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A SUFFICIENT CONDITION FOR 2-CONNECTED HAMILTON GRAPHS

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Abstract Let G be a graph of order $n \geqslant 3$. We show that if G is a 2-connected graph and $\sum_{i=1}^{3} d(x_i) + \sum_{1\leqslant j<k\leqslant 3} |N(x_j) \cup N(x_k)| \geqslant 3n-3$ for any 3-independent set $\{x_1, x_2, x_3\}$ with a pair of vertices x_s, x_t at distance two $(1 \leqslant s \neq t \leqslant 3)$, then G is Hamiltonian or $G \cong G_{\frac{n-1}{2}} \vee I_{\frac{n+1}{2}}$ $(n \equiv 1 \bmod 2)$ or $G \cong (K_p \cup K_q \cup K_r) \vee G_2$, where G_m is a simple graph of order m, I_m is an independent set of order m.

Key words Neighborhood, Hamiltonian cycle.

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1 Introduction

We use [1] for terminologies and notations and consider only simple graphs. If G has a cycle containing every vertex of G, then G is called Hámiltonian. The set of vertices adjacent to vertex v is denoted by $N(v)$; $d(v) = |N(v)|$ is the degree of the vertex v. If A, B are subgraphs of G and $U \subseteq V(G)$, we define $N(A) = \bigcup_{v \in V(A)} N(v)$, $N_B(A) = N(A) \cap V(B)$ and $\Delta(U) = \max\{d(u) \mid u \in U\}$. The distance, denoted by $d(u,v)$, between two vertices u and v of a connected graph is the minimum length of all paths joining u and v. Let I_k be a k-independent set, if $\min\{d(u,v) \mid u,v \in I_k\} = r$, then I_k is called (k,r)-independent set, denoted by I_k^r.

If C is a cycle of graph G, we let $\vec{C}$ denote the cycle C with a given orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices on C from u to v. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We use u^+ for the successor of u on $\vec{C}$ and u^- for its predecessor; $u^{++} = (u^+)^+$ and $u^{--} = (u^-)^-$, If $A \subseteq V(C)$, then $A^+ = \{v^+ \mid v \in A\}$. The set A^- is de-

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fined analogously.

The development of the theory on Hamiltonian graphs has seen a series of results based on controlling the degrees of the vertices of G. In this paper, we improve Theorem 3 in [4] to obtain a stronger result.

Theorem 1 [2] Let G be a graph of order $n\geqslant 3$ such that for each pair of nonadjacent vertices x and y, $d(x)+d(y)\geqslant n$, then G is Hamiltonian.

Theorem 2 [3] Let G be a 2-connected graph of order $n\geqslant 3$. If for each pair of nonadjacent vertices x and y, $|N(x)\cup N(y)|\geqslant\frac{2n-1}{3}$, then G is Hamiltonian.

Theorem 3 [4] Let G be a 2-connected graph of order $n\geqslant 3$. If for any 3-independent set $\{x_1,x_2,x_3\}$, $\sum_{i=1}^{3}d(x_i)+\sum_{1\leqslant j<k\leqslant 3}|N(x_j)\cup N(x_k)|>3n-3$, then G is Hamiltonian.

Theorem 4 Let G be a 2-connected graph of order $n\geqslant 3$. If for any (3,2)-independent set $\{x_1,x_2,x_3\}$, $\sum_{i=1}^{3}d(x_i)+\sum_{1\leqslant j<k\leqslant 3}|N(x_j)\cup N(x_k)|\geqslant 3n-3$, then G is Hamiltonian or $G\cong G_{\frac{n-1}{2}}\vee I_{\frac{n+1}{2}}(n\equiv 1 \bmod 2)$ or $G\cong(K_p\cup K_q\cup K_r)\vee G_2$, where G_m is a simple graph of order m, I_m is an independent set of order m.

It is easy to find Hamiltonian graphs that satisfy the conditions of Theorem 4, but not the conditions of Theorem 3. One such graph is $G_m\vee(I_{m+1}\cup e)$, where $m\geqslant 4$, G_m is a graph of order m, I_{m+1} is an $(m+1)$-independet set, e is an edge with two vertices in I_{m+1}.

2 Proof of Theorem 4

Let G be a graph satisfying the condition of Theorem 4, and let C be a longest cycle of G with a fixed orientation. Assume C is not an Hamilton cycle of G. Then $G-V(C)$ has a connected component B. Let $v_1,v_2,\cdots,v_m$ be the elements of $N_C(B)$ occuring on $\vec{C}$ in consecutive order. Since G is 2-connected, we have $m\geqslant 2$. For each $i\neq j$, let $v_iP_{i,j}v_j$ be a path of length at least 2 which joins v_i and v_j with all internal vertices of the path in B. Let x_j be a vertex of B which is adjacent to v_j (for $i\neq j$, possibly $x_i=x_j$). The indices are taken modulo m in the proof. Let $N^-=\{v_1^-,v_2^-,\cdots,v_m^-\}$, $N^+=\{v_1^+,v_2^+,\cdots,v_m^+\}$.

