A SUFFICIENT CONDITION FOR A GRAPH TO BE HAMILTONIAN*

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Hamilton 图的一个充分条件

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摘 要

设 G 是 $n(\geqslant 3)$ 阶的 2-连通无向图.若对任意的 $u,v\in V(G)$ 且 $d(u,v)=2$ 均满足条件 $\mathrm{Max}\{d(u),d(v)\}\geqslant \frac{n}{2}$ 或 $|N(u)\bigcup N(v)|\geqslant \frac{2}{3}(n-1)$, 则 G 是 hamiltonian.在类似的条件下,也讨论了图的 hamilton-connected 性质,得到了相应的结果.

Abstract

In this paper we prove that a 2-connected graph G of order $n\geqslant 3$ is hamiltonian if for all distinct vertices u and v, $d(u,v)=2$ implies that $\max\{d(u),d(v)\}\geqslant n/2$ or $|N(u)\bigcup N(v)|\geqslant(2n-2)/3$. We also demonstrate hamilton-connected property in graphs under similar conditions.

Introduction

This paper uses terms and notations of [2]. Throughout this paper G denotes an undirected 2-connected simple graph of order $n(\geqslant 3)$ with connectivity k and

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independence number α. Let L be a subset of $V(G)$, F a subgraph of G and V a vertex in G. Define $N_L(v)=\{u\in L|uv\in E(G)\}$, $N_L(F)=\bigcup_{v\in V(F)} N_L(v)$. Specifically, if $L=V(G)$, we simple write them as $N(v)$ and $N(F)$. If no ambiguity can arise we sometimes write F instead of $V(F)$.

The following results are the inspiration for the work in this paper.

Theorem A[3] Let G be a 2–connected graph on n vertices. If for all distinct vertices u,v, $d(u,v)=2$ implies that $\max\{d(u),d(v)\}\geqslant n/2$, then G is hamiltonian.

Theorem B[4] Let G be a 2–connected graph of order n. If for all distinct vertices u,v, $d(u,v)=2$ implies that $|N(u)\bigcup N(v)|\geqslant(2n-1)/3$, then G is hamiltonian.

In this paper, we shall prove a stronger result. Theorem A and B are corollaries of our result.

Main results

Theorem 1 Let G be a 2–connected graph of order $n\geqslant 3$. If for all distinct vertices u,v, $d(u,v)=2$ implies that $\max\{d(u),d(v)\}\geqslant n/2$ or $|N(u)\bigcup N(v)|\geqslant(2n-2)/3$. then G is hamiltonian.

Proof It is trivial for $n\leqslant 4$, so we assume that $n\geqslant 5$. Let $A=\{u\in V(G)|d(u)\geqslant n/2\}$, $E'=\{xy|x,y\in A,xy\notin E(G)\}$ and $H=G+E'$. Then $H[A]$ is complete and there exists a cycle containing A in H. By Bondy and Chvatal's Closure Theorem[1], G is hamiltonian if and only if H is hamiltonian. Thus, we only need to prove that H is hamiltonian. Let $C=v_1v_2\cdots v_tv_1$ (simply, $12\cdots t1$) be a longest cycle containing A in H. If H is not hamiltonian, let B be any component of $H\setminus V(C)$. Let $N_C(B)=\{i_1,i_2,\cdots,i_m\}$, $N^-=\{i_1-1,i_2-1,\cdots,i_m-1\}$ and $N^+=\{i_1+1,i_2+1,\cdots,i_m+1\}$ where$i_1<i_2<\cdots i_m$, and where and later on $i\pm j$ is taken modulo t. Since $\chi(H)\geqslant\chi(G)=k$, we have

Assertion 1. $m\geqslant k\geqslant 2$

Let x_j be some vertex in B which is adjacent to i_j, It is possible that $x_i=x_j$ for $x\neq j$.

Assertion 2. For any j with $1\leqslant j\leqslant m$, $d_H(x_j,i_j\pm 1)=2$, and if $d_H(i_j\pm 1)<n/2$, then $d_G(x_j,i_j\pm 1)=2$.

