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ON THE EXPONENT SET OF PRIMITIVE LOCALLY SEMICOMPLETE DIGRAPHS*

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Abstract. A locally semicomplete digraph is a digraph $D=(V,A)$ satisfying the following condition: for every vertex $x\in V$ the $D[O(x)]$ and $D[I(x)]$ are semicomplete digraphs. In this paper, we get some properties of cycles and determine the exponent set of primitive locally semicomplete digraphs.

1. Introduction

A digraph D is *primitive* if there exists an integer $k>0$ such that for all ordered pairs of vertices $u,v\in V(D)$ (not necessarily distinct), there is a walk from u to v with length k. The least such k is called the exponent of the digraph D, denoted by $\gamma(D)$.

The *exponent from vertex u to vertex v*, denoted by $\gamma(u,v)$, is the least integer γ such that there exists a walk of length m from u to v for all $m\geqslant\gamma$. Let $L(D)=\{r_1,r_2,\ldots,r_\lambda\}$ be the set of distinct lengths of the cycles of D and we say that $L(D)$ is the cycle length set of D. The following two results are well-known.

Lemma 1.1. ([3]) *A digraph D is primitive iff D is strong connected and* $\gcd(r_1,r_2,\ldots,r_\lambda)=1$, *where* $L(D)=\{r_1,r_2,\ldots,r_\lambda\}$.

Lemma 1.2. *If D is a primitive digraph, then*

$$\gamma(D)=\max\{\gamma(u,v)\,|\,u,v\in V(D)\}.$$

Let D be a primitve digraph and $R=\{r_{i_1},r_{i_2},\ldots,r_{i_t}\}\subseteq L(D)$ such that $\gcd(r_{i_1},r_{i_2},\ldots,r_{i_t})=1$. For any ordered pair of vertices u,v of D, we define that the *relative distance* with R from u to v, denoted by $d_R(u,v)$, is the length of the shortest walk from u to v which meets at

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least one cycle of length r_{i_j} for $j = 1,2,\ldots,t$.

Suppose $\{r_1,r_2,\ldots,r_\lambda\}$ is a set of distinct positive integers with gcd $(r_1,r_2,\ldots,r_\lambda\} = 1$. Then we define $\varphi(r_1,r_2,\ldots,r_\lambda)$ to be the least integer m such that every integer $k \geqslant m$ can be expressed in the form $k = c_1r_1 + c_2r_2 + \ldots + c_\lambda r_\lambda$, where $c_1,c_2,\ldots,c_\lambda$ are some nonnegative integers. A result due to Schur shows that $\varphi(r_1,r_2,\ldots,r_\lambda)$ is well defined if gcd $(r_1,r_2,\ldots,r_\lambda) = 1$. When $\lambda = 2$, $\varphi(r_1,r_2) = (r_1 - 1)(r_2 - 1)$, where gcd $(r_1,r_2) = 1$. Roberts [7] has shown that if $a_j = a_0 + jd$, $j = 0,1,2,\ldots,s$, $a_0 \geqslant 2$, then

$$\varphi(a_0,a_1,\ldots,a_s) = [\frac{a_0 - 2}{s} + 1]a_0 + (d - 1)(a_0 - 1), \tag{1.1}$$

where $[x]$ denotes the greatest integer $\leqslant x$.

The following result is well-known.

Lemma 1.3 ([8]) *Let D be a primitive digraph and $R = \{r_{i_1},r_{i_2},\ldots,r_{i_t}\} \subseteq L(D) = \{r_1, r_2,\ldots,r_\lambda\}$ with* gcd$(r_{i_1},r_{r_2},\ldots,r_{i_t}) = 1$. *Then for all ordered pairs of $u,v \in V(D)$, we have*

$$\gamma(u,v) \leqslant d_R(u,v) + \varphi(r_{i_1},r_{r_2},\ldots,r_{i_t})$$

and

$$\gamma(D) \leqslant \max_{u,v \in V(D)} d_R(u,v) + \varphi(r_{i_1},r_{r_2},\ldots,r_{i_t}).$$

Lemma 1.4. *Let x and y be any ordered pair of vertices of primitive digraph D. If there exist walks $P_1(x,y)$ and $P_2(x,y)$ with $l(P_1(x,y)) - l(P_2(x,y)) \equiv 1$* (mod 2) *where $l(P_i(x,y))$ is the length of $P_i(x,y)$, and $P_i(x,y)$ meets at least a 2-cycle for $i = 1,2$, then*

$$\gamma(x,y) \leqslant \max\{l(P_1(x,y)),l(P_2(x,y))\} - 1.$$

Proof. Let $a = l(P_1(x,y))$, $b = l(P_2(x,y))$ or $l \geqslant \max\{a,b\} - 1$, then $l - a$ or $l - b$ is an even integer, say $l - a$. We add the 2-cycle to $P_1(x,y)$ by $\frac{l-a}{2}$ times and get a new walk of length l from x to y. Hence

$$\gamma(x,y) \leqslant \max\{l(P_1(x,y)),l(P_2(x,y))\} - 1. \square$$

Corollary 1.5. *Let x and y be any ordered pair of vertices of primitive digraph D. There exist walks $P_i(x,y)$, of length $t+i$ from x to y for $i = 0,1,2,\ldots,m$, where $m \geqslant 2$. If there are two integers $a_0,b_0 \in \{t,t+1,\ldots,t+m\}$ such that $a_0 - b_0 \equiv 1$* (mod 2) *and both $P_{a_0-t}(x,y)$, $P_{b_0-t}(x,y)$ meet 2-cycle, and if there does not exist any walk of length $t-1$ from x to y, then we have*

$$\gamma(x,y) = t.$$

The proof of this corollary is obvious. $\square$

A *semicomplete digraph* is a digraph without nonadjacent vertices. A *Locally semicom-*

plete digraph is a digraph D satisfying the following condition: for every vertex $x \in V(D)$, $D[O(x)]$ and $D[I(x)]$ are semicomplete digraphs. We shall sometimes use the abbreviation Lsd to denote a locally semicomplete digraph. A *local tournament* is a locally semicomplete digraph without 2-cycles and loops.

Locally semicomplete digraphs were first introduced by Bang Jensen [1]. They are generalization of semicomplete digraphs and tournaments. Many of the classic theorems of tournaments have been generalized to Lsd. For example:

Lemma 1.6. ([1]) *Every connected Lsd has a directed Hamilton path and every strong Lsd has a Hamilton cycle.*

The properties of arc-pancyclicity and completely strong path-connectivity have been generalized to Lsd (see [2],[4]and [5]). Hence it is clear that Lsds form a new and interesting class. In this paper, we get some properties of cycles and determine the exponent set of primitive Lsds.

2. The Distribution of the Length of Cycles on LSDS

In the following we always suppose $D=(V,A)$ is a strong Lsd and $L(D)=\{r_1,r_2,\ldots,r_\lambda\}$ is a cycle length set of D where $r_1<r_2<\ldots<r_\lambda$. We say that a cycle C is *semicomplete* if $D[V(C)]$ is a semicomplete digraph. If $(x,y)\in A(D)$, then we say that x dominates y and we will use the notation $x\rightarrow y$ to denote this. If $S\subseteq V(D)$ such that $x\rightarrow y$ (resp., $y\rightarrow x$) for every $y\in S$ we will use the notation $x\rightarrow S$ (resp., $S\rightarrow x$) to denote this. For a walk $P(u_0,u_k)=u_0u_1u_2\ldots u_i\ldots u_k$ (resp., cycle $C=(u_0u_1\ldots u_ku_0)$), we will use the notation $u_iP(u_0,u_k)u_j$ (resp., u_iCu_j) to denote a walk along $P(u_0,u_k)$ (resp., C) from u_i to u_j, and $[m,n]^0$ to denote a set of integers $\{m,m+1,\ldots,n\}$.