Lemma 1 [5] Let x be any vertex of B. For $u,v\in N^-\cup\{x\}$, there exists no (u,v)-path with all internal vertex disjoint from C, particularly, $uv\notin E$.

For any $j(1\leqslant j\leqslant m)$ Lemma 1 implies $v_{j+1}^-\notin N(v_j^-)$. So there is a vertex $w_j\in\{v_j^+,v_j^{++},\cdots,v_{j+1}^-\}$ such that $w_j\notin N(v_j^-)$ and $v\in N(v_j^-)$ for any $v\in\{v_j,v_j^+,v_j^{++},\cdots,w_j^-\}$. Let $H_j=\{v_j^+,v_j^{++},\cdots,w_j\}$, $H=\{u_1,u_2,\cdots,u_m\}$, where u_j is a vertex of H_j.

Lemma 2 [5] Let $x\in B$ and $u,v\in H\cup\{x\}$, then there exists no (u,v)-path with all internal vertices disjoint from C, particularly, $uv\notin E$.

Lemma 3 [5] If $u_i,u_j\in H(i<j)$, then for any vertex $v\in u_i^+\vec{C}u_j^-$, we have $u_iv\notin E$ or $u_jv^-\notin E$; for $v\in u_j^+\vec{C}u_i^-$, we have $u_iv\notin E$ or $u_jv^+\notin E$.

Lemma 4 [5]　For $x \in B$ and $u_i, u_j \in H (i \neq j), d(x) + |N(u_i) \cup N(u_j)| \leqslant n-1$. The equality holds iff $V(C) = N_C(u_i) \cup N_C(u_j) \cup \{u_1, u_2, \cdots, u_m\}$; $V(B) = N_B(x) \cup \{x\}$; $N_C(x) = \{v_1, v_2, \cdots, v_m\}$; $V(R) = N_R(u_i) \cup N_R(u_j)$, where $R = G - V(B) - V(C)$.

Proof　We easily obtain

$$N_B(x) \subseteq V(B) - \{x\},$$
$$N_C(x) \subseteq \{v_1, v_2, \cdots, v_m\},$$
$$N_C(u_i) \cup N_C(u_j) \subseteq V(C) - \{u_1, u_2, \cdots, u_m\},$$
$$N_R(u_i) \cup N_R(u_j) \subseteq V(R).$$

Therefore, $d(x) + |N(u_i) \cup N(u_j)| \leqslant |V(B)| - 1 + m + |V(C)| - m + |V(R)| = n-1$. The equality holds iff the above sets are equal.

Using Lemma 1－3, We now derive an upper bound for $d(u_i) + |N(u_{i-1}) \cup N(x)|$. Let

$$R_1(u_i) = \{v \in u_i \vec{C} u_{i-1}^- \mid u_i v^+ \in E\},$$
$$S_1(u_{i-1}) = \{v \in u_i \vec{C} u_{i-1}^- \mid v \in N(u_{i-1}) \cup N(x)\},$$
$$R_2(u_i) = \{v \in u_{i-1} \vec{C} u_i^- \mid u_i v \in E\},$$
$$S_2(u_{i-1}) = \{v \in u_{i-1} \vec{C} u_i^- \mid u_{i-1} v^+ \in E\},$$
$$S_3(u_i) = R_3(u_i) = \{v \in V - V(C) \mid u_i v \in E\}.$$

By Lemma 2 and 3, $R_j(u_i) \cap S_j(u_{i-1}) = \varnothing$ for $j = 1, 2$ and 3. Let

$$\delta'_i = \begin{cases} 1, & \text{if } v_i x \in E \text{ and } v_i u_{i-1} \notin E, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta_i = \begin{cases} 1, & \text{if } \delta'_i = 1 \text{ and } u_i v_i^- \in E, \\ 0, & \text{otherwise.} \end{cases}$$

We get that $d(u_i) + |N(u_{i-1}) \cup N(x)| = |R_1(u_i)| + |S_1(u_{i-1})| + |R_2(u_i)| + |S_2(u_{i-1})| + |R_3(u_i)| + |S_3(u_{i-1})| + |N_B(x)| + \delta'_i \leqslant |R_1(u_i) \cup S_1(u_{i-1}) \cup R_2(u_i) \cup S_2(u_{i-1})| + (n - |V(C)| - |V(B)|) + |N_B(x)| + \delta'_i = n - |V(C) - (R_1(u_1) \cup S_1(u_{i-1}) \cup R_2(u_i) \cup S_2(u_{i-1}))| - |V(B) - N_B(x)| + \delta'_i \leqslant n - 1 - |V(C) - (R_1(u_i) \cup S_1(u_{i-1}) \cup R_2(u_i) \cup S_2(u_{i-1}))| + \delta'_i \leqslant n - 1 + \delta_i$.

