SOME LOCALIZATION CONDITIONS FOR LONG CYCLE IN GRAPHS

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Abstract In this note, we give some localization conditions for the existence of long cycle through any fixed vertex of G. Let G be a $\n$-connected, triangle-free graph of order $n$, and let $m$ be an integer $0 \leq m \leq n - 2$. If $G$ satisfies one of the following:

1. $\forall u \in V(G), d(u) = \frac{m}{2} = \frac{2m}{n} \in \{d(v) : v \in M'(u)\}$, where $M'(u) = \{v : d(v) = m, v \in V(G)\}$;

2. $\forall x \in V(G), d(x) = \frac{m}{2}, d(x) = \frac{2m}{n} \in \{d(v) : v \in M'(u)\}$, then, for every vertex $x \in V(G)$, there exists a cycle $C(x, 2)$ of length $2$ through $x$. Finally, we give some conjectures about localization conditions of long cycle.

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1 Introduction

Herzog and Khachatrian \cite{1,2} obtained some results concerning localization conditions for a graph to be hamiltonian. Tian Peng and Wang Wenliang \cite{3} gave two localization conditions concerning circumference of a graph. In this note, we discuss the long cycle through a fixed vertex under some localization conditions. Non-localization conditions concerning this can be found in \cite{4} and \cite{5}.

We use terminology and notation in \cite{1}. Let $G = (V(G), E(G))$, $Y \in V(G), N_{G}(y) = \{v \in V(G) : v \sim y, v \neq y\}$. If $m \in \mathbb{Z}$, then $y \in V(G)$, $d(v) = m, v \in N_{G}(y)$, then $y \in V(G)$, $d(v) = m, v \in N_{G}(y)$.

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Main Results

Lemma Let G be a 2-connected triangle-free graph of order n. If x is the fixed vertex of V(G), P = x, P, . . . , x, is the longest path through x with d(x) ≥ 2 (3 ≤ k ≤ n) in G, then there exists a cycle C(G) through x.

Proof Let P = x, x, . . . , x, be the longest path through x with d(x) ≥ 2. Let max (i; x, ∈ E(G)) = q = min (i; x, ∈ E(G)). Thus p > t and q > 1, otherwise we are done.

Case 1. p > q

Case 1. 1. x ∈ {x, x, . . . , x}. Since G is a triangle-free, N(x) is independent, for any x, ∈ N(x) \ {x, x, . . . , x} \ {x, x, . . . , x, x, . . . , x}, l = |{x, x, . . . , x, x, . . . , x, x, . . . , x}| ≥ 2.

Case 1. 2. x ∈ {x, x, . . . , x}. In this case, C = x, x, . . . , x is a desired cycle as the proof of case 1. 1.

Case 2. p = q

Case 2. 1. x ∈ {x, x, . . . , x}. The cycle C = x, x, . . . , x is desired.

Case 2. 2. x ∈ {x, x, . . . , x}. Since G is 2-connected, there is a path P = x, x, . . . , x, with l ≤ 1 + p − q < l(G) = l(x, x, . . . , x). If P = \O, let j = − max (j; x, ∈ E(G)). If x ∈ {x, x, . . . , x}, then C = x, x, . . . , x, . . . , x, . . . , x, . . . , x, . . . , x, . . . , x, . . . , x, . . . , x, . . . , x, . . . , x is a desired cycle respectively.

Case 3. p < q

Case 3. 1. x ∈ {x, x, . . . , x}, then C = x, x, . . . , x, x, x, .

Case 3. 2. x ∈ {x, x, . . . , x}. Since G is 2-connected, there are two disjoint paths P = u, u, . . . , u, P = v, v, . . . , v connecting the cycle C = x, x, . . . , x, to C = x, x, . . . , x, x, . . . , x. We may assume that one of them starts at x, say P., for otherwise we can walk on the path x, . . . , x till we hit either C or P, (i = 1 or 2), and replace an appropriate piece of P by this path. Similarly, we may assume that one of P or P ends at x. Let \{V(P), V(P)\} \ [x, x, . . . , x] = [x, x, . . . , x], j = max (i; x, ∈ E(G)) \ [x, x, . . . , x] \ [x, x, . . . , x], x = q (1 ≤ q)p and r = max (i; x, ∈ E(G)) \ [x, x, . . . , x] \ [x, x, . . . , x]. And let x < x < x.
where $P(P_r)$, resp.) is a section of $P$ from $v_i$ to $x_{n_i}$, $(x_i$ to $v_i$, resp.). If $x_i, x_{n_i}$ belong to different $P_t$'s, let $x_i, x_{n_i}$ be $P_t$'s, say $x_i \in P_t$, $x_{n_i} \in P_t$, thus we have a desired cycle $C_i = x_{n_i}x_{i-1}x_{i-2}\ldots x_{n}x_{i+1}x_{i+2}\ldots x_{n-1}$.  

Case 3.2. 2, $v_i = x_{n_i} \in P_t$. We can use an analogous method of Case 3.2.1 to get a desired cycle $C_i$.