Lemma 2.1. ([1]) *Let D be a strong Lsd on n vertices. If $D\not\cong C_n$ and has no loop, then there exists a vertex x of D such that $D-x$ is strong.*

Corollary 2.2. *Let D be a strong Lsd on $n\geqslant 3$ vertices, then D is a primitive Lsd iff $|A(D)|>n$.*

Proof. If D is a primitive Lsd, then D contains an n-cycle and there exists a r-cycle where $r<n$ by Lemma 1.6 and Lemma 1.1, therefore $|A(D)|>n$. Otherwise let $|A(D)|>n$, thus $D\not\cong C_n$. If D contains a loop, then $L(D)=\{1,r_2,\ldots,r_{\lambda-1},n\}$. So that gcd $(1,r_2,\ldots,r_{\lambda-1},n)=1$ and D is primitive by Lemma 1.1. Suppose $r_1>1$, there exists an $x\in V(D)$ such that $D-x$ is strong by Lemma 2.1. By the definition of Lsd, $D-x$ is a Lsd, thus we have $r_{\lambda-1}=n-1$ and gcd $(r_1,r_2,\ldots,r_{\lambda-1},n)=1$. So D is a primitive of Lsd. □

Lemma 2.3. *Let D be a strong Lsd on $n\geqslant 3$ vertices and $C_k=(u_0u_1\ldots u_{k-1}u_0)$ is a non-semicomplete k-cycle $(3<k<n)$. Then*

(1) *there exists an i_0, $1\leqslant i_0\leqslant k$ such that $u_{i_0-1}\rightarrow x\rightarrow u_{i_0}$ for any $x\in V(D)\backslash V(C_k)$, where $u_k=u_0$. Particularly, if C_k is a shortest cycle of D and $k\geqslant 5$, there are at most three arcs between*

x and C_k;

(2) *there exists a r-cycle C_r such that $V(C_k)\subset V(C_r)$ for $r=k+1,\dots,n$;*

(3) *for all ordered pairs of vertices $x,y\in V(D)$, there is a path $P(x,y)$ from x to y with length at most $k+1$ which meets at least one cycle of length r for $r=k$, $k+1,\dots,n$ and $V(P(x,y))-\{x,y\}\subseteq V(C_k)$.*

Proof. (1) Let $x_0\in V(D)\backslash V(C_k)$. If there is at least one arc between x_0 and C_k, without loss of generality, let $x_0\rightarrow u_{j_0}$ for some $j_0, 0\leqslant j_0\leqslant k-1$. Suppose (1) is false for x_0. By $u_{j_0-1}\rightarrow u_{j_0}$ and the definition of Lsd, x_0 and u_{j_0-1} are adjacent. If $u_{j_0-1}\rightarrow x_0$, (1) is true, this is a contradiction. So $x_0\rightarrow u_{j_0-1}$. Similarly, we can get that $x_0\rightarrow u_{j_0-2},\dots,x_0\rightarrow u_{j_0+1}$, where the subscript is module k. That is $x_0\rightarrow C_k$, thus C_k is semicomplete by the definition of Lsd. This contradicts the assumption of C_k. So there exists a $0\leqslant i_0\leqslant k-1$ such that $u_{i_0-1}\rightarrow x_0\rightarrow u_{i_0}$ in D.

Now, we suppose there is no arc between x_0 and C_k. Let $P=x_0x_1\dots x_t$ be a shortest path from x_0 to C_k where $x_t=u_j, 0\leqslant j\leqslant k-1$ and $t\geqslant 2$. Then x_{t-2} does not dominate u_i for $i=0,1,\dots,k-1$. Hence there is no arc between x_{t-2} and C_k; otherwise there is an i_0, $0\leqslant i_0\leqslant k-1$, such that $u_{i_0-1}\rightarrow x_{t-2}\rightarrow u_{i_0}$, a contradiction. On the other hand, since $x_{t-1}\rightarrow u_j$, we can get $0\leqslant i_1\leqslant k-1$ such that $u_{i_1-1}\rightarrow x_{t-1}\rightarrow u_{i_1}$ as above. Thus we have that x_{t-2} and u_{i_1-1} are adjacent by $x_{t-2}\rightarrow x_{t-1}, u_{i_1-1}\rightarrow x_{t-1}$ and the definition of Lsd, a contradiction. Hence there is no vertex x in $V(D)-V(C_k)$ such that there is no arc between x and C_k. So the first part of (1) is true.

Suppose C_k is a shortest cycle of D and there are at least four arcs between x and C_k for some $x\in V(D)-V(C_k)$. Then we easily get a r-cycle for a certain $r<k$. This is a contradiction. This completes the proof of (1).

(2) and (3) easily follow from (1). □

Theorem 2.4. *Let D be a primitive Lsd on n vertices without loop. $L(D)=\{r_1,r_2,\dots,r_\lambda\}$ is the cycle length set of D, where $r_1<r_2<\dots<r_\lambda$. Then the structure of $L(D)$ is only one of the following cases:*

(1) *$L(D)=\{s,s+1,\dots,n\}$, where $3\leqslant s\leqslant n-1$;*

(2) *$L(D)=\{2,s,s+1,\dots,n\}$, where $3\leqslant s\leqslant n-1$;*

(3) *$L(D)=\{s,s+1,\dots,t,k,k+1,\dots,n\}$, where $2\leqslant s\leqslant 3, 3\leqslant t\leqslant\left[\frac{n-1}{2}\right]$ and $t+2\leqslant k\leqslant n-t+1$.*

Proof.

Case 1. $r_1=s\geqslant 4$.

Let C_s be an s-cycle, then C_s is non-semicomplete. By Lemma 2.3 (2), there exists a r-cycle in D for $r=s,s+1,\dots,n$. Hence $L(D)=\{s,s+1,\dots,n\}$.

Case 2. $r_1=s=3$

If $L(D)\neq\{3,4,\dots,n\}$, let $t=\max\{l\mid$ there exists r-cycle in D for $r=3,4,\dots,l\}$ and

$k = \max\{l \mid l > t$ and there is a l- cycle in $D\}$.

Obviously, $\{3, \dots, t, k\} \subseteq L(D)$, $t+1 \notin L(D)$ and $k \geqslant t+2$. Let C_k be a k- cycle, then C_k is non-semicomplete since $k-1 \notin L(D)$. By Lemma 2.3 (2), D contains $(k+1)$-, $(k+2)$-, $\dots$, n- cycle. So $L(D) = \{3, \dots, t, k, k+1, \dots, n\}$. Now, we shall show that $t \leqslant \left[\frac{n-1}{2}\right]$ and $k \leqslant n-t+1$.

Let C_t be a t- cycle, by Lemma 2.3 (2) and $t+1 \notin L(D)$, C_t is semicomplete. Since D is strong, there exist x and y in $D - V(C_t)$ such that x is dominated by a vertex on C_t and y dominates a vertex on C_t. By the definition of Lsd and $t+1 \notin L(D)$, we have $C_t \to x$ and $y \to C_t$. Let $x_0, y_0 \in V(D)\backslash V(C_t)$ such that

$$d(x_0, y_0) = \min\{d(x, y) \mid y \to C_t \to x.\ x, y \in V(D)\backslash V(C_t)\}.$$

Let $P(x_0, y_0) = x_0 x_1 \dots x_m$ be a shortest path from x_0 to y_0, where $x_m = y_0$, and $d(x_0, y_0) = m$. Then $V(P(x_0, y_0)) \cap V(C_t) = \varnothing$. Otherwise, we suppose $\{x_0, x_1, \dots, x_{i_0}\} \cap V(C_t) = \varnothing$ and $x_{i_0+1} \in V(C_t)$, we may substitute x_{i_0} for y_0, a contradiction to the choice of y_0. Now, by $y_0 \to C_t \to x_0$ and $V(P(x_0, y_0)) \cap V(C_t) = \varnothing$, we can get r- cycle in D for $r = m+2, m+3, \dots, m+t+1$. Hence $t+2 \leqslant k \leqslant m+2$ and $m+1+t \leqslant n$, that is $t \leqslant \left[\frac{n-1}{2}\right]$ and $k \leqslant n-t+1$.

Case 3. $r_1 = 2$.

As shown above, we can prove that $L(D)$ will be the case (2) or (3).

This completes the proof of Theorem. □

In the following, we always suppose that the digraph has no loop.

Corollary 2.5. *Let D be a primitive Lsd on $n \geqslant 4$ vertices. Then $|L(D)| \leqslant 2$ iff D is $D_{n,n-1}$ or $\overline{D}_{n,n-1}$* (see Fig. 1).

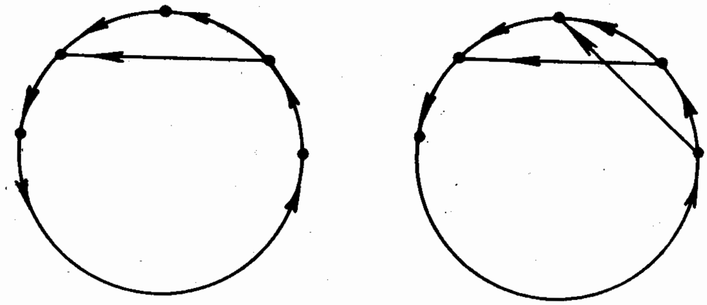
Proof. Clearly, if D is $D_{n,n-1}$ or $\overline{D}_{n,n-1}$, then $|L(D)| = 2$. Otherwise, suppose $|L(D)| \leqslant 2$. Since D is a primitive Lsd, $|L(D)| \geqslant 2$ and D contains a Hamiltonian cycle, that is $L(D) = \{r_1, n\}$ with $r_1 < n$. So $L(D) = \{n-1, n\}$ by Theorem 2.4. Thus D must be $D_{n,n-1}$ or $\overline{D}_{n,n-1}$. This completes the proof. □

Theorem 2.6. *Let D be a primitive Lsd on n vertices, $L(D) = \{r_1, r_2, \dots, r_\lambda\}$. Then*

(1) $\varphi(s, s+1, \dots, n) = s\left[\frac{n-2}{n-s}\right]$, $\quad 4 \leqslant s \leqslant n-1$;

(2) $\varphi(2, s, s+1) = \begin{cases} s, & \text{if } s \text{ is even}, \\ s-1, & \text{if } s \text{ is odd}, \end{cases} \quad (4 \leqslant s \leqslant n-1)$;

(3) $\varphi(3, 4, \dots, n) = \begin{cases} 3, & \text{if } n \geqslant 5, \\ 6, & \text{if } n = 4; \end{cases}$



$D_{n,n-1}$ $\overline{D}_{n,n-1}$

Fig. 1

(4) $\varphi(2,3,4,\ldots,n)=2$;

$$\text{(5) } \varphi(3,\ldots,t,k,k+1,\ldots,n)=\begin{cases}3, & \text{if } t\geqslant 5,\\ 6, & \text{if } t=4,\\ k-1, & \text{if } t=3 \text{ and } k\equiv 1 \pmod 3,\\ k, & \text{if } t=3 \text{ and } k\not\equiv 1 \pmod 3,\end{cases}$$

where $k\leqslant n-2$;

(6) $\varphi(2,3,\ldots,t,k,k+1,\ldots,n)=2$, $t\geqslant 3$.