Note that the last inequality follows since $v_i^- \notin R_2(u_i) \cup S_2(u_{i-1})$ if $\delta'_i = 1$ and $\delta_i = 0$. Thus we have:

Lemma 5[6]　(1) For $x \in B$ and $u_i \in H, d(u_i) + |N(u_{i-1}) \cup N(x)| \leqslant n - 1 + \delta_i$. If $d(u_i) + |N(u_{i-1}) \cup N(x)| = n$, then $x v_i \in E, u_i v_i^- \in E$ and $N_B(x) = V(B) - \{x\}$, $V(C) = R_1(u_i) \cup S_1(u_{i-1}) \cup R_2(u_i) \cup S_2(u_{i-1})$.

(2) If $u_i v_i^- \notin E$, then $d(u_i) + |N(u_{i-1}) \cup N(x)| \leqslant n-1$. When the equality holds, we have $N_B(x) = V(B) - \{x\}$ and $V(C) = R_1(u_i) \cup S_1(u_{i-1}) \cup R_2(u_i) \cup S_2(u_{i-1})$.

We prove Theorem 4 for two cases.

Case 1　$m \geqslant 3$.

Subcase 1.1 There is $i_0(1\leqslant i_0\leqslant m)$ such that $d(v_{i_0-1}^-)=n-|N(v_{i_0}^-)\cup N(x_{i_0})|$. Then by Lemma 5(1), we have $v_{i_0-1}^-v_{i_0-1}^+\in E$.

We consider (3,2)-independent set $\{v_{i_0}^-v_{i_0+1}^-,x_{i_0}\}$ in two cases.

Subcase 1.1.1 $d(v_{i_0+1}^-)=n-1-|N(v_{i_0}^-)\cup N(x_{i_0})|$.

By Lemma 5(2), for any $v\in v_{i_0+1}\vec{C}v_{i_0}^{--}$, we have $v\in N(v_{i_0}^-)\cup N(x_{i_0})$ or $v^+\in N(v_{i_0+1}^-)$. Particularly, since $v_{i_0-1}^-\notin N(v_{i_0}^-)\cup N(x_{i_0})$, we have $v_{i_0-1}\in N(v_{i_0+1}^-)$. Thus the cycle $v_{i_0+1}P_{i_0+1,i_0-1}v_{i_0-1}v_{i_0+1}^-\overleftarrow{C}v_{i_0-1}^+v_{i_0-1}^-\overleftarrow{C}v_{i_0+1}$ is longer than C; a contradiction.

Subcase 1.1.2 $d(v_{i_0+1}^-)\leqslant n-2-|N(v_{i_0}^-)\cup N(x_{i_0})|$.

By Lemmas 4 and 5, we have

$$d(v_{i_0}^-)+|N(v_{i_0+1}^-)\cup N(x_{i_0})|\leqslant n,$$
$$d(x_{i_0})+|N(v_{i_0}^-)\cup N(v_{i_0+1}^-)|\leqslant n-1.$$

By the conditions of Theorem 4, we have $3n-3\leqslant d(x_{i_0})+d(v_{i_0}^-)+d(v_{i_0+1}^-)+|N(v_{i_0}^-)\cup N(x_{i_0})|+|N(v_{i_0+1}^-)\cup N(x_{i_0})|+|N(v_{i_0}^-)\cup N(v_{i_0+1}^-)|\leqslant 3n-3$.

Hence we have $|N(v_{i_0}^-)\cup N(v_{i_0+1}^-)|=n-1-d(x_{i_0})$. By Lemma 4, we have $N(v_{i_0}^-)\cup N(v_{i_0+1}^-)=V(G)-V(B)-\{u_1,u_2,\cdots,u_m\}$. So $v_{i_0-1}\in N(v_{i_0}^-)$or $v_{i_0-1}\in N(v_{i_0+1}^-)$. If $v_{i_0-1}\in N(v_{i_0}^-)$, the cycle $v_{i_0}P_{i_0,i_0-1}v_{i_0-1}v_{i_0}^-\overleftarrow{C}v_{i_0-1}^+v_{i_0-1}^-\overleftarrow{C}\,v_{i_0}$is longer than C, a contradiction. If $v_{i_0-1}\in N(v_{i_0+1}^-)$, then cycle $v_{i_0+1}P_{i_0+1,i_0-1}v_{i_0-1}v_{i_0+1}^-\overleftarrow{C}v_{i_0-1}^+v_{i_0-1}^-\overleftarrow{C}v_{i_0+1}$ is longer than C, a contradiction.

