

On Value Sets of Polynomials over a Field

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Abstract

Let F be any field. Let $p(F)$ be the characteristic of F if F is not of characteristic zero, and let $p(F) = +\infty$ otherwise. Let A_1, \dots, A_n be finite nonempty subsets of F , and let

$$f(x_1, \dots, x_n) = a_1x_1^k + \dots + a_nx_n^k + g(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$$

with $k \in \{1, 2, 3, \dots\}$, $a_1, \dots, a_n \in F \setminus \{0\}$ and $\deg g < k$. We show that

$$|\{f(x_1, \dots, x_n) : x_1 \in A_1, \dots, x_n \in A_n\}| \geq \min \left\{ p(F), \sum_{i=1}^n \left\lfloor \frac{|A_i| - 1}{k} \right\rfloor + 1 \right\}.$$

When $k \geq n$ and $|A_i| \geq i$ for $i = 1, \dots, n$, we also have

$$\begin{aligned} & |\{f(x_1, \dots, x_n) : x_1 \in A_1, \dots, x_n \in A_n, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \\ & \geq \min \left\{ p(F), \sum_{i=1}^n \left\lfloor \frac{|A_i| - i}{k} \right\rfloor + 1 \right\}; \end{aligned}$$

consequently, if $n \geq k$ then for any finite subset A of F we have

$$|\{f(x_1, \dots, x_n) : x_1, \dots, x_n \in A, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \geq \min\{p(F), |A| - n + 1\}.$$

In the case $n > k$, we propose a further conjecture which extends the Erdős-Heilbronn conjecture in a new direction.

1 Introduction

For a field F we use $p(F)$ to denote the additive order of the multiplicative identity of F , which is either infinite or a prime. A field F is said to have characteristic zero if $p(F) = +\infty$, and have characteristic p if $p(F)$ is a prime p .

By the Chevalley–Warning theorem (cf. pp. 50–51 of [10]), for any polynomial $P(x_1, \dots, x_n)$ over a finite field F , if $n > \deg P$ then the characteristic of F divides the number of solutions

2000 *Mathematics Subject Classification*. Primary 11T06; Secondary 05A05, 11B75, 11P99, 12E10.
Supported by the National Science Fund (Grant No. 10425103) for Distinguished Young Scholars in China.

to the equation $P(x_1, \dots, x_n) = 0$ over F^n . However, this says nothing about the solvability of the equation over F^n unless there is an obvious solution.

Given a field F , we consider polynomials of the form

$$f(x_1, \dots, x_n) = a_1x_1^k + \dots + a_nx_n^k + g(x_1, \dots, x_n) \in F(x_1, \dots, x_n), \quad (1.1)$$

where

$$k \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}, \quad a_1, \dots, a_n \in F^* = F \setminus \{0\} \text{ and } \deg g < k. \quad (1.2)$$

What can we say about the solvability of the equation $f(x_1, \dots, x_n) = 0$ over F^n ?

Let F be the field $F_p = \mathbb{Z}/p\mathbb{Z}$ with p a prime, and assume (1.1) and (1.2). In 1956 Carlitz [3] proved that the equation $f(x_1, \dots, x_n) = 0$ has a solution in F_p^n when $k \mid p-1$ and $n \geq k$. In 2006 Felszeghy [7] extended this result by showing that the equation is solvable if $k < p$ and $n \geq \lceil (p-1)/\lfloor (p-1)/k \rfloor \rceil$. (For a real number α , $\lceil \alpha \rceil$ denotes the least integer not smaller than α while $\lfloor \alpha \rfloor$ represents the largest integer not exceeding α .) Note that the equation $f(x_1, \dots, x_n) = 0$ over F_p^n is solvable if and only if the value set

$$\{f(x_1, \dots, x_n) : x_1, \dots, x_n \in F_p\}$$

contains 0. In 1959 Chowla, Mann and Straus (cf. pp. 60-61 of [10]) used Vosper's theorem (cf. pp. 52-57 of [10]) to deduce that if $p > 3$, $1 < k < (p-1)/2$ and $k \mid p-1$, then

$$|\{a_1x_1^k + \dots + a_nx_n^k : x_1, \dots, x_n \in F_p\}| \geq \min \left\{ p, (2n-1)\frac{p-1}{k} + 1 \right\}.$$

