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SOME LOCALIZATION CONDITIONS FOR LONG CYCLE IN GRAPHS*

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Abstract In this note, we give some localization conditions for the existance of long cycle through any fixed vertex of G. Let G be a 2-connected, triangle-free graph of order n, and let s be an integer ($3\leqslant s\leqslant n$). If G satisfies one of the following:

(1) $\forall\, u\in V(G)$, $d(u)=k<\frac{s}{2}\Rightarrow|\{v\in M^2(u)\}|d(v)\leqslant k\}|\leqslant k-1$, where $M^i(u)=\{v|d(u,v)\leqslant i, v\in V(G)\}$;

(2) $\forall\, u,v\in V(G), d(u)<\frac{s}{2}, d(u,v)=2\Rightarrow d(v)\geqslant|M^3(u)|/2$,

then, for every vertex $x\in V(G)$, there exists a cycle $C_l(l\geqslant s)$ of length l through x. Finally, we give some conjectures about localization conditions of long cycle.

Key words Cycle, Localization conditions, Triangle - free graph.
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1 Introduction

Hasratian and Khachatrian[3,4] obtained some results concerning localization conditions for a graph to be hamiltonian. Tian Feng and Wang Wenliang[6] gave two localization conditions concerning circumference of a graph. In this note, we discuss the long cycle through a fixed vertex under some localization conditions. Non-localization conditions concerning this can be found in [2] and [5].

We use terminology and notation in [1]. Let $G=(V(G),E(G)),\forall\, u\in V(G), N_i(u)=\{v$

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$|d(u,v)=i$, $d(u,v)$ is the distance of u and v. $N(u)=N_1(u)$. $d(u)=|N(u)|$, $M^i(u)=\{v|d(u,v)\leqslant i\}$, $v\in V(G)\}$. A vertex set $V_1\subset V(G)$ is called independent, if $G[V_1]$ contains no edge, where $G[V_1]$ is an induced subgraph of G by V_1.

2　Main Results

Lemma　Let G be a 2 $-$connected triangle$-$free graph of order n. If x is the fixed vertex of $V(G)$, $P=x_1x_2\cdots x_t$ is the longest path through x with $d(x_t)\geqslant s/2$ $(3\leqslant s\leqslant n)$ in G, then there exists a cycle $C_l(l\geqslant s)$ through x.

Proof　Let $P=x_1x_2\cdots x_t$ be the longest path through x with $d(x_t)\geqslant s/2$, $p=\max\{i|x_1x_i\in E(G)\}$, $q=\min\{i|x_ix_t\in E(G)\}$. Thus $p<t$ and $q>1$, otherwise we are done.

Case 1. $p>q$

Case 1.1. $x\in\{x_1,x_2,\cdots,x_{q-1}\}$. Let $q_0=\max\{i|i<p,x_ix_t\in E(G)\}$, then $C_l=x_1x_2\cdots x_{q_0}x_tx_{t-1}\cdots x_px_1$ is a desired cycle. Indeed since G is triangle$-$free, $N(x_t)$ is independent, for any $x_i\in N(x_t)$, $\{x_{i-1},x_{i+1}\}\cap N(x_t)=\varnothing$ and $N(x_t)\subset\{x_q,x_{q+1},\cdots,x_{q_0},x_p,x_{p+1},\cdots,x_{t-1}\}$. thus $l=|\{x_1,x_2,\cdots,x_{q_0},x_tx_{t-1},\cdots,x_p\}|\geqslant s$.

Case 1.2. $x\in\{x_q,x_{q+1}\cdots x_t\}$. In this case, $C_l=x_qx_{q+1},\cdots,x_tx_g$ is a desired cycle as the proof of case 1.1.

Case 2. $p=q$

Case 2.1. $x\in\{x_q,x_{q+1},\cdots,x_t\}$. The cycle $C_l=x_qx_{q+1}\cdots x_tx_q$ is desired.

Case 2.2. $x\in\{x_1,x_2,\cdots,x_{p-1}\}$. Since G is 2$-$connected, there is a path $P_1=x_iu_1\cdots u_rx_j$ with $1\leqslant i<p,q<j\leqslant t,\{u_1,u_2,\cdots,u_r\}\cap V(P)=\varnothing$, Let $j_0=\max\ \{j'|j'<j,\ x_{j'}x_t\in E(G)\}$. If $x\in\{x_1,x_2,\cdots,x_i\}$. then $C_l=x_1x_2\cdots x_iP_1x_jx_{j+1}\cdots x_tx_{j_0}\cdots x_qx_1$. And if $x\in\{x_{i+1},\cdots,x_{p-1}\}$, then $C_l=x_ix_jx_{j+1}\cdots x_tx_{j_0}\cdots x_i$ is desired cycle respectively.