By the analogous proof of subcase 1.1, we can prove $d(v_i^+)\leqslant n-1-|N(v_{i-1}^+)\cup N(x_{i-1})|(i=1,2,\cdots,m)$. Hence

Subcase 1.2 For any $i(i=1,2,\cdots,m)$, $d(v_{i-1}^-)\leqslant n-1-|N(v_i^-)\cup N(x_i)|$ and $d(v_i^+)\leqslant n-1-|N(v_{i-1}^+)\cup N(x_{i-1})|$.

In the following, we will get 14 claims.

By Lemma 4,5(2) and the conditions of Theorem 4, we can easily obtain

(1)

$$d(v_i^-)+|N(v_{i-1}^-)\cup N(x_i)|=n-1,$$
$$d(v_{i-1}^+)+|N(v_i^+)\cup N(x_{i-1})|=n-1,\quad (i=1,2,\cdots,m),$$
$$d(x_i)+|N(v_i^-)\cup N(v_{i-1}^-)|=n-1,$$
$$d(x_{i-1})+|N(v_i^+)\cup N(v_{i-1}^-)|=n-1.$$

By Lemmas 5(2) and (1), we have

(2) For any $v\in v_i\vec{C}v_{i-1}^{--}$, we have $v\in N(v_{i-1}^-)\cup N(x_i)$ or $v^+\in N(v_i^-)$; for any $v\in v_{i-1}\vec{C}v_i^{--}$, $v\in N(v_{i-1}^-)\cup N(x_i)$ or $v^-\in N(v_i^-)$. Symmetrically, for any $v\in v_i^{++}\vec{C}v_{i-1}$, $v\in N(v_i^+)\cup N(x_{i-1})$ or $v^-\in N(v_{i-1}^+)$; for any $v\in v_{i-1}^{++}\vec{C}v_i$, $v\in N(v_i^+)\cup N(x_{i-1})$ or $v^+\in N(v_{i-1}^+)$.

Noting that $v_{i+1}^-\notin N(v_{i-1}^-)\cup N(x_i)$, and by (2) we have

(3) For any $i(i=1,2,\cdots,m)$, $v_i^-v_{i+1}\in E$. Symmetrically, $v_{i-1}v_i^+\in E$.

(4) $|V(B)|=1$.

Proof First we prove that $v_i^- v_i^{++} \in E$ for any $i \in \{1,2,\cdots,m\}$. Since $v_{i+1}^+ \notin N(v_i^+) \cup N(x_{i-1})$, we have $v_{i-1}^+ v_{i+1} \in E$ by (2). But $v_{i-1}^+ v_{i+1} \in E$ and $v_i^- v_{i+1}^+ \in E$ give the cycle $v_i \vec{C} v_{i+1} v_{i-1}^+ \vec{C} v_i^- v_{i+1}^+ \vec{C} v_{i-1} P_{i-1,i} v_i$, which is longer than C. Thus $v_i^- v_{i+1}^+ \notin E$ implies $v_{i+1}^+ \notin N(v_i^-) \cup N(x_{i-1})$, furthermore $v_{i+1}^- v_{i+1}^{++} \in E$ by (2). Suppose $|V(B)| \geqslant 2$, then we may assume $1 \leqslant s < t \leqslant m$. If $s+1 \neq t$, since $v_{s+1}^+ v_t^+ \notin E$ we have $v_s^+ v_t \in E$ by (2). The cycle $v_t v_s^+ \vec{C} v_s P_{s,t} v_t$ is longer than C, a contradiction. If $s+1=t$, since $v_{s-1}^- v_t^- \notin E$ we have $v_s^- v_t \in E$ by (2). The cycle $v_t v_s^- \overleftarrow{C} v_t^{++} v_t^- \overleftarrow{C} v_s P_{s,t} v_t$ is longer than C, a contradiction. So we have $|V(B)|=1$.

(5) $|V(C)|=n-1$.

Proof Assume $|V(C)|<n-1$, then there is another connected component $B' \in G-V(C)$ by (4). Let $w \in B'$, then we have $v_{i-1}^- w \in E$ or $v_i^- w \in E$ by the proof of Lemma 5. Since $m \geqslant 3$, there exist $s, t\ (s \neq t)$ such that $v_s^- w \in E$ and $v_t^- w \in E$. The cycle $v_s \vec{C} v_t^- w v_s^- \overleftarrow{C} v_t P_{t,s} v_s$ is longer than C, a contradiction. Therefore $|V(C)|=n-1$.

(6) For any $i\,(i=1,2,\cdots,m)$, $v_{i-1}^{++}=v_i$.