Case 3.3. \( x \in \{x_1, x_2, \ldots, x_n\} \). Using the notation of case 3.2, We have that:  

Case 3.3.1. $u_i = x_i \in P_t$. If there is a desired cycle $C_i = x_{n_i}x_{i-1}x_{i-2}\ldots x_{n}x_{i+1}x_{i+2}\ldots x_{n-1}x_i$, thus there is a desired cycle $C_i = x_{n_i}x_{i-1}x_{i-2}\ldots x_{n}x_{i+1}x_{i+2}\ldots x_{n-1}x_i$.

Case 3.3.2. $v_i = x_{n_i} \in P_t$. We can use similar arguments as case 3.3.1 to get a desired cycle $C_i$.

In all cases, $G$ is a cycle of length $l(V(G))$ through $x$. The proof of Lemma is completed.

**Theorem 1**  
Let $G$ be a 2-connected triangle-free graph of order $n$, and let $1 \leq k \leq n$ be an integer. If for any vertex $x \in V(G)$, $d(x) = k \leq l(x)$ then $G$ contains a cycle $C_i \subseteq V(G)$ through vertex $x$.

**Proof**  
Suppose $x \in V(G)$, $G$ has no cycle $C_i \subseteq V(G)$ through $x$. Let $A = \{P \mid P$ is a path with $x \in V(P)\}$. Choose a $P_i \in A$ satisfying: (1) $|P_i| = \max\{|P| : P \in A\}$; (2) Under (1), $P_i = x_{1}, x_{2}, \ldots, x_{n}$, such that $d(x_i) = d(x_{i-1}) = k$ as large as possible. Then, by Lemma, $N(x_{i}) \subseteq C(V(P_i)) \subseteq N(x_{i})$.

Let $N(x_{i}) = \{x_{n}, x_{n-1}, \ldots, x_{i}\}$ then $x_{n-i} \in \mathbb{M}(x_{i})$ and $d(x_{n-i}) \leq d(x_{i}) = k$ by (2), thus: $\{v \in \mathbb{M}(x_{i}) : d(v) \geq k\} \supseteq \{x_{n-i+1}, \ldots, x_{n-1}\} = k$, a contradiction.

**Theorem 2**  
Let $G$ be a 2-connected triangle-free graph of order $n$, and let $1 \leq k \leq n$ be an integer. If for any $v \in V(G)$, $d(v) = 2$, then $G$ contains a cycle $C_i \subseteq V(G)$ through $x$.

**Proof**  
Suppose $G$ satisfies the condition of the theorem, and for some $x \in V(G)$, there is no cycle of length $l(G)$ through $x$. Obviously, $G$ is not Hamiltonian. Let $A = \{P \mid P$ is a path with $x \in V(P)\}$. Choose $P_i = x_{1}, x_{2}, \ldots, x_{n} \in A$ such that $1 \leq |P_i| = \max\{|P| : P \in A\}$; (1) $P_i = x_{1}, x_{2}, \ldots, x_{n}$, such that $d(x_i) = d(x_{i-1}) = k$ as large as possible. Then, by Lemma, $d(x_{i}) \geq 2$, $N(x_{i}) \subseteq C(V(P_i)) \subseteq N(x_{i})$.

Then, for $P_i \in A$, $\{v \in V(G) : d(v) = k\}$, there is no cycle of length $l(G)$ through $x$. Otherwise, $G$ is not Hamiltonian. Let $A = \{P \mid P$ is a path with $x \in V(P)\}$. Choose $P_i = x_{1}, x_{2}, \ldots, x_{n} \in A$ such that $1 \leq |P_i| = \max\{|P| : P \in A\}$; (1) $P_i = x_{1}, x_{2}, \ldots, x_{n}$, such that $d(x_i) = d(x_{i-1}) = k$ as large as possible. Then, by Lemma, $d(x_{i}) \geq 2$, $N(x_{i}) \subseteq C(V(P_i)) \subseteq N(x_{i})$.

Indeed, if $d(v_{i+1}) = d(v_{i+2}) = \ldots = d(v_{n})$, then $P_{i} = v_{i+1}, v_{i+2}, \ldots, v_{n}$ is also one of the longest paths through $x$. By Lemma B, $G$ contains a cycle $C_i \subseteq V(G)$ through $x$. By contradiction, there is no cycle $C_i \subseteq V(G)$ through $x$.

Hence, $G$ satisfies the condition of the theorem, and for some $x \in V(G)$, there is no cycle of length $l(G)$ through $x$. Otherwise, $G$ is not Hamiltonian. Let $A = \{P \mid P$ is a path with $x \in V(P)\}$. Choose $P_i = x_{1}, x_{2}, \ldots, x_{n} \in A$ such that $1 \leq |P_i| = \max\{|P| : P \in A\}$; (1) $P_i = x_{1}, x_{2}, \ldots, x_{n}$, such that $d(x_i) = d(x_{i-1}) = k$ as large as possible. Then, by Lemma, $d(x_{i}) \geq 2$, $N(x_{i}) \subseteq C(V(P_i)) \subseteq N(x_{i})$.
Let \( d = \max(i | i \in [m, n], v_i \in E(G) \}) \); then for every \( i \), \( 1 \leq i \leq m \), \( v_i \in E(G) \), \( v_i \in E(G) \).