Proof. By (1.1), we can easily get that (1),(3) and (4) are true and check that (2),(5) and (6) are true too. □

3. The Gaps of Primitive LSDS

Theorem 3.1. *Let D be a primitive Lsd on n vertices, $L(D)=\{s,s+1,\ldots,n\}$, where $3\leqslant s\leqslant n-1$. Thus*

(1) *If $s\geqslant 4$, then $\gamma(D)\leqslant s+1+s\left[\frac{n-2}{n-s}\right]$ and there is a primitive Lsd $D_{n,s}$ such that $\gamma(D_{n,s})=s+1+s\left[\frac{n-2}{n-s}\right]$;*

(2) *If $s=3,n\geqslant 5$, then $\gamma(D)\leqslant n+4$.*

Proof. (1) Let C_s be an s-cycle of D. Since s is a shortest length of cycle, C_s is non-semicomplete. By Lemma 2.3 (3), for any ordered pair of vertices $x,y\in V(D)$, there exists a path $P(x,y)$ from x to y with length at most $s+1$ which meets at least one C_r for $r=s,s+1,\ldots,n$. Then

$$d_{L(D)}(x,y)\leqslant l(P(x,y))\leqslant s+1.$$

So

$$\gamma(D) \leqslant \max_{x,y\in V(D)} d_{L(D)}(x,y) + \varphi(s,s+1,\ldots,n) \leqslant s+1+s\left[\frac{n-2}{n-s}\right].$$

We denote $D_{n,s}$ to be the digraph with $V(D_{n,s}) = \{x_0, x_1, \ldots, x_{n-1}\}$ and the arc set as follows: $A(D_{n,s}) = \{(x_i, x_{i+1}), i = 0,1,\ldots,n-1\} \cup \{(x_i, x_j): i = 0,1,\ldots,n-s-1, j = i+2,\ldots,n-s+1\}$, where $x_n = x_0$.

We easily see that $D_{n,s}$ is a primitive Lsd with

$$L(D_{n,s}) = \{s, s+1, \ldots, n\} \text{ and } d_{L(D_{n,s})}(x_{n-s}, x_1) = d(x_{n-s}, x_1) = s+1.$$

Since there is a single path from x_{n-s} to x_1, we can easily check that

$$\gamma(x_{n-s}, x_1) = s+1+\varphi(s,s+1,\ldots,n)$$
$$= s+1+s\left[\frac{n-2}{n-s}\right].$$

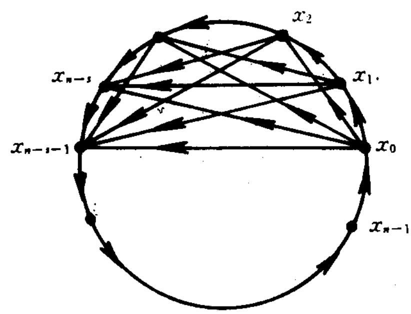


Fig. 2 $D_{n,s}$

Hence

$$\gamma(D_{n,s}) \geqslant \gamma(x_{n-s}, x_1) = s+1+s\left[\frac{n-2}{n-s}\right]$$

and

$$\gamma(D_{n,s}) = s+1+s\left[\frac{n-2}{n-s}\right].$$

(2) If $s = 3$, $L(D) = \{3,4,\ldots,n\}$, let $R = \{3,4,5\} \subseteq L(D)$. x and y are any ordered pair of vertices. Let $P(x,y)$ be a shortest (x,y)-path and let C_t be a maximal cycle which is semicomplete. Without loss of generality, we assume that $t < n-1$, then $D(V(C_t))$ is a semicomplete digraph and is vertex-pancyclic. Furthermore, any r-cycle is non-semicomplete with $r \geqslant t+1$.

If $d(x,y) \geqslant n-t$, then $P(x,y)$ meets C_t and at least one cycle of length r for $r = t+1, t+2, \ldots, n$. Hence

$$d_R(x,y) = l(P(x,y)) \leqslant n-1.$$

If $d(x,y) = n-t-1$, $P(x,y)$ meets a $(t+1)$-cycle C_{t+1}. Since C_{t+1} is non-semicomplete, by Lemma 2.3, we can extend C_{t+1} to $(t+2)$-cycle C_{t+2} containing a vertex u of C_t (if $V(C_t) \subseteq V(C_{t+1})$, we take $u \in V(D) - V(C_{t+1})$). Then C_{t+2} meets at least one cycle of length r for $r = 3,4,\ldots,t+1$ and $P(x,y)$. Hence

$$d_R(x,y) \leqslant l(P(x,y)) + l(C_{t+2}) = n-t-1+t+2 = n+1.$$

Now, we assume $d(x,y) = n-k \leqslant n-t-2$. Since $k-1 \geqslant t+1$, every $(k-1)$-cycle in D is non-semicomplete, we can get a $(k-1)$-cycle C_{k-1} such that $V(C_{t+1}) \subseteq V(C_{k-1})$ by Lemma 2.3. Hence C_{k-1} meets at least one cycle of length r for $r = t+1, \ldots, n$.

Case 1. C_{k-1} does not meet any t- cycle.

By Lemma 2.3, we can extend C_{k-1} to a k- cycle C_k containing a vertex of C_t. Then C_k meets at least one cycle of length r for $r = 3, 4, \ldots, n$ and $P(x, y)$. Hence

$$d_R(x, y) \leqslant l(P(x, y)) + l(C_k) = n.$$

Case 2. C_{k-1} meets a t- cycle C_t.

By Lemma 2.3, we can extend C_{k-1} to k- cycle C_k containing x, thus

$$d_R(x, y) \leqslant l(P(x, y)) + l(C_k) = n.$$

Hence for any ordered pair of vertices x and y, we have $d_R(x, y) \leqslant n + 1$, that is

$$\gamma(x, y) \leqslant d_R(x, y) + \varphi(3, 4, 5) \leqslant n + 4.$$

Thus

$$\gamma(D) \leqslant n + 4.$$

This completes the proof of Theorem. □

Theorem 3.2. *Let D be a primitive Lsd on n vertices with $L(D) = \{2, s, s+1, \ldots, n\}$, where $n \geqslant 6$ and $3 \leqslant s \leqslant n-1$.*

(1) *When $s \geqslant 4$, then $\gamma(D) \leqslant 2n-4$. Furthermore, there is a primitive Lsd $D'_{n,s}$ such that*

(*a*) *if $s \leqslant n-4$, then $\gamma(D'_{n,s}) = 2s+1$;*

(*b*) *if $s = n-i$, then $\gamma(D'_{n,s}) = n+s-3$ for $i = 1, 2, 3$.*

(2) *When $s = 3$, then $\gamma(D) \leqslant n+4$.*

Proof. (1) Let C_s be an s- cycle in D. Since $s - 1 \notin L(D)$, C_s is non-semicomplete.

If $s = n - 1$, then D must be $D'_{n,n-1}$ (see Fig. 3). It is easily to check $\gamma(D'_{n,n-1}) = 2n - 4$.

If $s \leqslant n - 2$, by Lemma 2.3, there are two cycles C_{s+1} and C_{s+2} which meet 2-cycle and $V(C_s) \subseteq (C_{s+i})$ for $i = 1, 2$. For any ordered pair of vertices x and y, let $\overline{P}(x, y)$ be an (x, y)- path as mentioned in Lemma 2.3 (3) on C_s. Then $\overline{P}(x, y)$ meet C_{s+i} for $i = 0, 1, 2$, and $l(\overline{P}(x, y)) \leqslant s + 1$. Let

$$P_i(x, y) = \overline{P}(x, y) \cup C_{s+i} \qquad \text{for } i = 0, 1, 2.$$

Then $P_i(x, y)$ is an (x, y) -walk of length $l(P(x, y)) + s + i$ for $i = 0, 1, 2$, and $P_1(x, y)$, $P_2(x, y)$ meet at least one cycle of length 2. By Corllary 1.5.