Proof Assume (6) is not true, then there is i such that $v_{i-1}^{++} \neq v_i$. Let $S=\{v_1^-, v_2^-, \cdots, v_m^-, x_2\}$. By (2) and $m \geqslant 3$, we may suppose $d(x_2)=\Delta(S)$ (if $d(x_2)<\Delta(S)$, we consider another longest cycle. By (1) and Lemma 4, we have $d(x_2)+|N(v_2^-) \cup N(v_1^-)|=n-1$, and $N_C(x_2)=\{v_1, v_2, \cdots, v_m\}$. Hence by (3) and the conditions of Theorem 4 we have

$$3n-3 \leqslant d(x_2)+|N(v_2^-) \cup N(v_1^-)|+d(v_1^-)+|N(v_2^-) \cup N(x_2)| + d(v_2^-)+|N(v_1^-) \cup N(x_2)| \leqslant n-1+2(\Delta(S)+2\Delta(S)-1),$$

i.e., $\Delta(S) \geqslant \frac{n}{3}$.

So there exists $k\,(1 \leqslant k \leqslant m)$ such that $v_{k-1}^{++}=v_k$. From the above discussion we may suppose $v_k^{++} \neq v_{k+1}$.

First we verify that there is a cycle longer than C in G when $m \geqslant 4$. Since $d(v_{k-2}^+, x_2)=2$, $\{v_{k-2}^+, v_{k-1}^+, x_2\}$ is a (3,2)-independent set. We can easily obtain $|N(v_{k-2}^+) \cup N(v_{k-1}^+)|+d(x_2)=n-1$ by Lemma 4,5 and the conditions of Theorem 4. Hence we have $v_k^{++} \vec{C} v_{k+1} \subseteq N(v_{k-1}^+) \cup N(v_{k-2}^+)$ by Lemma 4 which implies $v_{k-2}^+ v_{k+1}^- \in E$ by Lemma 1 and $v_k^- v_{k+1} \in E$ by (3). The cycle $v_{k-2} P_{k-2,k-1} v_{k-1} \overleftarrow{C} v_{k-2}^+ v_{k+1}^- \overleftarrow{C} v_k^- v_{k+1} \vec{C} v_{k-2}$ is longer than C, a contradiction.

Now we verify that there is a cycle longer than C in G when $m=3$. From the above, we may suppose $v_1^{++}=v_2$ and $v_2^{++} \neq v_3$. Since $d(v_1^+, x_2)=2$, $\{v_1^+, v_3^+, x_2\}$ is a (3,2)-independent set. We can easily obtain $|N(v_1^+) \cup N(v_3^+)|+d(x_2)=n-1$ by Lemma 4,5 and the conditions of Theorem 4. Hence we have $v_3^- \in N(v_1^+) \cup N(v_3^+)$ by Lemma 4 which implies $v_3^- \in N(v_3^+)$ by Lemma 1. But $v_3 v_1^+ \in E$ by (3). Thus the cycle $v_3 P_{3,2} v_2 \vec{C} v_3^- v_3^+ \vec{C} v_1^+ v_3$ is longer than C, a contradiction.

We can derive $G \cong G_{\frac{n-1}{2}} \vee I_{\frac{n+1}{2}}$ by (2) and (6).

Case 2 $m=2$.

Considering (3,2)-independent set $\{v_1^-, v_2^-, x_1\}$, by Lemma 4,5 and the conditions of Theroem 4, we can derive that:

$$d(v_2^-) + |N(v_1^-) \cup N(x_1)| = n-1.$$
$$d(v_1^-) + |N(v_2^-) \cup N(x_1)| = n-1,$$
$$d(x_1) + |N(v_1^-) \cup N(v_2^-)| = n-1.$$

Since $d(v_2^-)+|N(v_1^-)\cup N(x_1)|=n-1$, by Lemma 5(2) we have

(7) For any $v\in v_1\vec{C}v_2^{--}$, $v\in N(v_1^-)\cup N(x_1)$ or $v^-\in N(v_2^-)$; for any $v\in v_2\vec{C}v_1^{--}$, $v\in N(v_1^-)\cup N(x_1)$ or $v^+\in N(v_2^-)$. Symmetrically, for any $v\in v_1^{++}\vec{C}v_2$, $v\in N(v_2^+)\cup N(x_1)$ or $v^+\in N(v_1^+)$; for any $v\in v_2^{++}\vec{C}v_1$, $v\in N(v_2^+)\cup N(x_1)$ or $v^-\in N(v_1^+)$.

Since $d(x_1)+|N(v_1^-)+N(v_2^-)|=n-1$, by Lemma 4 we have

(8) $V(C)\subseteq N(v_1^-)\cup N(v_2^-)\cup\{v_1^-, v_2^-\}$, $V(C)\subseteq N(v_1^+)\cup N(v_2^+)\cup\{v_1^+, v_2^+\}$, $N_C(x_1)=\{v_1, v_2\}$, $N_B(x_1)=V(B)-\{x_1\}$, $V(R)=N_R(v_1^-)\cup N_R(v_2^-)$, $V(R)=N_R(v_1^+)\cup N_R(v_2^+)$.

