w,wu_2\}\cup P$ then both C' and C contradict Lemma 1.

2. 2 If there is a cycle $C_1 \subset C$, which contains exactly one of edges u_1w and u_2w, and let $C' = C - \{u_1w, u_2w\} + P$, then C' is a circuit with $|V(C')| > |V(C)|$, a contradiction. since C is a circuit, there is a shortest cycle $C_2 \supset P_0$.

2. 3. 1 If $x_1w \in E(G)$, then $x_1w \notin E(C)$ by 2. 2 thus there is $C' = C - \{x_1u_1, wu_2\} + \{x_1w\} + P$ with $|V(C')| > |V(C)|$, a contradiction. So $x_1w \notin E(G)$. Similarly, $x_tw \notin E(G)$.

Suppose that $w' \notin \{u_1, x_2\}$ and $w' \in N(x_1) \cap N(w)$, we have

Case 1 If $w' \in V(C_2) - \{u_1, x_2\}$, then $w'x_1, w'w \notin E(C)$, otherwise contradicting 2. 2. Thus we have $C' = C - \{x_1u_1, wu_2\} + \{x_1w', w'w\} + P$ with $|V(C')| > |V(C)|$, a contradiction.

Case 2 If $w' \in V(G) - V(C_2)$, by 2. 2, $\{w'x_1, w'w\} \not\subset E(C)$. Thus there exists a circuit C':

$$C' = \begin{cases} C - \{x_1u_1, wu_2\} + \{x_1w', w'w\} + P & \text{if } \{w'x_1,\ w'w\} \cap E(C) = \Phi \\ C - \{x_1u_1, x_1w', wu_2\} + \{ww'\} + P & \text{if } x_1w' \in E(C),\ ww' \notin E(C) \\ C - \{x_1u_1, w'w, wu_2\} + \{x_1w'\} + P & \text{if } x_1w' \notin E(C),\ ww' \in E(C) \end{cases}$$

with $|V(C')| > |V(C)|$, a contradiction. Similarly, we have $N(x_t) \cap N(w) \subset \{u_2, x_{t-1}\}$.

2. 3. 2 For any $v \in V(K)$, we have $\{vw, x_1v\} \cap E(G) = \Phi$ by Lemma 1.

If $v' \in N(x_1) \cap N(v) \cap (V(G) - \{u_1, x_2\} \neq \varnothing$, then $v' \notin V(C_2) \cup V(K)$ by Lemma 1, and $x_1v' \in E(C)$ by 2. 1. Let $C' = C - \{u_1x_1, x_1v'\} + P'$, with $|V(C')| > |V(C)|$, a contradiction, where P' is a $u_1 - v'$ path with $\varnothing \neq (V(P') - \{u_1v'\}) \subset V(K)$. Similarly, we have $N(x_t) \cap N(v) \subset \{u_2, x_{t-1}\}$.

Lemma 3 Let G be a connected graph and C a circuit of with maximum number of vertices, K a nontrivial component of $G - V(C)$. And there are $u_1, u_2 \in N_C(K)$ such that for any $u \neq v \in N_C(K)$: $3 = d_{G-V(K)}(u_1, u_2) \leqslant d_{G-V(K)}(u, v)$. The shortest $u_1 - u_2$ path in $G - V(K)$ is denoted by $u_1w_1w_2u_2$, then $N(u_1) \cap N(w_2) = \{w_1\}$ or $N(u_2) \cap N(w_1) = \{w_2\}$.

Proof Let P be a $u_1 - u_2$ path with $\varnothing \neq V(P) - \{u_1, u_2\} \subset V(K)$ such that $|V(P)|$ is minimum. If

$$v_1 \in N(u_1) \cap N(w_2) \cap (V(G) - \{w_1\}) \neq \varnothing$$

and

$$v_2 \in N(u_2) \cap N(w_1) \cap (V(G) - \{w_2\}) \neq \varnothing,$$

then $v_1 \neq v_2$, $v_i \notin V(K)$, $i = 1, 2$, by the definition of u_i and $w_i \in V(C)$, $i = 1, 2$. and Lemma $|\{u_1w_1, w_1w_2, w_2u_2\} \cap E(C)| \geqslant 2$. In the following, using symbol $ab \pm xy$ to denote

$$ab \pm xy = \begin{cases} ab + xy & \text{if } xy \notin E(C) \\ ab - xy & \text{if } xy \in E(C), \end{cases}$$

we divide the proof into three cases.

Case 1 $\{u_1w_1, w_1w_2, w_2u_2\} \subset E(C)$.