Let F_q be the finite field of q elements where $q > 1$ is a prime power. In 1993 Wan, Shiue and Chen [14] showed that if $P(x)$ is a polynomial over F_q and $l \in \mathbb{N} = \{0, 1, 2, \dots\}$ is the least nonnegative integer with $\sum_{x \in F_q} P(x)^l \neq 0$ then $|\{P(x) : x \in F_q\}| \geq l+1$; in 2004 Das [4] extended this to multi-variable polynomials over F_q . By modifying the proof of Theorem 1.5 of [4] slightly, one gets the following assertion: If $P(x_1, \dots, x_n) \in F_q[x_1, \dots, x_n]$, $\emptyset \neq S \subseteq F_q^n$, and l is the smallest nonnegative integer with $\sum_{(x_1, \dots, x_n) \in S} P(x_1, \dots, x_n)^l \neq 0$, then we have $|\{P(x_1, \dots, x_n) : (x_1, \dots, x_n) \in S\}| \geq l+1$. Here the lower bound depends heavily on values of $P(x_1, \dots, x_n)$.

In this paper we investigate two kinds of value sets of a polynomial in the form (1.1).

Here is our first theorem which includes Felszeghy's result as a special case.

Theorem 1.1. *Let $f(x_1, \dots, x_n)$ be a polynomial over a field F given by (1.1) and (1.2). Then, for any finite nonempty subsets A_1, \dots, A_n of F , we have*

$$|\{f(x_1, \dots, x_n) : x_1 \in A_1, \dots, x_n \in A_n\}| \geq \min \left\{ p(F), \sum_{i=1}^n \left\lfloor \frac{|A_i| - 1}{k} \right\rfloor + 1 \right\}. \quad (1.3)$$

This result will be proved in Section 2 where we also give an example to show that the inequality (1.3) is sharp when F is an algebraically closed field.

Corollary 1.1. (Cauchy–Davenport Theorem) *Let A_1, \dots, A_n be finite nonempty subsets of a field F . Then*

$$|A_1 + \dots + A_n| \geq \min\{p(F), |A_1| + \dots + |A_n| - n + 1\},$$

where the sumset $A_1 + \dots + A_n$ is given by

$$A_1 + \dots + A_n = \{x_1 + \dots + x_n : x_1 \in A_1, \dots, x_n \in A_n\}.$$

Proof. Simply apply Theorem 1.1 with $f(x_1, \dots, x_n) = x_1 + \dots + x_n$.

Remark 1.1. The original Cauchy–Davenport theorem (see p.44 of [10] or p.200 of [13]) is Corollary 1.1 in the case $n = 2$ and $F = \mathbb{Z}/p\mathbb{Z}$ with p a prime.

Corollary 1.2. *Let F be a field of characteristic zero, and assume (1.1) and (1.2). Then, for any finite nonempty subset A of F , we have*

$$|\{f(x_1, \dots, x_n) : x_1, \dots, x_n \in A\}| \geq n \left\lfloor \frac{|A| - 1}{k} \right\rfloor + 1.$$

Proof. Just apply Theorem 1.1 with $A_1 = \dots = A_n = A$.

Corollary 1.3. *Let F be a field with prime characteristic p , and let $f(x_1, \dots, x_n)$ be given by (1.1) and (1.2). If A is a finite subset of F satisfying $\lfloor (|A| - 1)/k \rfloor \geq (p - 1)/n$, then*

$$|\{f(x_1, \dots, x_n) : x_1, \dots, x_n \in A\}| \geq p.$$

Proof. It suffices to apply Theorem 1.1 with $A_1 = \dots = A_n = A$.