(9) $v_2^+\vec{C}v_1^{--}\cup\{v_1\}\subseteq N(v_1^-)$, $v_1^+\vec{C}v_2^{--}\cup\{v_2\}\subseteq N(v_2^-)$; $v_1^{++}\vec{C}v_2^-\cup\{v_1\}\subseteq N(v_1^+)$, $v_2^{++}\vec{C}v_1^-\cup\{v_2\}\subseteq N(v_2^+)$.

Proof　First if $a_i\in v_i^+\vec{C}v_{i+1}^-$ with $a_iv_i^+\in E$ and $a_i^+v_i^+\notin E$, then $a_iv_{i+1}^+\in E$ by (7) ($i=1,2$). Applying Lemma 3 and (8) we have $v_i^{++}\vec{C}a_i\subseteq N(v_i^+)$ and $a_i\vec{C}v_{i+1}\subseteq N(v_{i+1}^+)$. Thus there is a vertex $a_i\in v_i^+\vec{C}v_{i+1}^-$ such that $v_i^{++}\vec{C}a_i\subseteq N(v_i^+)$ and $a_i^+\vec{C}v_{i+1}\subseteq N(v_{i+1}^+)$ ($i=1,2$). Similarly, there is a vertex $b_i\in v_i^+\vec{C}v_{i+1}^-$ such that $v_i\vec{C}b_i^-\subseteq N(v_i^-)$ and $b_i\vec{C}v_{i+1}^{--}\subseteq N(v_{i+1}^-)$ ($i=1,2$).

If $b_1\neq v_1^+$, then $v_1^+v_2^-\notin E$, so that $v_1^-v_1^{++}\in E$ by (7). But $v_1^-v_1^{++}\in E$ and $v_2^+v_1^{+++}\in E$ give a cycle longer than C. Thus $v_2^+v_1^{+++}\notin E$ implies $a_1\in v_1^{+++}\vec{C}v_2^-$ and the cycle $v_1P_{1,2}v_2\overleftarrow{C}a_1^+v_2^+\vec{C}v_1^-v_2^{++}\vec{C}a_1v_1^+v_1$ is longer than C, a contradiction. Hence $b_1=v_1^+$. Similarly, $b_2=v_2^+$, $a_1=v_2^-$, $a_2=v_1^-$. Hence (9) holds.

(9) implies (10).

(10) $[v_1^{++}\vec{C}v_2^{--}, v_2^{++}\vec{C}v_2^{--}]=\varnothing$, $[v_1^+, v_2^+\vec{C}v_1^{--}]=\varnothing$, $[v_1^-, v_1^{++}\vec{C}v_2^-]=\varnothing$, $[v_2^+, v_1^+\vec{C}v_2^{--}]=\varnothing$, $[v_2^-, v_2^{++}\vec{C}v_1^-]=\varnothing$.

(11) $V(G)=V(C)\cup V(B)$.

Proof　Assume (11) is not true, then there is another connected component B' in $G-V(C)$.

If $v_1^{++}=v_2$ and $v_2^{++}=v_1$, then $|B|=1$ and $|B'|=1$. Let the vertex of B' be b'. By Lemma 2 and (8), we have $b'\in N(v_1^-)$ or $b'\in N(v_2^-)$, without loss of generality, say $b'\in N(v_1^-)$. Since G is 2-connected, b' is adjacent to a vertex of C except v_1^-. But this produces a cycle longer than C, a contradiction. Therefore $v_1^{++}\neq v_2$ or $v_2^{++}\neq v_1$. In the following, we suppose $v_2^{++}\neq v_1$.

By Lemma 2 and (8), we have $V(B')\subseteq N(v_1^-)$ or $V(B')\subseteq N(v_2^-)$, without loss of generality, say $V(B')\subseteq N(v_1^-)$. We have $v_1^{--}v_2^-\notin E$ by (10) and $v_2x_1\in E$ by (8), so that $\{v_1^{--}, v_2^-, x_1\}$ is a (3,2)-independent set. Since $v_1^{--}v_2^-\notin E$, we have $|N_C(v_1^{--})\cup N_C(v_2^-)|$

$\leqslant |C|-2$. Clearly, $d(x_1)\leqslant |V(B)|+1$, $|N_R(v_1^{--})\cup N_R(v_2^-)|\leqslant |V(G)-V(C)-V(B)-V(B')|$, where $R=G-V(B)-(C)$. Thus $d(x_1)+|N(v_1^{--})\cup N(v_2^-)|=d(x_1)+|N_C(v_1^{--})\cup N_C(v_2^-)|+|N_R(v_1^{--})\cup N_R(v_2^-)|\leqslant |V(B)|+1+|V(C)|-2+|V(G)-V(C)-V(B)-V(B')|\leqslant |V(G)|-|V(B')|-1\leqslant n-2$. Similarly, $d(v_2^-)+|N(v_1^{--})\cup N(x_1)|\leqslant n-2$, $d(v_1^{--})+|N(v_2^-)\cup N(x_1)|\leqslant n-2$. Thus $d(x_1)+d(v_1^{--})+d(v_2^-)+|N(v_1^{--})\cup N(v_2^-)|+|N(v_1^{--})\cup N(x_1)|+|N(v_2^-)\cup N(x_1)|\leqslant 3n-6$. This contradicts the condition of Theorem 4.