(iii) There is \( \exists k \in [m, n] \) such that \( v_k \) \( v_{k+1} \in E(G) \) and \( v_k \) \( v_{k+1} \in E(G) \).

In fact, if \( v_k \) \( v_{k+1} \in E(G) \) then \( 2 \leq k < n \). And by \( v_k \) \( v_{k+1} \in E(G) \), there is a path \( P = v_1 \rightarrow \ldots \rightarrow v_k \rightarrow \ldots \rightarrow v_n \in E \). Without loss of generality, let \( \ell(P) = m + 1 \geq 3 \). A contradiction.

Now, since \( d(v_{k+1}) = 2 \), \( d(v_k) = 2 \) and \( d(v_i) \leq 3 \), we have \( d(v_{k+1}) \geq |M(v_{k+1})|/2 \), \( d(v_k) \geq |M(v_k)|/2 \), \( N(v_{k+1}) \cup N(v_k) \subseteq M(v_{k+1}) \) and \( v_k \in E(G) \) by (iii), then \( d(v_k) \leq 2 \).

We also have \( d(v_{k+1}) \geq |M(v_{k+1})| - d(v_{k+1}) \) by (ii). Since \( d(v_{k+1}) \geq |M(v_{k+1})|/2 \), \( N(v_{k+1}) \subseteq M(v_{k+1}) \), we have \( d(v_{k+1}) \geq |M(v_{k+1})|/2 \).

Thus, \( d(v_k) \geq |M(v_k)|/2 \), \( N(v_k) \subseteq M(v_k) \), \( d(v_k) \geq |M(v_k)|/2 \) and \( d(v_{k+1}) \geq |M(v_{k+1})|/2 \). Thus we have \( d(v_i) \geq |M(v_i)|/2 \).

On the other hand, \( d(v_{k+1}) \geq |M(v_{k+1})|/2 \).

This contradicts (iii). Therefore the proof of Theorem 1 is completed.

**Remark**

Let us consider the following graphs:

\[ G = (V_1, E_1) \]

where \( V_1 = U_1 \cup \{ u \} \) with \( U_1 = \{ x \} \), \( U_2 = \{ u \} \) and \( E_1 = \{(x, x), (x, u) \} \) with \( \ell(G) = |x| \).

Next, we consider the following graphs:

\[ G = (V_2, E_2) \]

where \( V_2 = U_1 \cup \{ u \} \) with \( U_1 = \{ x \} \), \( U_2 = \{ u \} \) and \( E_2 = \{(x, x), (x, u) \} \) with \( \ell(G) = |x| \).

Clearly, \( G \) is connected. Using the results of [23, 24], we can conclude that there exists a cycle \( C \) in \( G \), \( C \) has length \( 2 \).

Finally, we have the following conjectures.

**Conjecture 1**

Let \( G \) be a \( 2 \)-connected graph of order \( n \) and let \( n \geq 5 \) be an integer. If for every vertex \( u \in V(G) \), \( d(u) = 2 \), then \( G \) is connected.

**Conjecture 2**

Let \( G \) be a \( 2 \)-connected graph of order \( n \) and let \( n \geq 5 \) be an integer. If for every \( u \in V(G) \), \( d(u) = 2 \), then \( G \) is connected.

**Conjecture 3**

Let \( G \) be a \( 2 \)-connected graph of order \( n \) and let \( n \geq 5 \) be an integer.
ger. If for every \( u \in V(G) \), \( d(u) \geq \min\left(\frac{k}{2}, \left\lfloor \frac{1}{2} M^*(u) \right\rfloor / 2 \right) \), then for any fixed vertex \( x \in G \) there is a cycle of length \( l \) through \( x \).

Conjectures 1, 2 and 3 are supported by the results of Theorems 1 and 2.

References


图中长圈的几个局部化条件

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摘要: 本文给出了图G中任意点的长圈存在性的几个局部化条件。设\( G \)是\( n \)阶无三角形的\( n \)阶图，\( d(v) \geq \frac{n}{2} \)为正整数，且\( G \)还满足下列条件之一：

(1) \( \forall u \in V(G), \ d(u) = k \geq \frac{5}{2} \Rightarrow \exists v \in N^*(u) \) \( d(v) \leq k - 1 \), 其中 \( M^*(u) = \min\{d(u), \ d(v) \} \) \( \forall u, v \in V(G) \);

(2) \( \forall u, v \in V(G), \ d(u) < \frac{k}{2} \Rightarrow d(u, v) = 2 \Rightarrow d(v) \leq M^*(u) / 2 \).

则\( x \in V(G) \), 存在长为\( (\left\lfloor \frac{1}{2} M^*(x) \right\rfloor / 2) \)的圈\( C \)过\( x \)。最后我们给出几个关于长圈的局部化条件的猜想。

关键词: 长圈, 局部化条件, 无三角形图。

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