We have

$$C' = \begin{cases} C + P \pm u_1v_1 \pm v_1w_2 - w_2u_2 & \text{if } \{u_1v_1, v_1w_2\} \not\subset E(C) \\ C + P - u_1w_1 \pm w_1v_2 \pm v_2u_2 & \text{if } \{w_1v_2, v_2u_2\} \not\subset E(C) \\ C + P - u_1w_1 - w_1w_2 - w_2u_2 & \text{if } \{u_1v_1, v_1w_2\} \cup \{w_1v_2, v_2u_2\} \subset E(C) \end{cases}.$$

with $|V(C')| > |V(C)|$, a contradiction.

Case 2 $|\{u_1w_1, u_2w_2\} \cap E(C)| = 1$.

Without loss of generality, we assume $u_1w_1 \notin E(C)$, then $\{w_1w_2, w_2u_2, w_1v_2, v_2u_2\} \subset E(C)$ by Lemma 1. Thus we have

$$C' = \begin{cases} C + P + u_1w_1 - w_1w_2 - w_2u_2 & \text{if } \{u_1v_1, v_1w_2\} \subset E(C) \\ C + P \pm u_1v_1 \pm v_1w_2 - w_2u_2 & \text{if } \{u_1v_1, v_1w_2\} \not\subset E(C) \end{cases}$$

with $|V(C')| > |V(C)|$, a contradiction.

Case 3 $w_1w_2 \notin E(C)$.

We have

$$C' = \begin{cases} C + P - u_1w_1 + w_1w_2 - w_2u_2 & \text{if } \{u_1v_1, v_1w_2\} \text{or } \{w_1v_2, v_2u_2\} \subset E(C) \\ C + P \pm u_1v_1 \pm v_1w_2 + w_1w_2 \pm w_1v_2 \pm v_2u_2 & \text{otherwise} \end{cases}$$

with $|V(C')| > |V(C)|$, a contradiction.

3 The Proof of the Main Theorem

Suppose that G contains no D-circuit, we will find a triple independent set $\{u, v, w\}$ of G such that $\deg(u) + \deg(v) + \deg(w) \leqslant n$. Obviously, $G \neq K_{1,n-1}$, G contains a circuit. Let C be a circuit of G such that $|V(C)|$ is maximum. Since C is not a D-circuit, $G - V(C)$ has a non-trivial component K. Since G is almost bridgeless, K has at least two neighbors on C. Let $N_c(K) = \{u_1, u_2, \cdots, u_t\}$. By Lemma 1 we have $d_{G-V(K)}(u_i, u_j) \geqslant 2$ $(i \neq j)$. We take $u_1, u_2 \in N_c(K)$ with $d_{G-V(K)}(u_1, u_2) \leqslant d_{G-V(K)}(u_i, u_j)$ $(i \neq j)$. Now the following three cases must be considered:

Case 1 $d_{G-V(K)}(u_1, u_2) = 2$.

Let P and $\{w, x_0, x_1, \cdots, x_{s+1}\}$ as in Lemma 2, $v_1 \in N_P(u_1)$, $v'_1 \in N_K(v_1)$. By Lemma 1 and 2, we have $v'_1w \notin E(G)$, $v'_1x_1 \notin E(G)$ and $x_1w \notin E(G)$. Thus there is a triple independent set $\{v'_1, x_1, w\}$. First, we have

$$N(v'_1) \cap N(w) \subset \{u_2\} \tag{1}$$

In fact, w has no neighbors in $V(K)$ and v'_1 no neighbors in $V(G) - (V(K) \cup V(C))$. If $w' \in V(C)$ with $u_2 \neq w' \in N(v'_1) \cap N(w)$, we have

$$C' = \begin{cases} C - u_1w + \{u_1v_1, v_1v'_1, v'_1w', ww'\} & \text{if } ww' \notin E(C) \\ C - \{u_1w, ww'\} + \{u_1v_1, v_1v'_1, v'_1w'\} & \text{if } ww' \in E(C) \end{cases}$$

with $|V(C')| > |V(C)|$, a contradiction. (1) follows. And then, by Lemma 2 and $v'_1u_1 \notin E(G)$, we have

$$N(x_1) \cap N(w) \subset \{x_2, u_1\} \tag{2}$$

$$N(v'_1)\cap N(x_1)\subset\{x_2\} \tag{3}$$

From (1), (2) and (3), we obtain

$$\deg(v'_1)+\deg(x_1)+\deg(w)\leqslant n-3+4=n+1.$$

If $\deg(v'_1)+\deg(x_1)+\deg(w)=n+1$. then all equalities of (1), (2) and (3) hold and $P=u_1v_1v'_1u_2$.

Case 1.1 $x_2 \ne u_2$. $x_2w\notin E(C)$ by Lemma 2. Thus there is a $C'=u_1v_1v'_1x_2wu_1$, which contradicts Lemma 1.

Case 1.2 $x_2=\cdots$ In this case,

$$\deg(v'_1)+\deg(x_1)+\deg(w)\leqslant n-5+5=n.$$

Thus, in case 1. we always have $\deg(v'_1)+\deg(x_1)+\deg(w)\leqslant n$.