We prove case 2 for the following three subcases.

Subcase 2.1　$v_1^-v_1^+\in E$ and $v_2^-v_2^+\in E$.

Since C is the longest cycle in G, we can derive $v_1^{++}\vec{C}v_2^{--}\neq\varnothing$, $v_2^{++}\vec{C}v_1^{--}\neq\varnothing$. Let $u_1\in v_1^{++}\vec{C}v_2^{--}$, then $v_1^-u_1\notin E$ by (10). Thus $\{v_1^-,u_1,x_1\}$ is a (3,2)-independent set. Since $v_1^-v_1^+\in E$, $v_2^-v_2^+\in E$ and C is the longest cycle in G, we have $N(v_1^-)\subseteq v_2^+\vec{C}v_1^{--}\cup\{v_1,v_1^+\}$, $N(u_1)\subseteq v_1^+\vec{C}v_2^- -\{u_1\}$, and $N(x_1)\subseteq(V(B)\cup\{v_1,v_2\})-\{x_1\}$ by (10). Thus $d(u_1)+|N(v_1^-)\cup N(x_1)|\leqslant|v_1^+\vec{C}v_2^-|-1+|v_2^+\vec{C}v_1^{--}\cup\{v_1,v_1^+\}\cup V(B)\cup\{v_1^-,v_2\}|-1=|(V(C)-\{v_1^-\})\cup V(B)|+|\{v_1^+\}|-2=n-2$. Similarly, $d(v_1^-)+|N(u_1)\cup N(x_1)|\leqslant n-2$, $d(x_1)+|N(u_1)\cup N(v_1^-)|\leqslant n-2$. Therefore $d(x_1)+d(v_1^-)+d(u_1)+|N(u_1)\cup N(v_1^-)|+|N(u_1)\cup N(x_1)|+|N(v_1^-)\cup N(x_1)|\leqslant 3n-6$. This contradicts the condition of Theorem 4.

Subcase 2.2　$v_1^-v_1^+\in E$ and $v_2^-v_2^+\notin E$ or $v_1^-v_1^+\notin E$ and $v_2^-v_2^+\in E$.

We discuss the latter, so do the former. Since $v_1^-v_1^+\notin E$, $\{v_1^-,v_1^+,x_1\}$ is a (3,2)-independent set. Since $v_2^-v_2^+\in E$ and C is the longest cycle, we have $v_1^-v_2\notin E$ and $v_1^+v_2\notin E$. By (10), we have $N(v_1^-)\subseteq v_2^+\vec{C}v_1^{--}\cup\{v_1\}$, $N(v_1^+)\subseteq v_1^{++}\vec{C}v_2^-\cup\{v_1\}$, $N(x_1)\subseteq V(B)\cup\{v_1,v_2\}-\{x_1\}$. Thus $d(v_1^-)+|N(v_1^+)\cup N(x_1)|\leqslant|v_2^+\vec{C}v_1^{--}\cup\{v_1\}|+|(v_1^{++}\vec{C}v_2^-\cup\{v_1\})\cup(V(B)\cup\{v_1,v_2\}-\{x_1\})|=|v_2^+\vec{C}v_1^{--}|+1+|v_1^{++}\vec{C}v_2\cup\{v_1\}|+|V(B)|-1=|V(C)|-2+|V(B)|=n-2$. Similarly, $d(v_1^+)+|N(v_1^-)\cup N(x_1)|\leqslant n-2$, $d(x_1)+|N(v_1^-)\cup N(v_1^+)|\leqslant n-2$. Therefore $d(x_1)+d(v_1^-)+d(v_1^+)+|N(v_1^-)\cup N(v_1^+)|+|N(v_1^-)\cup N(x_1)|+|N(v_1^+)\cup N(x_1)|\leqslant 3n-6$. This contradicts the condition of Theorem 4.

Subcase 2.3　$v_1^-v_1^+\notin E$, $v_2^-v_2^+\notin E$.

(12) $v_1^+v_2\in E$, $v_1^-v_2\in E$, $v_2^+v_1\in E$, and $v_2^-v_1\in E$.

Proof　Since $d(v_1^-)+|N(v_2^-)\cup N(x_1)|=n-1$ and (10), we have

$$\begin{aligned}
n-1 &= d(v_1^-)+|N(v_2^-)\cup N(x_1)|\\
&=|N_C(v_1^-)|+|N_R(v_1^-)|+|N_C(v_2^-)\cup N_C(x_1)|+|N_R(v_2^-)|+|N_B(x_1)|\\
&=|N_C(v_1^-)\cup N_C(v_2^-)\cup N_C(x_1)|+|N_C(v_1^-)\cap(N_C(v_2^-)\cup N_C(x_1))|\\
&\quad+|N_R(v_1^-)\cup N_R(v_2^-)|+|N_B(x_1)|\\
&\leqslant|V(C)-\{v_1^-,v_2^-\}|+|\{v_1,v_2\}|+|V(R)|+|V(B)-\{x\}|\\
&=n-1,
\end{aligned}$$

where $R=G-V(C)-V(B)$.