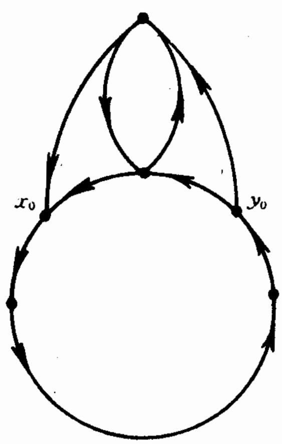


Fig. 3 $D_{n,n-1}$

$$\gamma(x, y) \leqslant l(P_0(x, y)) = l(\overline{P}(x, y)) + s. \tag{3.1}$$

Since $l(\overline{P}(x, y) \leqslant s + 1$, we have $\gamma(x, y) \leqslant 2s + 1$.

Hence $\gamma(D) \leqslant 2s + 1$.

Case 1. $s \leqslant n - 3$. Then $\gamma(D) \leqslant 2n - 5 < 2n - 4$.

Case 2. $s = n - 2$. We shall show that $\gamma(x,y) \leqslant 2s - 1$ for all ordered pairs of vertices $x, y \in V(D)$. In fact, if $l(\bar{P}(x,y)) \leqslant s - 1$, then $\gamma(x,y) \leqslant l(\bar{P}(x,y)) + s \leqslant 2s - 1$ by the form (3,1).

Hence, in the following we suppose $l(\bar{P}(x,y) \geqslant s$.

When $x, y \in V(C_s)$, then $x = y$. By Lemma 2.3 and Corollary 1.5, we easily check that

$$\gamma(x,y) = s < 2s - 1.$$

Hence, without loss of generality, we assume that $x \notin V(C_s)$, then there exists $i_0 (1 \leqslant i_0 \leqslant s)$ such that $u_{i_0 - 1} \to x \to u_{i_0}$ by Lemma 2.3.

Subcase 2.1. C_s does not meet a cycle of length 2.

Let $C_2 = (uvu)$ be a cycle of length 2 in D. Without loss of generality, we assume that $u_{s-1} \to u \to u_0$, then $u_{s-1} \to v \to u_0$ by the definition of Lsd and $3 \notin L(D)$. Since $x \notin V(C_s)$ and $x \in \{u, v\}$, we easily obtain a walk $P_1(x,y)$ from x to y of length $l(\bar{P}(x,y)) + 1$. By Lemma 1.4

$$\gamma(x,y) \leqslant l(\bar{P}(x,y)) \leqslant s + 1 < 2s - 1.$$

Subcase 2.2. C_s meets a cycle of length 2.

Let $C_2 = (uvu)$ be a cycle of length 2 and meets C_s. Without loss of generality, we assume $u = u_0$, then $u_{s-1} \to v \to u_1$ by the definition of Lsd and $3 \notin L(D)$.

If $x = v$ or $y = v$ or $x = y$, we easily get that a walk $P_1(x,y)$ of length $l(\bar{P}(x,y)) + 1$ meets a cycle $C_2 = (uvu)$. Thus by Lemma 1.4, we have

$$\gamma(x,y) \leqslant l(\bar{P}(x,y)) \leqslant s + 1.$$

If $x \neq v, y \neq v$ and $x \neq y$, then $y \in V(C_s)$. Since $l(\bar{P}(x,y)) \geqslant s, y = u_{i_0 - 1}$ and $\bar{P}(x,y) = xu_{i_0}u_{i_0+1}\ldots u_{i_0-1}$. Put that if $i_0 \neq 0, 1, P_1(x,y) = xu_{i_0}u_{i_0+1}\ldots u_0vu_1\ldots u_{i_0-1}$, if $i_0 = 1, P_1(x,y) = xu_1u_2\ldots u_{s-1}vu_0$, where $u_0 = y$ or if $i_0 = 0, P_1(x,y) = xu_0vu_1u_2\cdots u_{s-1}$, where $u_{s-1} = y$.

Thus $P_1(x,y)$ meets a cycle $C_2 = (uvu)$ with $l(P(x,y)) = l(\bar{P}(x,y)) + 1$. So by Lemma 1.4, we have

$$\gamma(x,y) \leqslant l(\bar{P}(x,y)) \leqslant s + 1 < 2s - 1.$$

Hence, for all ordered pairs of vertices $x, y \in V(D)$ we have

$$\gamma(x,y) \leqslant 2s - 1 = 2n - 5.$$

Thus

$$\gamma(D) \leqslant 2n - 5.$$

So the first part of (1) is true.

When $s \leqslant n-4$, let $D'_{n,s}$ be the resulting digraph of $D_{n,s}$ in Fig. 2 with an adding arc (x_3, x_2), then $D'_{n,s}$ is a primitive Lsd with $L(D'_{n,s}) = \{2, s, s+1, \ldots, n\}$. By Corollary 1.5, it is easy to check that

$$\gamma(x_{n-s}, x_1) = 2s+1 \text{ in } D'_{n,s}.$$

Hence $\gamma(D'_{n,s}) \geqslant 2s+1$ and $\gamma(D'_{n,s}) = 2s+1$.

When $s = n-3$ or $n-2$, $D'_{n,n-i}, i = 2,3$, are described in Fig. 4. By Corollary 1.5, it is easy to check that

$$\gamma(x,y) \leqslant \gamma(x_0, y_0) = 2n - i - 3,$$

for any ordered pair of vertices $x, y \in V(D'_{n,n-i}), i = 2,3$. Hence by Lemma 1.2 $\gamma(D'_{n,n-i}) = 2n - i - 3$ for $i = 2,3$.

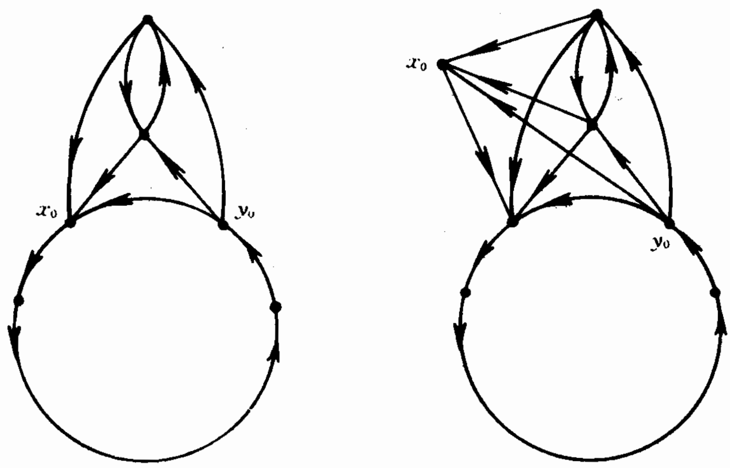


Fig. 4

(2) When $s = 3$, the proof is similar to the proof of Theorem 3.1 (2).

This completes the proof of Theorem. □

In order to consider the exponent of D with the structure of $L(D)$ of Theorem 2.4 (3), we need the following Lemma 3.3.

Lemma 3.3. *Let D be a primitive digraph with a 3-cycle. For $x, y \in V(D)$, there exist (x, y)-walks $P_i(x,y)$ such that $l(P_i(x,y)) = r_0 + i$ for $i = 0, 1, \ldots, t$ and $r_0 > 0$. If $t \geqslant 2$ and $P_i(x,y)$ meet at least one 3-cycle for $i = t-2, t-1, t$, then*

$$\gamma(x,y) \leqslant l(P_0(x,y)) = r_0.$$

Proof. For any integer $m \geqslant r_0$, D has an (x,y)-walk of length m as $m \leqslant t + r_0$. If $m > r_0 +$

t, let

$$m - (r_0 + t - 2) = 3k + b,$$

where $0 \leqslant b \leqslant 2$, that is

$$m = 3k + (r_0 + t - 2 + b).$$

Now, adding a 3-cycle C_3 to $P_{t-2+b}(x, y)$ by k times, we get a new walk of length m from x to y. So

$$\gamma(x, y) \leqslant r_0 = l(P_0(x, y)). \square$$

Theorem 3.4. *Let D be a primitive Lsd on n vertices, $L(D) = \{s, s+1, \ldots, t, k, k+1, \ldots, n\}$, where $2 \leqslant s \leqslant 3$, $3 \leqslant t \leqslant \left[\frac{n-1}{2}\right]$, $t+2 \leqslant k \leqslant n-t+1$ and $n \geqslant 6$, then*

$$\gamma(D) \leqslant 2n - 4.$$

Furthermore, when $k + t = n - 1$, there is a primitive Lsd $D_{n,t,k}$ such that $L(D_{n,t,k}) = \{3, \ldots, t, k, k+1, \ldots, n\}$ and

$$\gamma(D_{n,t,k}) = 2k.$$

Proof. First we consider $s = 3$.