Remark 1.2. In the case $A = F = F_p$, Corollary 1.3 yields Felszeghy’s result.

Now we state our second theorem.

Theorem 1.2. *Let $f(x_1, \dots, x_n)$ be a polynomial over a field F given by (1.1) and (1.2) with $n \leq k = \deg f$. And let A_1, \dots, A_n be finite subsets of F with $|A_i| \geq i$ for $i = 1, \dots, n$. Then, for the restricted value set*

$$V = \{f(x_1, \dots, x_n) : x_1 \in A_1, \dots, x_n \in A_n, \text{ and } x_i \neq x_j \text{ if } i \neq j\}, \quad (1.4)$$

we have

$$|V| \geq \min \left\{ p(F), \sum_{i=1}^n \left\lfloor \frac{|A_i| - i}{k} \right\rfloor + 1 \right\}. \quad (1.5)$$

Corollary 1.4. *Let A be a finite subset of a field F , and let $f(x_1, \dots, x_n)$ be a polynomial given by (1.1) and (1.2). Write $|A| = kq + r$ with $q, r \in \mathbb{N}$ and $r < k$. Then we have*

$$\begin{aligned} & |\{f(x_1, \dots, x_n) : x_1, \dots, x_n \in A, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \\ & \geq \begin{cases} \min\{p(F), n(q - 1) + \min\{n, r\} + 1\} & \text{if } n \leq k, \\ \min\{p(F), |A| - n + 1\} & \text{if } n \geq k. \end{cases} \end{aligned} \quad (1.6)$$

In Section 3 we shall prove Theorem 1.2 and Corollary 1.4, and give an example to illustrate that the inequality (1.5) is essentially best possible when F is an algebraically closed field.

In 1964 Erdős and Heilbronn [6] conjectured that if A is a subset of $\mathbb{Z}/p\mathbb{Z}$ with p a prime then

$$|\{x_1 + x_2 : x_1, x_2 \in A \text{ and } x_1 \neq x_2\}| \geq \min\{p, 2|A| - 3\}.$$

Thirty years later this deep conjecture was confirmed by Dias da Silva and Hamidoune [5] who used the representation theory of groups to show that if A is a finite subset of a field F then

$$\begin{aligned} & |\{x_1 + x_2 + \dots + x_n : x_1, \dots, x_n \in A, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \\ & \geq \min\{p(F), n(|A| - n) + 1\}. \end{aligned}$$

This suggests that Corollary 1.4 in the case $n > k$ might be further improved. In Section 4 we will discuss our following conjecture which extends the Dias da Silva–Hamidoune result in a new way. (The reader may consult [2], [8], [9] and [12] for other generalizations of the Erdős–Heilbronn conjecture.)

Conjecture 1.1. *Let $f(x_1, \dots, x_n)$ be a polynomial over a field F given by (1.1) and (1.2), and let A be any finite subset of F . Provided $n > k$, we have*

$$\begin{aligned} & |\{f(x_1, \dots, x_n) : x_1, \dots, x_n \in A, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \\ & \geq \min \left\{ p(F) - \delta, \frac{n(|A| - n)}{k} - k \left\{ \frac{n}{k} \right\} \left\{ \frac{|A| - n}{k} \right\} + 1 \right\}, \end{aligned} \quad (1.7)$$

where $\{\alpha\}$ denotes the fractional part $\alpha - \lfloor \alpha \rfloor$ of a real number α , and

$$\delta = \begin{cases} 1 & \text{if } n = 2 \text{ and } a_1 = -a_2, \\ 0 & \text{otherwise.} \end{cases}$$

2 Proof of Theorem 1.1

We need a useful tool of algebraic nature.

