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Several Results on Systems of Residue Classes

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Let (m,n) and $a(n)$ denote the g.c.d. of m, n and the residue class $\{x\in Z: x\equiv a \pmod{n}\}$ respectively. Any period of the characteristic function of $\bigcup_{i=1}^{k} a_i(n_i)$ is called a covering period of $\{a_i(n_i)\}_{i=1}^{k}$.

Theorem 1 Let $A=\{a_i(n_i)\}_{i=1}^{k}$ be a disjoint system (i.e. $a_1(n_1),\cdots,a_k(n_k)$ are pairwise disjoint). Let $[n_1,\cdots,n_k]$ (the l.c.m. of $n_1,\cdots,n_k$) have the prime factorization $[n_1,\cdots,n_k]=\prod_{i=1}^{r}p_i^{\alpha_i}$ and $T=\prod_{i=1}^{r}p_i^{\beta_i}\ (\beta_i\geqslant 0)$ be the smallest positive covering period of A. Then

$$p_r^{\min(\delta_r,\ \alpha_r-\beta_r)}\leqslant\max_{\substack{1\leqslant s\leqslant k\\ p_r^{\alpha_r}\mid n_s}}|\{1\leqslant i\leqslant k:\ n_i=n_s\}|\prod_{i=1}^{r-1}\frac{p_i}{p_i-1},$$

where

$$\delta_r=\min\{\delta\geqslant 1:\ p_r^{\alpha_r-\delta}\|n_i \text{ for some } 0\leqslant i\leqslant k\} \text{ and } n_0=1.$$

In the case $T=1$, our theorem gives an improvement to Burshtein's conjecture which is better than that of R. J. Simpson (1986).

Theorem 2 Let $\{a_i(n_i)\}_{i=1}^{k}$ be a disjoint covering system (i.e. $a_i(n_i)$, $1\leqslant i\leqslant k$, form a partition of Z). Let $n_0=1$. If $d>1$ divides some n_i then

$$|\{1\leqslant i\leqslant k: d\mid n_i\}|\geqslant 1\{a_i \bmod d:\ d\mid n_i,\ 1\leqslant i\leqslant k\}|\geqslant\min_{\substack{0\leqslant i\leqslant k\\ d\nmid n_i}}\frac{d}{(d,n_i)}.$$

This improves the Newman–Znám result. The improvement is better than that of Berger, Felzenbaum and Fraenkel (1988).

In the following $\boldsymbol{F}$ will denote the class of all complex functions defined on $D=\{\langle r,n\rangle:\ 0\leqslant r<n,\ r,n\in Z\}$.

Theorem 3 For any $f\in\boldsymbol{F}$ the following statements are equivalent:

(a) If $0\leqslant a_i<n_i$, $a_i,n_i\in Z$ and λ_i is complex for each $i=1,\cdots,k$, then

$$\sum_{\substack{i=1\\ x\equiv a_i \pmod{n_i}}}^{k}\lambda_i=0 \text{ for all } x\in Z\Rightarrow\sum_{i=1}^{k}\lambda_i f(a_i,n_i)=0.$$

(b) There exists a function $g\in\boldsymbol{F}$ such that

$$f(a,n)=\frac{1}{n}\sum_{m=0}^{n-1} g\left(\frac{m}{(m,n)},\ \frac{n}{(m,n)}\right)e^{2\pi i\frac{m}{n}a} \text{ for all } \langle a,n\rangle\in D.$$

The key idea of the proof is that both (a) and (b) are equivalent to

$$(*)\qquad \sum_{j=0}^{n-1} f(a+jd,\ nd)=f(a,d) \text{ for all } n\in Z^+ \text{ and } \langle a,d\rangle\in D.$$

Theorem 4 If $f\in \boldsymbol{F}$ satisfies (b) or (∗) and

$$\sum_{r=0}^{n_j-1} f(r,n_j)e^{2\pi i\frac{r}{n_j}s}\neq 0 \text{ for } s=0,1,\cdots,n_j-1,\ j=1,2,\cdots,k,$$

then for any numbers $\lambda_1,\cdots,\lambda_k$ we have

$$\sum_{\substack{i=1\\ x\equiv a_i(\mathrm{mod}\ n_i)}}^{k}\lambda_i\equiv 0 \iff \sum_{i=1}^{k}\lambda_i f((x-a_i)\mathrm{mod}\ n_i,n_i)\equiv 0.$$

(By $g(x)\equiv 0$ we mean $g(x)=0$ for each x.)

Corollary If positive integers $n_1,\cdots,n_k$ are all squarefree then

$$\sum_{\substack{i=1\\ n_i\mid x-a_i}}^{k}\lambda_i\equiv 0 \iff \sum_{\substack{i=1\\ (x-a_i,n_i)=1}}^{k}\frac{\lambda_i}{\varphi(n_i)}\equiv 0$$

where φ denotes Euler's totient function.