Let $C_k = (u_0 u_1 \ldots u_{k-1} u_0)$, and let x and y be any ordered pair of vertices in D. $\bar{P}(x, y)$ is an (x, y)-path as mentioned in Lemma 2.3 on C_k, Then $\bar{P}(x, y)$ meets at least one cycle of length r for $r = k, k+1, \ldots, n$ and $d_0 = l(\bar{P}(x, y)) \leqslant k + 1$.

Case 1. C_k does not meet any 3-cycle.

Let $C_3 = (x_1 x_2 x_3 x_1)$ be a 3-cycle. Since C_k in non-semicomplete, by Lemma 2.3 we can extend C_{k+i-1} to a $(k+i)$- cycle C_{k+i} containing x_i for $i = 1, 2, 3$. Thus

$$P_i(x, y) = \bar{P}(x, y) \cup C_{k+i}$$

is an (x, y)- walk of length $d_0 + k + i$ for $i = 0, 1, 2, 3$. Clearly, $P_1(x, y)$, $P_2(x, y)$ and $P_3(x, y)$ meet the 3-cycle C_3. By Lemma 3,3

$$\gamma(x, y) \leqslant d_0 + k \leqslant 2k + 1.$$

Case 2. C_k meets at least one 3-cycle.

Since $k \leqslant n - 2$ and C_k is non-semicomplete, similarly, we can prove that

$$\gamma(x, y) \leqslant d_0 + k \leqslant 2k + 1.$$

Hence

$$\gamma(D) \leqslant 2k + 1.$$

When $k \leqslant n - 3$, then $\gamma(D) \leqslant 2k + 1 \leqslant 2n - 5$.

In the following we shall show that $\gamma(D) \leqslant 2n - 4$ when $k = n - 2$.

We first prove that:

$$\text{there is no chord on } C. \tag{3.2}$$

Since $k = n - 2, t = 3$. If there is a chord on C_k, without loss of generality, let $u_{r-1} \to u_0$, where $3 \leqslant r \leqslant k-1$. Since $L(D) = \{3, n-2, n-1, n\}$, $r = 3$ and $u_2 \to u_0$. By the definition of Lsd, u_2 and u_{k-1}, u_0 and u_3 are adjacent in D, it must be $u_{k-1} \to u_2$ and $u_0 \to u_3$ since $4 \notin L(D)$. Thus the length of cycle $(u_0u_3u_4\cdots u_{k-1}u_2u_0)$ is $k-1$, this contradicts $k-1 \notin L(D)$. Hence there is no chord in C_k. Thus every arc in C_k does not lie on 3-cycle. Otherwise, there is a $C_3 = (u_iu_{i+1}uu_i)$, thus $u \notin C_k$. By (3.2) and the definition of Lsd, $u_{i+2}u \in A(D)$. Thus we get a $C_4 = (u_iu_{i+1}u_{i+2}uu_i)$, a contradiction. Since $k = n-2$, without loss of generality, we may assume that a 3-cycle is $C_3 = (u_0vwu_0)$. Thus $u_{k-1} \to w$ or $u_{k-2} \to w \to u_{k-1}$ (similarly, $v \to u_1$ or $u_1 \to v \to u_2$) by (3.2). If $u_{k-2} \to w \to u_{k-1}$, then u_{k-2} and v are adjacent and $v \to u_{k-2}$ by (3.2). Thus $(u_0vu_{k-2}u_{k-1}u_0)$ is a 4-cycle in D. This contradicts $4 \notin L(D)$. Therefore $u_{k-1} \to w$. Similarly, we have that $v \to u_1, u_{k-1} \to v, w \to u_1$ and $D \supseteqq D''$ (see Fig. 5). We easily check that

$$\gamma(D) \leqslant \gamma(D'') = 2k - 2 = 2n - 6.$$

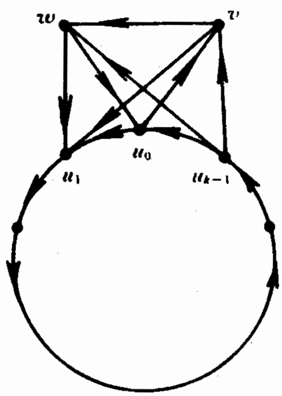


Fig. 5 D''

When $s = 2$. Let $T(D)$ be a maximal local tournament as a subdigraph of D, then $L(T(D)) = L(D)\backslash\{2\} = \{3, 4, \ldots, t, k, k+1, \ldots, n\}$ and

$$\gamma(D) \leqslant \gamma(T(D)) \leqslant 2n - 4.$$

So the first part of Theorem is true.

When $k + t = n - 1$, $D_{n,t,k}$ (see Fig. 6) is defined to be the digraph with the veretx set $V(D_{n,t,k}) = \{u_1, u_2, \ldots, u_t, x_0, x_1, \ldots, x_k\}$ and the following set of arcs:

(a) Let $D_{n,t,k}[u_1, u_2, \ldots, u_t]$ be a strong tournament T_t;

(b) $x_i \to x_{i+1}$ for $i = 0, 1, \ldots, k-1$;

(c) $x_k \to \{u_1, u_2, \ldots, u_t, x_0, x_1\}$ and $\{u_1, u_2, \ldots, u_t\} \to \{x_0, x_1\}$.

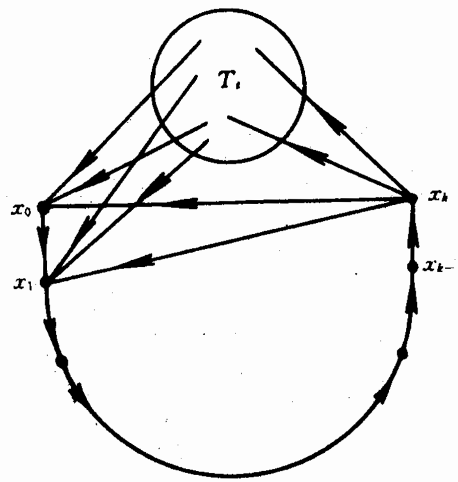


Fig. 6 $D_{n,t,k}$

Clearly, $D_{n,t,k}$ is a primitive Lsd and

$$L(D_{n,t,k}) = \{3, \ldots, t, k, k+1, \ldots, n\}.$$

By Lemma 3.3, we easily check that

$$\gamma(x,y) \leqslant \gamma(x_0,x_k) = 2k$$

for any ordered pair of vertices $x,y \in V(D_{n,t,k})$. Hence

$$\gamma(D_{n,t,k}) = 2k. \square$$

Lemma 3.5. *Let D be a primitive Lsd on $n \geqslant 6$ vertices. If $|L(D)| \geqslant 3$, then*

$$\gamma(D) \leqslant n - 1 + (n-2)\left[\frac{n-2}{2}\right].$$

Proof. By Theorems 3.1, 3.2 and 3.4, we easily check that Lemma 3.5 is true. □

Theorem 3.6. *For any primitive Lsd D on $n \geqslant 6$ vertices, $\gamma(D) \notin \left[[\frac{1}{2}w_n]+1, w_n-2\right]^0$, where $w_n = (n-1)^2+1$.*

Proof. Let D be a primitive Lsd on n vertices, then $|L(D)| \geqslant 2$. If $|L(D)| \geqslant 3$, by Lemma 3.5,

$$\gamma(D) \leqslant n - 1 + (n-2)\left[\frac{n-2}{2}\right] \leqslant \left[\frac{1}{2}w_n\right].$$

Hence, by Corollary 2.5, for any primitive Lsd D, we have

$$\gamma(D) \notin \left[[\frac{1}{2}w_n] + 1, w_n - 2\right]^0. \square$$

4. The Exponent Set of Primitive Locally Semicomplete Digraphs

Let LE_n be the exponent set of primitive Lsds on n vertices, and let $E_n(s)$ be the set of all primitive Lsds on n vertices with the length $s(\geqslant 4)$ of the shortest cycle.

Theorem 4.1. *For $n \geqslant 6$, $[2, 2n-4]^0 \cup \{w_n - 1, w_n\} \subseteq LE_n$, where $w_n = (n-1)^2+1$.*

Proof. By Corollary 2.5, we easily get $w_n - 1, w_n \in LE_n$. Since tournaments are Lsds, we have $[3, n+2]^0 \subseteq LE_n$ by [6]. Let K_n^* be a complete symmetric digraph with n vertices, then $\gamma(K_n^*) = 2$, that is $2 \in LE_n$.

From Theorems 3.2 and 3.4 we can get $[n+3, 2n-4]^0 \subseteq LE_n$.

Hence

$$[2, 2n-4]^0 \cup \{w_n - 1, w_n\} \subseteq LE_n. \square$$

Lemma 4.2. *Let D be a primitive Lsd on n $(\geqslant 8)$ vertices. If $L(D) \neq \{s, s+1, \ldots, n\}$ or*

$L(D)=\{s,s+1,\ldots,n\}$ with $2\leqslant s\leqslant\frac{n+2}{2}$, then $\gamma(D)\leqslant 2n-4$.