Combinatorial Nullstellensatz (Alon [1]). *Let A_1, \dots, A_n be finite subsets of a field F , and let $P(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$. Suppose that $\deg P = k_1 + \dots + k_n$ where $0 \leq k_i < |A_i|$ for $i = 1, \dots, n$. Then $P(x_1, \dots, x_n) \neq 0$ for some $x_1 \in A_1, \dots, x_n \in A_n$ if*

$$[x_1^{k_1} \dots x_n^{k_n}]P(x_1, \dots, x_n) \neq 0,$$

where $[x_1^{k_1} \dots x_n^{k_n}]P(x_1, \dots, x_n)$ denotes the coefficient of $x_1^{k_1} \dots x_n^{k_n}$ in $P(x_1, \dots, x_n)$.

Proof of Theorem 1.1. Let m be the largest nonnegative integer not exceeding n such that $\sum_{0 < i \leq m} \lfloor (|A_i| - 1)/k \rfloor < p(F)$. For each $0 < i \leq m$ let A'_i be a subset of A_i with cardinality $k \lfloor (|A_i| - 1)/k \rfloor + 1$. In the case $m < n$, $p = p(F)$ is a prime and we let A'_{m+1} be a subset of A_{m+1} with

$$\begin{aligned} |A'_{m+1}| &= k \left(p - 1 - \sum_{0 < i \leq m} \left\lfloor \frac{|A_i| - 1}{k} \right\rfloor \right) + 1 \\ &< k \left\lfloor \frac{|A_{m+1}| - 1}{k} \right\rfloor + 1 \leq |A_{m+1}|. \end{aligned}$$

If $m + 1 < j \leq n$ then we let $A'_j \subseteq A_j$ be a singleton. Whether $m = n$ or not, we have $\sum_{i=1}^n (|A'_i| - 1) = k(N - 1)$, where

$$N = \min \left\{ p(F), \sum_{i=1}^n \left\lfloor \frac{|A_i| - 1}{k} \right\rfloor + 1 \right\}.$$

Set

$$C = \{f(x_1, \dots, x_n) : x_1 \in A'_1, \dots, x_n \in A'_n\}.$$

Suppose that $|C| \leq N - 1$. Then

$$\begin{aligned} & [x_1^{|A'_1|-1} \cdots x_n^{|A'_n|-1}] f(x_1, \dots, x_n)^{N-1-|C|} \prod_{c \in C} (f(x_1, \dots, x_n) - c) \\ &= [x_1^{|A'_1|-1} \cdots x_n^{|A'_n|-1}] (a_1 x_1^k + \cdots + a_n x_n^k)^{N-1} \\ &= \frac{(N-1)!}{\prod_{i=1}^n ((|A'_i|-1)/k)!} a_1^{(|A'_1|-1)/k} \cdots a_n^{(|A'_n|-1)/k} \neq 0. \end{aligned}$$

By the Combinatorial Nullstellensatz, for some $x_1 \in A'_1, \dots, x_n \in A'_n$ we have

$$f(x_1, \dots, x_n)^{N-1-|C|} \prod_{c \in C} (f(x_1, \dots, x_n) - c) \neq 0$$

which contradicts the fact $f(x_1, \dots, x_n) \in C$.

In view of the above,

$$|\{f(x_1, \dots, x_n) : x_1 \in A_1, \dots, x_n \in A_n\}| \geq |C| \geq N$$

and this concludes the proof.

Example 2.1. Let $a_1, \dots, a_n \in F^* = F \setminus \{0\}$, where F is an algebraically closed field. For each $i = 1, \dots, n$ let

$$A_i = \{x \in F : f_k(x) \in \{a_i^{-1}, 2a_i^{-1}, \dots, q_i a_i^{-1}\}\} \setminus R_i,$$

where k and $q_i < p(F)$ are positive integers,

$$f_k(x) = \begin{cases} x^k - x & \text{if } p(F) \text{ is a prime divisor of } k, \\ x^k & \text{otherwise,} \end{cases} \quad (2.1)$$