Case 3. $p<q$

Case 3.1. $x\in\{x_q,x_{q+1},\cdots,x_t\}$, then $C_l=x_qx_{q+1}\cdots x_tx_q$.

Case 3.2. $x\in\{x_{p+1},x_{p+2},\cdots,x_{q-1}\}$, Since G is 2$-$connected, there are two disjoint paths $P_1=u_1u_2\cdots u_g,P_2=v_1v_2\cdots v_f$ with $u_1\neq v_1,u_g\neq v_f$ connecting the cycle $C_1=x_1x_2\cdots x_px_1$ to $C_2=x_qx_{q+1}\cdots x_tx_q$, We may assume that one of them starts at x_p, say P_1, for otherwise we can walk on the path $x_px_{p+1}\cdots x_q$ till we hit either C_2 or $P_i(i=1$ or $2)$, and replace an appropriate piece of P_i by this path. Similarly, we may assume that one of P_1 or P_2 ends at x_q. Let

$$[V(P_1\cup V(P_2)]\cap\{x_p,x_{p+1},\cdots,x_q\}=\{x_{i_1},x_{i_2},\cdots,x_{i_m}\},$$

$$j=\max\ \{k|x_k\in[V(P_1)\cup V(P_2)]\cap\{x_q,x_{q+1},\cdots,x_t\},v_1=x_i(1\leqslant i<p)$$

and

$$r=\max\ \{k|x_kx_t\in E,k<j\},\text{And let } x_{i_h}<x\leqslant x_{i_{h+1}}.$$

Case 3.2.1. $u_g=x_g\in P_1$. If $x_{i_h},x_{i_{h+1}}\in P_i$, say $i=2$, otherwise we can use a similar argument to get a desired cycle $C_l=x_pP_1x_q\cdots x_rx_t\cdots x_j(=v_f)P'_2x_{i_h+1}\cdots x_{i_h}P''_2v_1(=x_i)\ x_{i+1}\cdots x_p$.

where P'_2 (P''_2, resp.) is a section of P_2 from v_f to $x_{i_{h+1}}$ (x_{i_h} to $v_1=x$, resp.). If x_{i_h}, x_{i_h+1} belong to different $P_i (i=1,2)$, say $x_{i_h}\in P_1$, $x_{i_h+1}\in P_2$, thus we have a desired cycle $C_l = x_{i_h}\cdots x_{i_h+1}P'_2x_j\cdots x_tx_r\cdots x_q(=u_g)u_{g-1}\cdots x_{i_h}$.

Case 3. 2. 2. $v_f=x_q\in P_2$, We can use an analogous method of Case 3. 2. 1 to get a desired cycle C_l.

Case 3. 3. $x\in\{x_1,x_2,\cdots,x_p\}$. Using the notation of case 3. 2, We have that:

Case 3. 3. 1. $u_g=x_q\in P_1$. If $x\in\{x_1,x_2,\cdots,x_i\}$, thus there is a desired $C_l=x_1x_2\cdots x_iP_2x_j\cdots x_tx_r\cdots x_qP_1x_px_1$; If $x\in\{x_i,x_{i+1},\cdots,x_p\}$, thus there is a desired $C_l=x_i\cdots x_pP_1x_q\cdots x_rx_t\cdots x_jP_2x_i$.

Case 3. 3. 2. $v_f=x_q\in P_2$. We can use similar argument as case 3. 3. 1 to get a desired cycle C_l. #

In all cases, C_l is a cycle of length $l(l\geqslant s)$ through x. The proof of Lemma is completed.

Theorem 1 Let G be a 2-connected triangle-free graph of order n, and let s $(3\leqslant s\leqslant n)$ be an integer. If for any vertex $u\in V(G)$, $d(u)=k<s/2$ implies $|\{v\in M^2(u)|d(v)\leqslant k\}|\leqslant k-1$ then for any fixed vertex x, there is a cycle $C_e(l\geqslant s)$ through vertex x.

Proof Suppose $x\in V(G)$, G has no cycle $C_l(l\geqslant s)$ through x. Let $A=\{P|P$ is a path with $x\in V(P)\}$. Choose a $P_0\in A$ satisfying: (1) $|P_0|=\max\{|P|\,|P\in A\}$; (2) Under (1), $P_0=x_1x_2\cdots x_t$ such that $d(x_1)+d(x_t)$ as larger as possible. Then, $d(x_1)<s/2$ by Lemma, $N(x_1)\subset V(P_0)$ by (1). Let $N(x_1)=\{x_{t_1},x_{t_2},\cdots,x_{t_k}\}$, then $x_{t_{i-1}}\in M^2(x_1)$ and $d(x_{t_i-1})\leqslant d(x_1)=k$ by (2), thus $|\{v\in M^2(x_1)|d(v)\leqslant k\}|\geqslant|\{x_{t_1-1},x_{t_2-1},\cdots,x_{t_k-1}\}|=k$, a contradiction.