Proof. If $L(D)\neq\{s,s+1,\ldots,n\}$, then $\gamma(D)\leqslant 2n-4$ by Theorems 3.1, 3.2 and 3.4.

If $L(D)=\{s,s+1,\ldots,n\}$ with $2\leqslant s<\frac{n+2}{2}$, by Theorems 3.1 and 3.2,

$$\gamma(D)\leqslant\max\left\{s+1+s\left[\frac{n-2}{n-s}\right],n+4\right\}=n+4.$$

Hence

$$\gamma(D)\leqslant n+4\leqslant 2n-4.\ \square$$

By Lemma 4.2 and Corollary 2.5, we only need to consider the exponent of a primitive Lsd D with $L(D)=\{s,s+1,\ldots,n\}$ and $\frac{n+2}{2}\leqslant s\leqslant n-2$.

In the following, let $n\geqslant 6$, $\frac{n+2}{2}\leqslant s\leqslant n-2$ and $N_s=\{a_1s+a_2(s+1)+\cdots+a_{n-s+1}n\mid a_1,a_2,\ldots,a_{n-s+1}$ are nonnegative integers $\}$. And let $\varphi_s=\varphi(s,s+1,\ldots,n)$;

$s-2=k_1(n-s)+r_1$, where $0\leqslant r_1\leqslant n-s$;

$n=k_2(n-s)+r_2$, where $0\leqslant r_2<n-s$,

then $n-s=(k_2-k_1)(n-s)+r_2-r_1-2$. Hence $r_2=r_1+2$ or $r_2=r_1+2-(n-s)$.

Lemma 4.3. *For any n and s satisfying the above condition, we have*

(1) $\{\varphi_s-s,\varphi_s-s+1,\ldots,\varphi_s-s+\left[\frac{s-2}{n-s}\right](n-s)\}\subseteq N_s$;

(2) $\varphi_s-s+\left[\frac{s-2}{n-s}\right](n-s)+i\notin N_s$ *for* $i=1,2,\ldots,r_1+1=s-\left[\frac{s-2}{n-s}\right](n-s)-1$.

Proof. (1) First, $\varphi_s-s=s\left[\frac{n-2}{n-s}\right]-s=s\left[\frac{s-2}{n-s}\right]\in N_s$ since $s\geqslant\frac{n+2}{2}$.

For any integer i, $1\leqslant i\leqslant\left[\frac{s-2}{n-s}\right](n-s)$, there exists j_0, $1\leqslant j_0\leqslant\left[\frac{s-2}{n-s}\right]$ such that $i=(j_0-1)(n-s)+j$, where $0<j\leqslant n-s$. Then

$$\varphi_s-s+i=(\left[\frac{s-2}{n-s}\right]-j_0)s+(j_0-1)n+(s+j)\in N_s.$$

Hence

$$\{\varphi_s-s,\varphi_s-s+1,\ldots,\varphi_s-s+\left[\frac{s-2}{n-s}\right](n-s)\}\subseteq N_s.$$

(2) If there exists an integer i_0 with $1\leqslant i_0\leqslant s-\left[\frac{s-2}{n-s}\right](n-s)-1$ such that $\varphi_s-s+\left[\frac{s-2}{n-s}\right](n-s)+i_0=\left[\frac{s-2}{n-s}\right]n+i_0\in N_s$, then there are nonnegative integers $a_1,a_2,\ldots,a_{n-s+1}$ such that $\left[\frac{s-2}{n-s}\right]n+i_0=a_1s+a_2(s+1)+\cdots+a_{n-s+1}n$.

If $\sum_{j=1}^{n-s+1}a_j\leqslant\left[\frac{s-2}{n-s}\right]$, then $\left[\frac{s-2}{n-s}\right]n+i_0\leqslant(a_1+a_2+\cdots+a_{n-s+1})n\leqslant\left[\frac{s-2}{n-s}\right]n$. This contradicts $i_0 \geqslant 1$. Hence $\sum_{j=1}^{n-s+1} a_j \geqslant \left[\frac{s-2}{n-s}\right] + 1$, thus

$$\varphi_s - s + \left[\frac{s-2}{n-s}\right](n-s) + i_0 \geqslant (a_1 + a_2 + \cdots + a_{n-s+1})s$$

$$\geqslant s\left(\left[\frac{s-2}{n-s}\right] + 1\right) = s\left[\frac{s-2}{n-s}\right] + s = \varphi_s.$$

On the other hand,

$$\varphi_s - s + \left[\frac{s-2}{n-s}\right](n-s) + i_0$$

$$\leqslant \varphi_s - s + \left[\frac{s-2}{n-s}\right](n-s) + s - \left[\frac{s-2}{n-s}\right](n-s) - 1 = \varphi_s - 1.$$

This is a contradiction. So (2) is true. □

Lemma 4. 4. *Let $D \in E_n(s)$ and let x, y be any ordered pair of vertices. Then we have:*

(1) If there exist walks $P_l, P_{l+1}, \ldots, P_{l+r_1+1}$ from x to y of length $l, l+1, \ldots, l+r_1$ and $l+r_1+1$ respectively, and P_i meets at least one s-cycle for $i=l, l+1, \ldots, l+r_1+1$, then

$$\gamma(x,y) \leqslant \varphi_s - s + l.$$

(2) Let $L_D(x,y) = \{l(P(x,y)) \mid P(x,y)$ is a path from x to y in $D\}$ and $d = d(x,y)$. If $L_D(x,y) \subseteq \{d, d+1, \ldots, d+r_1\}$ and D has a path $P_d(x,y)$ of length d which meets at least one s-cycle, then

$$\gamma(x,y) = \varphi_s + d.$$

Proof. (1) By Lemma 2. 3, P_i meets at least one r-cycle for $r = s, s+1, \ldots, n$ and $i = l, l+1, \ldots, l+r_1+1$. Thus D has a walk from x to y of length $i+m$ for any $m \in N_s$ and $i = l, l+1, \ldots, l+r_1+1$. By Lemma 4. 3 (1), D has walks from x to y of length $\varphi_s - s + l, \varphi_s - s + l + 1, \ldots, \varphi_s - s + l + r_1 + 1, \ldots, \varphi_s - s + \left[\frac{s-2}{n-s}\right](n-s) + l + r_1 + 1 = \varphi_s - 1 + l$, respectively. On the other hand, for $i \geqslant 1, \varphi_s - 1 + i \in N_s, D$ has a walk from x to y of length $\varphi_s - 1 + i + l$. So

$$\gamma(x,y) \leqslant \varphi_s - s + l.$$

(3) By the condition of (2) and Lemma 2. 3, we have

$$\gamma(x,y) \leqslant \varphi_s + d.$$

If $\gamma(x,y) < \varphi_s + d$, then there exists a walk from x to y of length $\varphi_s + d - 1$. By the assumption of (2), there exist integers $i_0, 0 \leqslant i_0 \leqslant r_1$ and $m \in N_s$ such that $\varphi_s + d - 1 = (d + i_0) + m$. That is, $\varphi_s - 1 - i_0 = m \in N_s$. By Lemma 4. 3, we have

$$\varphi_s - 1 - i_0 \leqslant \varphi_s - s + \left[\frac{s-2}{n-s}\right](n-s) = \varphi_s - (r_1 + 2).$$

So $r_1 + 1 \leqslant i_0$, this contradicts $i_0 \leqslant r_1$. Hence $\gamma(x,y) = \varphi_s + d$. □

For $n \geqslant 6$ and $\frac{n+2}{2} \leqslant s \leqslant n-2$, let $C_s = (x_1, x_2, \ldots, x_s, x_1)$ be a cycle with the length s in $D \in E_n(s)$. By Lemma 2.3 (1), for any $u \in V'(C_s) = V(D) \backslash V(C_s)$, u is adjacent to C_s only in following three cases: there exists an i_0, $1 \leqslant i_0 \leqslant s$, such that $x_{i_0} \to u \to x_{i_0-1}$ or $\{x_{i_0-1}, x_{i_0}\} \to u \to x_{i_0+1}$ or $x_{i_0} \to u \to \{x_{i_0+1}, x_{i_0+2}\}$, satisfying that the subscript is taken modulo s. For convenience, the vertices, which are adjacent to u, are called the root vertices of u. Especially, x_{i_0}, x_{i_0+1} are called the main root vertices of u. Since C_s is the minimal cycle in D, $D[V'(C_s)]$ is an acyclic digraph. Thus by the definition of Lsd, the arcs of $D[V'(C_s)]$ must be in the same direction as with C_s.

Let $E'(s) = \{D | D \in E_n(s)$ and for any $u \in V(D) \backslash V(C_s)$, u has exactly three root vertices on $C_s\}$. Note that if $D \in E_n(s)$ and $D \notin E'_n(s)$, we can change D for $D' \in E'_n(s)$ by adding some arcs to D. Clearly, $\gamma(D) \geqslant \gamma(D')$. So $\min\{\gamma(D) | D \in E'_n(s)\} = \min\{\gamma(D) | D \in E_n(s)\}$. Hence it is enough to consider $D \in E'_n(s)$ if we only consider the minimal exponent problem in $E_n(s)$.