and R_i is a subset of $\{x \in F : f_k(x) = q_i a_i^{-1}\}$ with $|R_i| = r_i \leq k - 1$. For each $c \in F^*$, the equation $f_k(x) = c$ has exactly k distinct solutions in F since there is no $x \in F$ satisfying $f_k(x) - c = 0 = f'_k(x)$, where f'_k is the formal derivative of f_k . So $|A_i| = kq_i - r_i$ and hence

$$\left\lfloor \frac{|A_i| - 1}{k} \right\rfloor = \left\lfloor \frac{k(q_i - 1) + (k - 1 - r_i)}{k} \right\rfloor = q_i - 1.$$

For every $i = 1, \dots, n$ we have

$$\{f_k(a) : a \in A_i\} = \{a_i^{-1}, 2a_i^{-1}, \dots, q_i a_i^{-1}\}.$$

Thus

$$\begin{aligned} & \{a_1 f_k(x_1) + \cdots + a_n f_k(x_n) : x_1 \in A_1, \dots, x_n \in A_n\} \\ &= \{a_1 (y_1 a_1^{-1}) + \cdots + a_n (y_n a_n^{-1}) : y_i \in \{1, \dots, q_i\} \text{ for } i = 1, \dots, n\} \\ &= \{(y_1 + \cdots + y_n)e : y_i \in \{1, \dots, q_i\} \text{ for } i = 1, \dots, n\} \\ &= \{ne, (n+1)e, \dots, (q_1 + \cdots + q_n)e\}, \end{aligned}$$

where e denotes the multiplicative identity of the field F . Observe that

$$q_1 + \cdots + q_n - n = \sum_{i=1}^n (q_i - 1) = \sum_{i=1}^n \left\lfloor \frac{|A_i| - 1}{k} \right\rfloor.$$

Therefore

$$\begin{aligned} & |\{a_1 f_k(x_1) + \cdots + a_n f_k(x_n) : x_1 \in A_1, \dots, x_n \in A_n\}| \\ &= \min \left\{ p(F), \sum_{i=1}^n \left\lfloor \frac{|A_i| - 1}{k} \right\rfloor + 1 \right\}. \end{aligned}$$

3 Proofs of Theorem 1.2 and Corollary 1.4

Proof of Theorem 1.2. Let $q_i = \lfloor (|A_i| - i)/k \rfloor$ for $i = 1, \dots, n$. And let m be the largest nonnegative integer not exceeding n such that $\sum_{0 < i \leq m} q_i < p(F)$. For each $0 < i \leq m$ let A'_i be a subset of A_i with cardinality $kq_i + i$. In the case $m < n$, $p = p(F)$ is a prime and we let A'_{m+1} be a subset of A_{m+1} with

$$\begin{aligned} |A'_{m+1}| &= k \left(p - 1 - \sum_{0 < i \leq m} q_i \right) + m + 1 \\ &< kq_{m+1} + m + 1 \leq |A_{m+1}|. \end{aligned}$$

If $m + 1 < j \leq n$ then we let $A'_j \subseteq A_j$ with $|A'_j| = j$. Whether $m = n$ or not, we have $\sum_{i=1}^n (|A'_i| - i) = k(N - 1)$, where

$$N = \min\{p(F), q_1 + \cdots + q_n + 1\}.$$

Set

$$C = \{f(x_1, \dots, x_n) : x_1 \in A'_1, \dots, x_n \in A'_n, \text{ and } x_i \neq x_j \text{ if } i \neq j\}.$$

