Several Results on Systems of Residue Classes

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Let \((m,n)\) and \(a(n)\) denote the g.c.d. of \(m, n\) and the residue class \(\{x \in \mathbb{Z} : x \equiv a \pmod{n}\}\) respectively. Any period of the characteristic function of \(\bigcup_{i=1}^{k} a_i(n_i)\) is called a covering period of \(\{a_i(n_i)\}_{i=1}^{k}\).

**Theorem 1** Let \(A = \{a_i(n_i)\}_{i=1}^{k}\) be a disjoint system (i.e., \(a_i(n_i), \ldots, a_k(n_k)\) are pairwise disjoint). Let \([n_1, \ldots, n_k]\) (the l.c.m. of \(n_1, \ldots, n_k\)) have the prime factorization \([n_1, \ldots, n_k] = \prod_{i=1}^{r} p_i^{t_i} i\) and \(T = \prod_{i=1}^{r} p_i^{t_i} + (\beta_i \geq 0)\) be the smallest positive covering period of \(A\). Then

\[
p_{r, \min\{\delta, \beta, \alpha, \gamma, \varphi\}} \leq \max_{1 \leq i \leq k \leq s} \left| \left\{ 1 \leq i \leq k : n_i = n_s \right\} \right| \prod_{i=1}^{r-1} \frac{p_i}{p_i - 1},
\]

where

\[
\delta_r = \min\{\delta \geq 1 : p_r^{\delta} \| n_i \text{ for some } 0 \leq i \leq k\} \text{ and } n_0 = 1.
\]

In the case \(T = 1\), our theorem gives an improvement to Burshtein's conjecture which is better than that of R. J. Simpson (1986).

**Theorem 2** Let \(\{a_i(n_i)\}_{i=1}^{k}\) be a disjoint covering system (i.e., \(a_i(n_i), 1 \leq i \leq k, \) form a partition of \(Z\)). Let \(n_0 = 1\). If \(d > 1\) divides some \(n_i\) then

\[
\left| \left\{ 1 \leq i \leq k : d | n_i \right\} \right| \geq 1 \left\{ a_i \pmod{d} : d | n_i, 1 \leq i \leq k \right\} \geq \min_{d \leq k+1} \frac{d}{(d, n_i)}.
\]

This improves the Newman–Znám result. The improvement is better than that of Berger, Felzenbaum and Fraenkel (1988).

In the following \(F\) will denote the class of all complex functions defined on \(D = \{ \langle r, n \rangle : 0 \leq r < n , r, n \in \mathbb{Z}\} \).

**Theorem 3** For any \(f \in F\) the following statements are equivalent:

(a) If \(0 \leq a_i < n_i, a_i, n_i \in \mathbb{Z} \) and \(\lambda_i \) is complex for each \(i = 1, \ldots, k\), then

\[
\sum_{i=1}^{k} \lambda_i = 0 \text{ for all } x \in \mathbb{Z} \Rightarrow \sum_{i=1}^{k} \lambda_if(a_i, n_i) = 0.
\]

(b) There exists a function \(g \in F\) such that
\[ f(a,n) = \frac{1}{n} \sum_{m=0}^{s-1} g\left(\frac{m}{(m,n)}, \frac{n}{(m,n)}\right)e^{2\pi i \frac{m}{n}} \quad \text{for all } (a,n) \in D. \]

The key idea of the proof is that both (a) and (b) are equivalent to

(*) \[ \sum_{j=0}^{s-1} f(a+jd, nd) = f(a, d) \quad \text{for all } n \in \mathbb{Z}^+ \text{ and } (a, d) \in D. \]

**Theorem 4.** If \( f \in F \) satisfies (b) or (*) and

\[ \sum_{r=0}^{s-1} f(r, n_j)e^{2\pi i \frac{s_j}{n_j}} \neq 0 \quad \text{for } s = 0, 1, \ldots, n_j - 1, \ j = 1, 2, \ldots, k, \]

then for any numbers \( \lambda_1, \ldots, \lambda_k \) we have

\[ \sum_{x \equiv a_i \pmod{n_i}} \lambda_i \equiv 0 \iff \sum_{x \equiv a_i \pmod{n_i}} \lambda_i f((x - a_i) \pmod{n_i}) \equiv 0. \]

(By \( g(x) \equiv 0 \) we mean \( g(x) = 0 \) for each \( x \).)

**Corollary.** If positive integers \( n_1, \ldots, n_k \) are all squarefree then

\[ \sum_{i=1}^{s} \lambda_i \equiv 0 \iff \sum_{i=1}^{s} \frac{\lambda_i}{\varphi(n_i)} \equiv 0 \]

where \( \varphi \) denotes Euler's totient function.