Assertion 3. For any j with $1\leqslant j\leqslant m$, we have

(1) $d(i_j-1)\leqslant n-|N(i_{j+1}-1)\bigcup N(x_{j+1})|-\varepsilon$,

(2) $d(i_j+1)\leqslant n-|N(i_{j-1}+1)\bigcup N(x_{j-1})|-\varepsilon$,

where $\varepsilon=0$, if $(i_j-1)(i_j+1)\in E(H)$, and $\varepsilon=1$, if $(i_j-1)(i_j+1)\notin E(H)$.

Let $u\in N(i_{j+1}-1)\bigcup N(x_{j+1})$. A bijection f is defined by:

$$f(u)=\begin{cases} u & \text{if } u\notin V(C). \\ i-1 & \text{if } u=i\in V(C) \text{ and } i_{j+1}\leqslant i\leqslant i_j-1, \\ i+1 & \text{if } u=i\in V(C) \text{ and } i_j+1\leqslant i< i_{j+1}-2, \\ i_j-1 & \text{if } u=i_j\in V(C), \\ x_{j+1} & \text{if } u=i_{j+1}-2\in V(C) \text{ and } i_{j+1}-2\neq i_j. \end{cases}$$

Since C is a longest cycle in H, it is easy to check that $N(i_j-1)\bigcap f(N(i_{j+1}-1)\bigcup N(x_{j+1}))=\varnothing$ (For example, see [4]). So $d(i_j-1)\leqslant n-|N(i_{j+1}-1)\bigcup N(x_{j+1})|$. And note that when $(i_j-1)(i_j+1)\notin E(H)$, $x_{j+1}\notin f(N(i_{j+1}-1)\bigcup N(x_{j+1}))$ if $i_{j+1}-2=i_j$ and $i_j+1\notin f(N(i_{j+1}-1)\bigcup N(x_{j+1}))$ if $i_{j+1}-2>i_j$. Hence (1) is true. Similarly, (2) is true too.

Assertion 4. For any $u,v\in N^+$ or $u,v\in N^-$, $d(u)+d(v)<n$.

Let $u=i, v=j (i<j)$ and $i,j\in N^-$. At most one of $\{ik,(k-1)j\}$ belongs to $E(H)$, if $i<k<j$. So does $\{ik,(k+1)j\}$, if $j<k<i$. So $d_C(i)+d_C(j)\leqslant |C|$. On the other hand. $N(i)\bigcap N(j)\subseteq V(C)$. Hence $d(i)+d(j)=d(u)+d(v)<n$. Similarly, we have that $d(u)+d(v)<n$ for any $u,v\in N^+$.

Assertion 5. For any $u\in N^-\bigcup N^+$, $d(u)<n/2$.

If not, there is $u\in N^-\bigcup N^+$ such that $d(u)\geqslant n/2$, without loss of generality, we assume that $d(i_1-1)\geqslant n/2$. By assertion 4 for any $2\leqslant j\leqslant m$, $d(i_j-1)<n/2$. Hence, by assertion 2, $d_G(i_j-1,x_j)=2$. And then by the hypothesis of Theorem and assertion 1, $|N(i_2-1)\bigcup N(x_2)|\geqslant(2n-2)/3$. Hence by assertion 3, $d(i_1-1)\leqslant(n+2)/3<n/2$, a contradiction.

By assertions 1,2,5 and the hypothesis of Theorem, we have:

Assertion 6. For any j with $1\leqslant j\leqslant m$, $|N(i_j\pm 1)\bigcup N(x_j)|\geqslant(2n-2)/3$.

If there exists an integer j, $1\leqslant j\leqslant m$, with $(i_j-1)(i_j+1)\notin E(H)$, then $d_G((i_j-1)(i_j+1))=2$ and, by assertions 3, 6, $d(i_j-1)\leqslant(n-1)/3$, $d(i_j+1)\leqslant(n-1)/3$. this implies that $|N(i_j-1)\bigcup N(i_j+1)|\leqslant 2(n-1)/3-1=(2n-5)/3$. This contradicts the hypothesis of the Theorem. Therefore,

Assertion 7. $(i_j-1)(i_j+1)\in E(H)$ $(1\leqslant j\leqslant m)$.