Theorem 2 Let G be a 2-connected triangle-free graph of order n, and let s $(3\leqslant s\leqslant n)$ be an integer. If for any u, $v\in V(G)$, $d(u,v)=2$, $d(u)<s/2$ implying $d(v)\geqslant|M^3(u)|/2$. Then, for any fixed vertex x, G contains a cycle $C_l(l\geqslant s)$ through x.

Proof Suppose G satisfies the condition of the theorem, and for some $x\in V(G)$, there is no cycle of length l $(l\geqslant s)$ through x. Obviously, G is not hamiltonian. Let $A=\{P|P$ is a path with $x\in V(P)\}$. Choose $P_0=v_1\cdots v_m\cdots v_t\in A$ such that: (1) $|V(P_0)|=\max\{|V(P)|\,|P\in A\}$; (2) Let $f(P_0)=m=\max\{i|v_1v_i\in E(G)\}$. Under (1), $f(P_0)$ is as large as possible. By Lemma, $d(v_1)<s/2$. $N(v_1)\subset V(P)$ by (1). Furthermore, we have

(i) If $v_1v_i\in E(G)$, then $d(v_{i-1})<s/2$ and $N(v_{i-1})\subset\{v_1,v_2,\cdots,v_m\}$.

Indeed, if $d(v_{i-1})\geqslant s/2$, then $v_{i-1}v_{i-2}\cdots v_1v_i\cdots v_t$ is also one of the longest paths through x. By Lemma G contains a cycle $C_l(l\geqslant s)$ through x, a contradiction. If there is $v\notin V(P)$, $v_{i-1}v\in E$, then $P'=vv_{i-1}\cdots v_1v_i\cdots v_t$ is longer than P_0. This contradicts the choice of P_0. If $v_{i-1}v_j\in E(G)$, $m<j\leqslant t$, thus there is $P''=v_{i-1}v_{i-2}\cdots v_1v_i\cdots v_t$ with $f(P'')>f(P_0)$. This contradicts the choice of P_0 yet.

(ii) There exists some i, $2\leqslant i<m$, such that $v_1v_i\notin E(G)\}$. Otherwise, by (i), we have $\bigcup_{i=1}^{m-1}N(V_i)\subset\{v_1,v_2,\cdots,v_m\}$. This contradicts the 2-connectivity of G.

Let $k=\max\{i \mid i\leqslant m, v_1v_i \notin E(G)\}$, then for every i: $k<i\leqslant m$, $v_1v_i\in E(G)$, $v_1v_k\notin E(G)$.

(iii) There is no $i(2\leqslant i<m)$ such that $v_iv_{m+1}\in E(G)$ and $v_kv_{i-1}\in E(G)$.

In fact, if $v_{m+1}v_i\in E(G)$, then $2\leqslant i<k$ by (i). And by $v_kv_{i-1}\in E(G)$, there is a path $P=v_iv_{i+1}\cdots v_kv_{i-1}v_{i-2}\cdots v_1v_{k+1}\cdots v_t\in A$ with $f(P)=m+1>f(P_0)=m$, a contradiction.

Now, since $d(v_1,v_{m+1})=2$, $d(v_1,v_k)=2$ and $d(v_1)<s/2$, we have $d(v_{m+1})\geqslant |M^3(v_1)|/2$, $d(v_k)\geqslant |M^3(v_1)/2$, $N(v_k)\cup N(v_{m+1})\subset M^3(v_1)$ and $v_kv_{m+1}\notin E(G)$ by (i), thus $d(v_k,v_{m+1})=2$. We also have $d(v_{m+1})<|M^3(v_1)|-d(v_1)$ by (i). Since $d(v_{m+1})\geqslant|M^3(v_1)|/2, d(v_1)<|M^3(v_1)|/2$. Hence, $d(v_{m+1})>d(v_1)$. Similarly, we have $d(v_k)>d(v_1)$. Since $d(v_k)<s/2$ and $d(v_k,v_1)=d(v_k,v_{m+1})=2$, $d(v_1)\geqslant|M^3(v_k)|/2$ and $d(v_{m+1})\geqslant|M^3(v_k)|/2$. Thus we have

$$d(v_k)+d(v_{m+1})>d(v_1)+d(v_{m+1})\geqslant|M^3(v_k)|. \qquad (*)$$

On the other hand, since (i), (iii) and $v_1v_{m+1}\notin E(G)$, $v_1,v_{m+1}\in M^3(v_k)$, We have $d(v_{m+1})\leqslant|M^3(v_k)|-d(v_k)-1$.