In the following, we always assume that $D \in E'_n(s)$. If a pair of $\{x, y\}$ does not lie on a common s-cycle in D, then x and y are adjacent with $d(x,y) = s$, and there is an s-cycle C_s such that $x \notin V(C_s)$ and $y \in V(C_s)$ or $x \in V(C_s)$ and $y \notin V(C_s)$ (see Fig. 7).

When $r_1 = n - s - 1$, thus $\{x, y\}$ satisfies the condition of Lemma 4.4 (2), and then $\gamma(x,y) = \varphi_s + s$, i.e. $\gamma(D) \geqslant \varphi_s + s$. By Lemma 4.5, this kind of graph is impossible to have the minimal exponent. So, we set them aside. For the remaining cases between x and y, they satisfy Lemma 4.4 (1) with $l = s$. Hence we only need to consider $\gamma(x,y)$, satisfying that the pair of $\{x, y\}$ lies on a common s-cycle C_s.

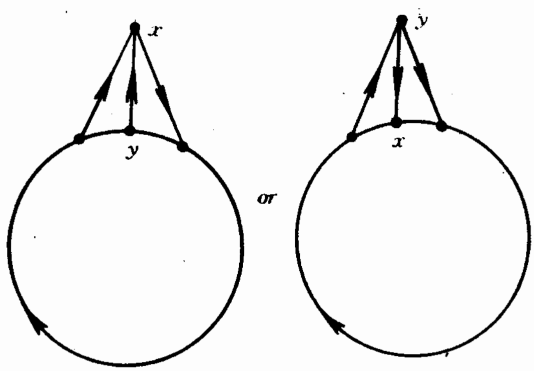


Fig. 7

Now, let $x, y \in V(C_s)$, thus the length of $x C_s y = d(x,y) = d_{L(D)}(x,y)$ by C_s being a shortest cycle in D. Let $A' = \{\{x,y\} | x, y \in D, x, y$ satisfy the condition of Lemma 4.4 (2) or the condition of Lemma 4.4 (1) with the least $l \geqslant s+2\}$, and let $\bar{d}_s(D) = \max_{\{x,y\} \in A'} \{d(x,y)\}$. Thus there is a $\{x_0, y_0\} \in A'$ with $d(x_0, y_0) = \bar{d}_s(D)$. If $\{x_0, y_0\}$ satisfies the condition of Lemma 4.4 (2), then we have $\gamma(x_0, y_0) = \varphi_s + d(x_0, y_0)$. Otherwise, $\{x_0, y_0\}$ satisfies the condition of Lemma 4. 4 (1) with the least $l \geqslant s+2$. If there is a (x_0, y_0)-walk with the length $\varphi_s + d(x_0, y_0) - 1$, then there exist a $m \in N_s$ and a (x_0, y_0)- walk $P(x_0, y_0)$ such that $\varphi_s + d(x_0, y_0) - 1 = l(P(x_0, y_0)) + m$. Let $l(P(x_0, y_0)) = d(x_0, y_0) + i_0$, thus

$$\varphi_s - 1 - i_0 \in N_s. \tag{4. 1}$$

By Lemma 4. 3, we have $i_0 > r_1$. Since $\{x_0, y_0\} \in A'$ and $d(x_0, y_0) = \bar{d}_s(D)$, the number of vertices of $V'(C_s)$, whose main root vertices lie on $x_0 C_s y_0$, is no more than γ_1. Hence i_0 must be the length of some (y_0, y_0)- closed walk. Thus we have $i_0 \in N_s$. By (4. 1), we also have $\varphi_s - 1 = \varphi_s - 1 - i_0 + i_0 \in N_s$, a contradiction. So, we always have $\gamma(x_0, y_0) = \varphi_s + d(x_0, y_0)$, i.e. $\gamma(D) \geqslant \varphi_s + d(x_0, y_0) = \varphi_s + \bar{d}_s(D)$. On the other hand, by Lemma 1. 3, we have $\gamma(D) = \max_{\{x,y\} \in A'} \{\gamma(x, y)\} \leqslant \varphi_s + \bar{d}_s(D)$. Therefore

$$\gamma(D) = \varphi_s + \bar{d}_s(D) \tag{4. 2}$$

for any $D \in E_n'(s)$.

Now, for n and s with $n \geqslant 6$ and $\frac{n+2}{2} \leqslant s \leqslant n-2$, we need to construct a special primitive Lsd with the minimal exponent in $E_n(s)$. Now, let $C_s = (x_1 x_2 \ldots x_s x_1)$ be an s- cycle, and we divide C_s into $n-s$ pieces such that the number of vertices of any piece is $\left[\frac{s}{n-s}\right]$ or $\left[\frac{s}{n-s}\right] + 1$. Since $r_2 = s - \left[\frac{s}{n-s}\right](n-s)$, the number of pieces with $\left[\frac{s}{n-s}\right] + 1$ vertices is exactly r_2. When $r_2 > 0$, r_2 pieces with $\left[\frac{s}{n-s}\right] + 1$ vertices are distributed in $n-s$ pieces on C_s as evenly as possible. This process is called $n-s$ well-distributed on C_s. Without loss of generality, we assume that the number of vertices of first piece is $\left[\frac{s}{n-s}\right] + 1$ as $r_2 > 0$.

In the $n-s$ well-distributed on C_s, let the vertex set of ith piece be $\{x_{k_i}, x_{k_i+1}, \ldots, x_{k_i+j_i}\}$, where $\left[\frac{s}{n-s}\right] - 1 \leqslant j_i \leqslant \left[\frac{s}{n-s}\right]$, $k_{i-1} + j_{i-1} + 1 = k_i$ for $i = 1, 2, \ldots, n-s$, and $k_0 = k_{n-s}$, $j_0 = j_{n-s}$, $k_1 = 1$, $k_{n-s} + j_{n-s} = s$.

Now let C_s have an $n-s$ well-distributed, and let $D(s)$ be the digraph with vertex set $V(D(s)) = V(C_s) \cup \{u_1, u_2, \ldots, u_{n-s}\}$ and the following set of arcs:

$A(D(s)) = A(C_s) \cup \{(x_{k_{i-1}+j_{i-1}}, u_i), (x_{k_i}, u_i), (u_i, x_{k_i+1}) \mid i = 1, 2, \ldots, n-s\} \cup \{(u_i, u_{i+1}) \mid \text{if } j_i = 0 \text{ for } 1 \leqslant i \leqslant n-s\}$, where $u_{n-s+1} = u_1$.

By (4. 2), it is easy to see that $D(s) \in E_n(S)$ and for every $D \in E_n(s)$,

$$\gamma(D(s)) \leqslant \gamma(D). \tag{4. 3}$$

In the following we consider the exponent of $D(s) \in E_n(s)$.

Lemma 4. 5. *For $D(s) \in E_n(s)$ with $\frac{n+2}{2} \leqslant s \leqslant n-2$ and $n \geqslant 6$, we have that*

(1) *If* $r_2 = r_1 + 2 - (n-s)$, *then* $\gamma(D(s)) = \varphi_s + (r_1+1)\left[\frac{s}{n-s}\right] + r_2$;

(2) *If* $r_2 = r_1 + 2$, *then* $\gamma(D(s)) = \varphi_s + (r_1+1)\left\{\frac{s}{n-s}\right\} - r_1 + i$ *as*

$$\frac{i(n-s)}{r_1+1} < r_2 \leqslant \frac{(i+1)(n-s)}{r_1+1} \text{ for } 0 \leqslant i \leqslant r_1.$$

Proof. (1) $r_2 = r_1 + 2 - (n-s)$. Since $r_1 \leqslant n-s-1$, $r_2 = r_1 + 2 - (n-s) \leqslant 1$ and $r_2 = 0$ or 1.

If $r_2 = 0$, then $r_1 = n-s-2$ and $(n-s)|s$. Hence the number of vertices of any piece in the $n-s$ well-disrtubted on C_s is $\frac{s}{n-s}$. Since $s \geqslant \frac{n+2}{2}$, $\frac{s}{n-s} > 1$. It is easy to check

$$\bar{d}_s(D(s)) = d(x_1, u_{n-s}) = \frac{s}{n-s}(r_1+1).$$

By (4.2) we have

$$\gamma(D(s)) = \varphi_s + (r_1+1)\frac{s}{n-s} = \varphi_s + (r_1+1)\left[\frac{s}{n-s}\right] + r_2.$$

If $r_2 = 1$, then $r_1 = n-s-1$, $s = \left[\frac{s}{n-s}\right](n-s) + 1$. It is easy to see that

$$\bar{d}_s(D(s)) = d(x_1, u_1) = s = \left[\frac{s}{n-s}\right](n-s) + 1.$$

Hence by (4.2) we have

$$\gamma(D(s)) = \varphi_s + (n-s)\left[\frac{s}{n-s}\right] + 1 = \varphi_s + (r_1+1)\left[\frac{s}{n-s}\right] + r_2.$$

(2) $r_2 = r_1 + 2$

Since $r_1 + 2 = r_2 \leqslant n-s-1$, $r_1 + 3 \leqslant n-s$. In the $n-s$ well-distributed on C_s, the number of pieces with $\left[\frac{s}{n-s}\right] + 1 = \left\{\frac{s}{n-s}\right\}$ vertices is exactly r_2.