There exists a vertex h, $i_2+1\leqslant h\leqslant i_3-1$ if $m\geqslant 3$ or $i_2+1\leqslant h\leqslant i_1-1$ if $m=2$ such that $h(i_2-1)\notin E(H)$, $i(i_2-1)\in E(H)$ for all $i_2\leqslant i\leqslant h-1$. This is true since $(i_2-1)(i_3-1)\notin E(H)$. or a cycle longer than C exists. Assertion 7 implies that $h\geqslant i_2+2$. Let $u\in N(i_1+1)\bigcup N(x_1)$. It is easy to prove that $u\notin\{i_2+1,i_2+2,\cdots,h\}$. A bijection g is defined by:

$$g(u)=\begin{cases} u & \text{if } u\notin V(C), \\ i-1 & \text{if } u=i\in V(C) \text{ and } i_1+3\leqslant i\leqslant i_2, \\ i+1 & \text{if } u=i\in V(C) \text{ and } h+1\leqslant i< i_1, \\ h & \text{if } u=i_1+2, \end{cases}$$

since C is a longest cycle in H, $g(N(i_1+1)\bigcup N(x_1))\bigcap N(h)=\phi$. Note that $x_1\notin g(N(i_1+1)\bigcup N(x_1))\bigcup N(h)$. Thus by assertions 6, 7 and the hypothesis of Theorem, $d(h)\leqslant n-2-|N(i_1+1)\bigcup N(x_1)|\leqslant(n-4)/3$, if $i_2h\notin E(H)$; $d(h)\leqslant n-1-|N(i_1+1)\bigcup N(x_1)|\leqslant(n-1)/3$ and $|N(i_2-1)\bigcap N(h)|\geqslant 2$, if $i_2h\in E(H)$. On the other hand, by assertion 3, $d(i_2-1)\leqslant(n+2)/3$. Hence we have: (a) $\max\{d(i_2-1),d(h)\}<n/2$; (b) $|N(i_2-1)\bigcup N(h)|\leqslant(n-4)/3+(n+2)/3-1=(2n-5)/3$ if $i_2h\notin E(H)$; or $|N(i_2-1)\bigcup N(h)|\leqslant(n-1)/3+(n+2)/3-2=(2n-5)/3$ if $i_2h\in E(H)$. This is contrary to the hypothesis of Theorem.

Consider the graph G_0, which consists of three copies of K_r graphs G_1,G_2,G_3 and the set edges $\{x_1x_2,x_2x_3,x_3x_1,y_1y_2,y_2y_3,y_3y_1\}$, where $x_i,y_i\in V(G_i)$ for any i, $1\leqslant i\leqslant 3$. If $r\geqslant 3$, then it is easy to check that for any $u,v\in V(G_0)$ with $d(u,v)=2$, $\max\{d(u),d(v)\}<n/2$, $|N(u)\bigcup N(v)|\geqslant(2n-3)/3$ and G_0 is nonhamiltonian. So the conditions of Theorem 1 is the best possible in a sense.

We now consider hamilton-connected property. The graph G being 3-connected must be necessary, since there does not exist any u-v hamiltonian path for any vertex cut $\{u,v\}$ in G.

Theorem 2 Let G be a 3-connected graph of order $n(\geqslant 3)$, and let u and v be distinct vertices of G. If $d(u,v)=2$ implies that $\max\{d(u),d(v)\}\geqslant(n+1)/2$ or $|N(u)\bigcup N(v)|\geqslant 2n/3$, then G is hamilton-connected.