Thus $N_C(v_1^-)\cap(N_C(v_2^-)\cup N_C(x_1))=\{v_1,v_2\}$ implies $v_1^-v_2\in E$. Similarly, we have $v_1^+v_2$

$\in E, v_2^+v_1 \in E$ and $v_2^-v_1 \in E$.

(13) For any $a_i \in v_i^+\vec{C}v_{i+1}^-$, we have $N(a_i)\cup\{a_i\}=v_i\vec{C}v_{i+1}(i=1,2)$.

Proof By (10) and (11), we have $N(a_i)\cup\{a_i\}\subseteq N_i\vec{C}v_{i+1}$. Thus

$$d(x_1)+|N(v_i^-)\cup N(a_i)|\leqslant|V(G)-\{v_i^-,a_i,x_1\}|+|\{v_1,v_2\}|=n-1,$$
$$d(v_i^-)+|N(a_i)\cup N(x_1)|\leqslant|V(G)-\{v_i^-,a_i,x_1\}|+|\{v_1,v_2\}|=n-1,$$
$$d(a_i)+|N(v_i^-)\cup N(x_1)|\leqslant|V(G)-\{v_i^-,a_i,x_1\}|+|\{v_1,v_2\}|=n-1.$$

Since $\{v_i^-,a_i,x_1\}$ is a $(3,2)$-independent set, the above three equalities hold by the conditions of Theorem 4. Therefore $N(a_i)\cup\{a_i\}=v_i\vec{C}v_{i+1}(i=1,2)$.

(14) For any $x\in V(B)$, we have $N(x)=V(B)\cup\{v_1,v_2\}-\{x\}$.

Proof Since $N(v_1^-)\subseteq v_2\vec{C}v_1-\{v_1^-\}$, $N(v_1^+)\subseteq v_1\vec{C}v_2-\{v_1^+\}$ and $N(x)\subseteq(V(B)\cup\{v_1,v_2\}-\{x\})$, we have $d(v_1^-)+|N(v_1^+)\cup N(x)|\leqslant|v_2\vec{C}v_1-\{v_1^-\}|+|(v_1\vec{C}v_2-\{v_1^+\})\cup(V(B)\cup\{v_1,v_2\}-\{x\})|\leqslant|V(C)-\{v_1^-,v_1^+\}|+|\{v_1,v_2\}|+|V(B)-\{x\}|=n-1$.

Similarly, we have:

$$d(v_1^+)+|N(v_1^-)\cup N(x)|\leqslant n-1,$$
$$d(x)+|N(v_1^-)\cup N(v_1^+)|\leqslant n-1.$$

Since $\{v_1^-,v_1^+,x\}$ is a $(3,2)$-independent set, the conditions of Theorem 4 imply that the above three equalities hold. Thus $N(x)=V(B)\cup\{v_1,v_2\}-\{x\}$.

For (13) and (14) it is readily seen that $G\cong(K_p\cup K_q\cup K_r)\vee G_2$. This completes the proof of Theorem 4.

Corollary 1(Tian [8]) If G is a 2-connected graph of order n satisfying $|N(u)\cup N(v)|+|N(u)\cup N(w)|+|N(v)\cup N(w)|\geqslant 2n-1$ for any independent set $\{u,v,w\}$ in G, then G is Hamiltonian.

Corollary 2(Chen [7]) Let G be 2-connected graph of order n. If $2|N(u)\cup N(v)|+d(u)+d(v)\geqslant 2n-1$ for each pair of nonadjacent vertices $u,v\in V(G)$, then G is Hamiltonian.

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一个二连通 Hamilton 图的充分条件

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摘要　设 G 是 $n(\geqslant 3)$阶的 2-连通图. 本文证明如果对于 G 中任一其中有两顶点距离为 2 的 3-独立集$\{x_1,x_2,x_3\}$成立 $\sum_{i=1}^{3} d(x_i)+\sum_{1\leqslant j<k\leqslant 3}|N(x_j)\cup N(x_k)|\geqslant 3n-3$,则 G 是 Hamilton 图,除非 $G\cong G_{\frac{n-1}{2}} \vee I_{\frac{n+1}{2}}(n\equiv 1 \bmod 2)$或者 $G\cong(K_p\cup K_q\cup K_r)\dot{\vee} G_2$. 这里 G_m 表示 m 阶的简单图,I_m 表示 m 阶的独立集.

关键词　邻域,Hamilton 图.
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