When $\frac{i(n-s)}{r_1+1} < r_2 \leqslant \frac{(i+1)(n-s)}{r_1+1}$ for some $1 \leqslant i \leqslant r_1$, the number of pieces with $\left\{\frac{s}{n-s}\right\}$ vertices is i or $i+1$ in any r_1+1 pieces in succession. Furthermore, there exist $r_1 + 1$ pieces in succession which exactly contains $i+1$ pieces with $\left\{\frac{s}{n-s}\right\}$ vertices, say 1 to $r_1 + 1$ pieces on C_s. Thus

$$\bar{d}_s(D(s)) = d(x_1, u_{r_1+2}) = (r_1+1)\left(\left\{\frac{s}{n-s}\right\} - 1\right) + i + 1 = (r_1+1)\left\{\frac{s}{n-s}\right\} - r_1 + i.$$

Hence by (4.2) we have

$$\gamma(D(s)) = \varphi_s + (r_1+1)\left\{\frac{s}{n-s}\right\} - r_1 + i.$$

This completes the proof of Lemma. □

Theorem 4. 6. *For $n \geq 8$, we have*

$$LE_n = [2, 2n-4]^0 \cup \{w_n - 1, w_n\} \bigcup_{\frac{n+2}{2} \leq s \leq n-2} [\gamma(D(s)), \varphi_s + s + 1]^0,$$

where $\gamma(D(s))$ is defined in Lemma 4. 5.

Proof. By Theorem 4. 1, $[2, 2n-4]^0 \cup \{w_n - 1, w_n\} \subseteq LE_n$.

For $\frac{n+2}{2} \leq s \leq n-2$, we will prove that $[\gamma(D(s)), \varphi_s + s + 1]^0 \subseteq LE_n$.

We take an $n-s$ well-distributed on C_s. Now, we define a digraph $D'(s)$ as follows:

$$V(D'(s)) = V(C_s) \cup \{u_1, u_2, \ldots, u_{n-s}\} \text{ and } A(D'(s)) = A(C_s) \cup \{(x_{k_{i-1}+j_{i-1}}, u_i),$$
$$(u_i, x_{k_i}) \mid i = 1, 2, \ldots, n-s\}.$$

It is easy to see that $D'(s) \in E_n(s)$, and $\gamma(D'(s)) = \gamma(s)) + 1$, where $\bar{d}_s(D'(S)) = d_{D'(s)}(u_1, u_{r_1+2}) = \bar{d}_s(D(S)) + 1$.

Now, if $\gamma(D(S)) + 1 < \varphi_s + s + 1$, for any integer k with $1 < k \leq \varphi_s + s - \gamma(D'(s))$, let $D'_k(s)$ denote the following resulting Lsd: u_{r_1+2} with its root vertices in $D'(s)$ moves to k positions along C_s, where the subscript is modulo $n-s$. And for $i, r_1 + 2 < i \leq n-s, u_i$ with its root vertices also moves correspondingly along C_s but do not surpass the root vertices of u_1. After this process, let $\{u_{i_1}, u_{i_2}, \ldots, u_{i_t}\}$, which have the same root vertices, induce a transitive subtournament on $D'_k(s)$. It is easy to check that $D'_k(s) \in E_n(s)$ and $\bar{d}_s(D'_k(s)) = \bar{d}_s(D'(s)) + k = \bar{d}_s(D(s)) + k + 1$. By Lemma 4. 4 we have

$$\gamma(D'_k(s)) = \varphi_s + \bar{d}_s(D(s)) + k + 1 = \gamma(D(s)) + 1 + k.$$

Hence $[\gamma(D(s)), \varphi_s + s + 1]^0 \subseteq LE_n$. So that

$$[2, 2n-4]^0 \cup \{w_n - 1, w_n] \bigcup_{\frac{n+2}{2} \leq s \leq n-s} [\gamma(D(s)), \varphi_s + s + 1]^0 \subseteq LE_n.$$

On the other hand, for any primitive Lsd D on n vertices, if the length of the shortest cycle of D is s, then $\gamma(D) \in [2, 2n-4]^0 \cup \{w_n - 1, w_n\}$ as $2 \leq s < \frac{n+2}{2}$ or $s = n-1$. And by Theorems 3. 1(1) and (4. 3), $\gamma(D) \in [\gamma(D(s)), \varphi_s + s + 1]^0$ when $\frac{n+2}{2} \leq s \leq n-2$. Hence

$$LE_n = [2, 2n-4]^0 \cup \{w_n - 1, w_n] \bigcup_{\frac{n+2}{2} \leq s \leq n-2} [\gamma(D(s)), \varphi_s + s + 1]^0.$$

This completes the proof of Theorem. □

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Letters

ON THE EXPONENT SET OF PRIMITIVE LOCALLY SEMICOMPLETE DIGRAPHS*

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A digraph D is *primitive* if there exists an interger $k>0$ such that for all ordered pair of vertices u, $v \in V(D)$ (not necessarily distinct), there is a walk from u to v with length k. The least such k is called the exponent of the digraph D, denoted by $\gamma(D)$.

A *semicomplete digraph* is a digraph without nonadjacent vertices. A *locally semicomplete digraph* is a digraph D that satisfies the following condition: for every vertex $x \in V(D)$, the D[O(x)] and D[I(x)] are semicomplete digraphs. We shall sometimes use the abbreviation Lsd to denote a locally semicomplete digraph. A *local tournament* is a locally semicomplete digraph without 2-cycles and loops.

Locally semicomplete digraphs, which is a generalization of semicomplete digraphs and tournaments, were first introduced by J. Bang-Jensen [1]. Many of classic theorems of tournaments have been generalized to Lsd. For example: (see [1]、[2] and [3]).

Every connected Lsd has a directed Hamilton path and every strong Lsd has a directed Hamilton cycle.

The arc-pancyclicity and completely strong path-connectivity have been generalized to Lsd. Therefore it is clear that Lsd form a new and interesting class. In this paper, we get some properties of cycles and determinate the exponent set of primitive Lsds.

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Zhang Kemin, male, in July 1935, Professor, Combinatorial Mathematics and Graph Theory, "On Lewin and Vitek's Conjecture about the exponent set of primitive matrices" etc papers have been published

THEOREM 1. Let D be a primitive Lsd on n vertices without loop. $L(D)=\{r_1,r_2,\cdots,r_\lambda\}$ is the cycle length set of D with $r_1<r_2<\cdots<r_\lambda$. Then the structure of L(D) is only one of the following cases:

(1) $L(D)=\{s,s+1,\cdots,n\}$, where $3\leqslant s\leqslant n-1$;

(2) $L(D)=\{2,s,s+1,\cdots,n\}$, where $3\leqslant s\leqslant n-1$;

(3) $L(D)=\{s,s+1,\cdots,t,k,k+1,\cdots,n\}$, where $2\leqslant s\leqslant 3, 3\leqslant t\leqslant \left[\frac{n-1}{2}\right]$ and $t+2\leqslant k\leqslant n-t+1$.

Let LE_n be the exponent set of primitive Lsds on n vertices. And let $E_n(s)$ be the set of all primitive Lsds on n vertices with the length $s(\geqslant 4)$ of the shortest cycle.

For $D(s)\in E_n(s)$ with $\frac{n+2}{2}\leqslant s\leqslant n-2$ and $n\geqslant 6$, we get that

(1) $\gamma(D(s)) = \psi_s + (r_1+1)\left[\frac{s}{n-s}\right] + r_2$ if $r_2 = r_1 + 2 - (n-s)$;

(2) $\gamma(D(s)) = \psi_s + (r_1+1)\left\{\frac{s}{n-s}\right\} - r_1 + i$ as

$$\frac{i(n-s)}{r_1+1} < r_2 \leqslant \frac{(i+1)(n-s)}{r_1+1} \text{ for } 0\leqslant i\leqslant r_1 \qquad \text{if } r_2 = r_1+2.$$

whtere ψ_s is Frobenius number $\psi(s,s+1,\cdots,n)$.

THEOREM 2. For $n\geqslant 8$, we have that.

$$LE_n=[2,2n-4]^0\cup\{w_n-1,w_n\}\quad\bigcup_{\frac{n+2}{2}\leqslant s\leqslant n-2}[\gamma(D(s)),\psi_s+s+1]^0,$$

where $[m,n]^0$ denotes a set of integers $\{m,m+1,\cdots,n\}$.

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