Proof It is trivial for $n\leqslant 4$. So we assume that $n\geqslant 5$. Let $A=\{u\in V(G)|d(u)\geqslant(n+1)/2\}$, $E'=\{xy|x,y\in A,xy\notin E(G)\}$ and $H=G+E'$. Then $H[A]$ is complete. It is easy to prove that there exists a $u-v$ path containing A for any $u,v\in V(G)$ in H. By Bondy and Chvatal's Closure Theorem[1], G is hamilton-connected if and only if H is hamilton-connected. Thus, we only need to prove that H is hamilton-connected. Suppose that H is not hamilton-connected. Then there exists a pair of vertices u and v such that no hamiltonian $u-v$ path exists in H. Consider a longest $u-v$ path P containing A, denoted $P=v_1v_2\cdots v_t$ (simply, 1 2 ⋯ t) in H, where $v_1=u,v_t=v$. let B be any component of $H\setminus V(P)$. Let $N_p(B)=\{i_1,i_2,\cdots,i_m\}$, $N^-=\{i_1-1,i_2-1,\cdots,i_m-1\}$ and $N^+=\{i_1+1,i_2+1,\cdots,i_m+1\}$, where $i_1<i_2<\cdots<i_m$. We can use an analogous arguments of Theorem 1 and have several assertions as follows:

Assertion 1. $m\geqslant k\geqslant 3$.

Let x_j be some vertex in B which is adjacent to i_j. It is possible that $x_i = x_j$ for $i \neq j$.

Assertion 2. For any j with $1 \leqslant j \leqslant m$, $d_H(x_j, i_j \pm 1) = 2$, and if $d_H(i_j \pm 1) \leqslant n/2$, then $d_G(x_j, i_j \pm 1) = 2$.

Assertion 3. For any j with $1 \leqslant j \leqslant m$, we have

(1) $d(i_j - 1) \leqslant n + 1 - |N(i_{j+1} - 1) \bigcup N(x_{j+1})| - \varepsilon$, where $i_{j+1} = i_2$ when $j = m$ and $i_1 = 1$; and $i_{j+1} = i_1$ when $j = m$ and $i_1 > 1$;

(2) $d(i_j + 1) \leqslant n + 1 - |N(i_{j-1} + 1) \bigcup N(x_{j-1})| - \varepsilon$, where $i_{j-1} = i_{m-1}$ when $j = 1$ and $i_m = t$ and $i_{j-1} = i_m$ when $j = 1$ and $i_m < t$, where $\varepsilon = 1$ for $2 \leqslant j \leqslant m-1$ and $(i_j - 1)(i_j + 1) \notin E(H)$, and $\varepsilon = 0$ for other cases,

Assertion 4. For any $u, v \in N^+$ or $u, v \in N^-$, $d(u) + d(v) \leqslant n$.

Assertion 5. For any $u \in N^- \bigcup N^+$, $d(u) \leqslant n/2$.

Assertion 6. For any j with $1 \leqslant j \leqslant m$, $|N(i_j \pm 1) \bigcup N(x_j)| \geqslant 2n/3$, if $i_j \pm 1 \in P$.

Assertion 7. For any j with $2 \leqslant j \leqslant m - 1$, $(i_j - 1)(i_j + 1) \in E(H)$.

Thus using assertions 1–7, we can obtain a contradiction by a similar argument of Theorem 1.

Consider 3–connected graph $G = 3K_r \vee K_3$, which is no hamilton–connected. It is easy to check that for all distinct vertices u and v of G with $d(u,v) = 2$ implies that $\max\{d(u), d(v)\} < (n+1)/2$ and $|N(u) \bigcup N(v)| = 2n/3 - 1$. So, the conditions of Theorem 2 is best possible in a sence.

Corollary 2.1[3] Let G be a 3–connected graph of order $n(\geqslant 3)$ and let u and v be distinct vertices of G. If $d(u,v) = 2$ implies that $\max\{d(u), d(v)\} \geqslant (n+1)/2$, then G is hamilton–connected.

Corollary 2.2[4] Let G be a 3–connected graph of order $n(\geqslant 3)$, and let u and v be distinct vertices of G. If $d(u,v) = 2$ implies that $|N(u) \bigcup N(v)| \geqslant 2n/3$, then G is hamilton–connected.

References

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