This contradicts (*). Therefore the proof of Theorem is completed. #

Remark Let us to consider the following graphs:

$G_1=(V_1,E_1)$, where $V_1=U_0\cup(\bigcup_{i=1}^{t}U_i)W$ with $U_0=\{x_1,x_2,\cdots,x_t\}$ $t\geqslant 2$, $U_i=\{u_i,v_i,x_{i,1},x_{i,2},\cdots,x_{i,s}\}$, $i=1,2,\cdots,t$. $W=\{y_1,y_2,\cdots,y_{4s-[s/4]}\}$ and $E_1=\{x_iu_i,x_iv_i \mid i=1,2,\cdots,t\}\cup(\bigcup_{i=1}^{t}\{u_ix_{i,j},v_ix_{i,h} \mid 1\leqslant j\leqslant[s/2],[\frac{s}{2}]+1\leqslant h\leqslant s\})\cup(\bigcup_{i=1}^{t}\{x_{i,j},y_h \mid 1\leqslant j\leqslant s,1\leqslant h\leqslant 4s-[s/4]\})$;

$G_2=(V_2,E_2)$, where $V_2=U_1\cup U_2\cup U_3\cup U_4$ with $|U_1|=[\frac{s}{2}]-2$, $|U_2|=[\frac{s}{2}]-1$, $|U_3|=s$, $|U_4|=s$ and $E_2=\bigcup_{i=1}^{3}\{x_ix_{i+1} \mid \forall\ x_i\in U_i,\forall\ x_{i+1}\in U_{i+1}\}$;

$G_3=(V_3,E_3)$, where $V_3=(\bigcup_{i=1}^{[s/4]}U_i)\cup W$ with $U_i=\{u_i,v_i,x_i,y_i\}$, $W=\{w_1,w_2,\cdots,w_h\}$, $[s/4]<h\leqslant\frac{s}{2}-2$ and $E_3=\{u_iv_i,u_ix_i,v_iy_i \mid v=1,2,\cdots,[s/4]\}\cup\{x_iw_j,y_iw_j \mid i=1,2,\cdots,[s/4],j=1,2,\cdots,h\}$. Clearly, $G_i(i=1,2,3)$ are nonhamiltonian. Using the results of [2],[5], we can't judge if there exists a cycle $C_l(l\geqslant s)$ through the fixed vertex x in $G_i(i=1,2,3)$, but we can do by Theorem 1 for G_1 or G_2 and by Theorem 2 for G_1 or G_3.

Finally, we have following conjectures:

Conjecture 1 Let G be a 2-connected graph of order n, and let $s(3\leqslant s\leqslant n)$ be an integer. If for every vertex $u\in V(G)$, $d(u)=k<s/2$ implies $|\{v\in M^2(u) \mid d(v)\leqslant k\}|\leqslant k-1$, then for any fixed vertex x, there is a cycle $C_l(l\geqslant s)$ through vertex x.

Conjecture 2 Let G be a 2-connected graph of order n, and let s $(3\leqslant s\leqslant n)$ be an integer, If for every u, $v\in V(G)$, $d(u,v)=2$, $d(u)<s/2$ implyies $d(v)\geqslant|M^3(u)|/2$, then for any fixed vertex x in G there is a cycle $C_l(l\leqslant s)$ through x.

Conjecture 3 Let G be a 2-connected graph of order n, and let s $(3\leqslant s\leqslant n)$ be an integer, If for every $u\in V(G)$, $d(u)\geqslant\min\{\frac{s}{2}, |M^3(u)|/2\}$, then for any fixed vertex x in G there is a cycle of length l through x.

Conjectures 1, 2 and 3 are supported by the results of Theorems 1 and 2.

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图中长圈的几个局部化条件

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摘要 本文给出过图 G 中任给定点的长圈存在性的几个局部化条件。设 G 是 2－连通无三角形的 n 阶图，$s(3\leqslant s\leqslant n)$为正整数，若 G 还满足下列条件之一：

(1) $\forall\ u\in V(G)$, $d(u)=k<\frac{s}{2}\Rightarrow|\{v\in M^2(u)\,|\,d(v)\leqslant k\}|\leqslant k-1$，这里 $M^i(u)=\{v|d(u,v)\leqslant i, v\in V(G)\}$；

(2) $\forall\ u,v\in V(G)$, $d(u)<\frac{S}{2}, d(u,v)=2\Rightarrow d(v)\geqslant|M^3(u)|/2$.

则$\forall\ x\in V(G)$，存在长为 $l(l\geqslant s)$的圈 C_l 过 x。最后我们给出几个关于长圈的局部化条件的猜测。

关键词 圈，局部化条件，无三角形图。

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