Case 2. $d_{G-V(K)}(u_1,u_2)=3$.

Let P be a shortest u_1-u_2 path with $\varnothing\neq V(P)-\{u_1,u_2\}\subset V(K)$, $v_1\in N_P(u_1)$.

Case 2.1 There exist $v'_1\in N_K(v_1)$ and $v'_1\notin V(P)$ or $v'_1\in N_K(v_1)$, $v'_1\in V(P)$ and $|V(P)|\geqslant 5$. In this case, $\{v'_1,u_1,u_2\}$ is a triple independent set, and $N(v'_1)\cap N(u_1)=\{v_1\}$, $N(v'_1)\cap N(u_2)\subset\{v_2\}=N_K(u_2)$ and $N(u_1)\cap N(u_2)\subset V(K)$ with $|N(u_1)\cap N(u_2)|\leqslant 1$. Then

$$\deg(v'_1)+\deg(u_1)+\deg(u_2)\leqslant n-3+3=n.$$

By the symmetry of u_1 and u_2, the remains is that:

Case 2.2 $K=K_2=v_1v'_1$ and $P=u_1v_1v'_1u_2$. Let $u_1w_1w_2u_2$ be a shortest u_1-u_2 path in $G-V(K)$, by Lemma 3, we have

$$N(u_1)\cap N(w_2)=\{w_1\} \tag{4}$$

or

$$N(u_2)\cap N(w_1)=\{w_2\} \tag{5}$$

without loss of generality, we assume that (4) holds. Thus there is a triple independent set $\{v'_1,u_1,w_2\}$. *Furthermore, we have*

$$N(v'_1)\cap N(u_1)=\{v_1\} \tag{6}$$

$$N(w_2)\cap N(v'_1)=\{u_2\} \tag{7}$$

In fact, $N(v'_1)\subset V(K)\cup V(C)$, $N(u_1)\cap(V(K)-\{v_1\})=\varnothing$. If $x\in N(v'_1)\cap N(u_1)$, $x\neq v_1$, then $x\in N_C(K)$, This contradicts $d_{G-V(K)}(u_1,u_2)=3$. Similarly, we can get (7). From (4), (6) and (7), we have

$$\deg(v'_1)+\deg(u_1)+\deg(w_2)\leqslant n-3+3=n.$$

Case 3 $d_{G-V(K)}(u_1,u_2)\geqslant 4$.

Suppose the shortest u_1-u_2 path in $G-V(K)$ is $P_2=u_1w_1w_2\cdots u_2$. P is defined as case 1.

Case 3.1 $P=u_1v_1u_2$. In this case, $\{v'_1,u_1,u_2\}$ *is a triple independent set, where* $v'_1\in K$ with $v_1v'_1\in E(G)$. And $N(u_1)\cap N(v'_1)=N(u_2)\cap N(v'_1)=\{v_1\}$, $N(u_1)\cap N(u_2)=\{v_1\}$. Thus we have:

$$\deg(v'_1)+\deg(u_1)+\deg(u_2)\leqslant n-3+3=n.$$

Case 3.2　$P=u_1v_1v_2\cdots u_2$. In this case, $\{v_1, w_1, u_2\}$ is a triple independent set, and by $d_{G-V(K)}(u_1, u_2)\geqslant 4$, we have $N(w_1)\cap N(u_2)=\varnothing$, $N(v_1)\cap N(w_1)=\{u_1\}$, $N(v_1)\cap N(u_2)\subset\{v_2\}$. Thus we get:

$$\deg(v_1)+\deg(w_1)+\deg(u_2)\leqslant n-3+2=n-1.$$

Up to now, we exhaust all cases. They alwayr contradict the hypothesis of Main Theorem. Therefore the proof is completed.

From a result of Harary and Nash-William[3]: A line graph $L(G)$ of a graph G is Hamiltonian if and only if G contains a D-circuit or $G=K_{1,s}$ $(s\geqslant 3)$. So we have

Corollary　A connected, almost bridgeless graph G of order $n\geqslant 3$. If $\deg(u)+\deg(v)+\deg(w)>n$ for every triple independent set $\{u, v, w\}$ of G, then $L(G)$ is Hamiltonien.

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关于 D-闭迹的一个新充分条件

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摘　要　A. Benhocine 等人证明了当 G 为几乎无桥的阶≥3 的连通图且对任意不相邻的两点 u, v 有 $\deg(u)+\deg(v)\geqslant(2n+1)/3$ 时，有 D-闭迹存在．我们推广了这一结果，并得到：若 G 为连通的几乎无桥的阶 $n>3$ 的图且对任意三点独立集 $\{x, y, z\}$ 有 $\deg(x)+\deg(y)+\deg(z)>n$，则 G 含 D—闭迹．

关键词　无桥、闭迹、D—闭迹．
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