Suppose that $|C| \leq N - 1$ and let $P(x_1, \dots, x_n)$ denote the polynomial

$$f(x_1, \dots, x_n)^{N-1-|C|} \prod_{c \in C} (f(x_1, \dots, x_n) - c) \times \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Then

$$\deg P \leq k(N - 1) + \binom{n}{2} = \sum_{i=1}^n (|A'_i| - 1).$$

By linear algebra,

$$\prod_{1 \leq i < j \leq n} (x_j - x_i) = \det(x_i^{j-1})_{1 \leq i, j \leq n} = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n x_i^{\sigma(i)-1},$$

where S_n is the symmetric group of all permutations on $\{1, \dots, n\}$, and $\text{sign}(\sigma)$ is 1 or -1 according as σ is even or odd. Recall that

$$|A'_i| = \begin{cases} kq_i + i & \text{if } 1 \leq i \leq m, \\ k(p(F) - 1 - q_1 - \cdots - q_m) + i & \text{if } i = m + 1 \leq n, \\ i & \text{if } m + 1 < i \leq n. \end{cases}$$

For each $i = 1, \dots, n$, we clearly have $|A'_i| - 1 \equiv i - 1 \pmod{k}$ and $\sigma(i) - 1 < n \leq k$ for all $\sigma \in S_n$. Thus

$$\begin{aligned} & \left[x_1^{|A'_1|-1} \cdots x_n^{|A'_n|-1} \right] P(x_1, \dots, x_n) \\ &= \left[x_1^{|A'_1|-1} \cdots x_n^{|A'_n|-1} \right] (a_1 x_1^k + \cdots + a_n x_n^k)^{N-1} \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n x_i^{\sigma(i)-1} \\ &= \left[\prod_{i=1}^n x_i^{|A'_i|-1} \right] \sum_{j_1 + \cdots + j_n = N-1} \frac{(N-1)!}{j_1! \cdots j_n!} a_1^{j_1} \cdots a_n^{j_n} \prod_{i=1}^n x_i^{j_i k + i - 1} \\ &= \frac{(N-1)!}{q_1! \cdots q_m! q!} a_1^{q_1} \cdots a_m^{q_m} a \neq 0, \end{aligned}$$

where $q = a = 1$ if $m = n$, and $q = (p(F) - 1 - q_1 - \cdots - q_m)!$ and $a = a_{m+1}^q$ if $m < n$. In light of the Combinatorial Nullstellensatz, there are $x_1 \in A'_1, \dots, x_n \in A'_n$ such that $P(x_1, \dots, x_n) \neq 0$. Obviously this contradicts the fact $f(x_1, \dots, x_n) \in C$.

By the above, $|V| \geq |C| \geq N$ and hence (1.5) holds.

Example 3.1. Let F be any algebraically closed field, and let $a_1, \dots, a_n \in F^*$. For each $i = 1, \dots, n$ let

$$A_i = S_i \cup \{x \in F : f_k(x) \in \{j a_i^{-1} : 1 < j \leq q_i\}\},$$

where $k \geq n$ and $q_i < p(F)$ are positive integers, $f_k(x)$ is given by (2.1), and S_i is a subset of $\{x \in F : f_k(x) = a_i^{-1}\}$ with $|S_i| \geq i$. Observe that $|A_i| = k(q_i - 1) + |S_i|$ and hence

$$\left\lfloor \frac{|A_i| - i}{k} \right\rfloor = q_i - 1.$$

For every $i = 1, \dots, n$, we have

$$\{f_k(a) : a \in A_i\} = \{a_i^{-1}, 2a_i^{-1}, \dots, q_i a_i^{-1}\}.$$

If $y_i \in \{1, \dots, q_i\}$ for $i = 1, \dots, n$, we can find distinct $x_1 \in A_1, \dots, x_n \in A_n$ such that $f_k(x_i) = y_i a_i^{-1}$ for $i = 1, \dots, n$; in fact, if $x_1 \in A_1, \dots, x_{i-1} \in A_{i-1}$ are distinct with $i \leq n$, and $f_k(x_j) = y_j a_j^{-1}$ for $j = 1, \dots, i-1$, then we can choose $x_i \in A_i \setminus \{x_1, \dots, x_{i-1}\}$ satisfying $f_k(x_i) = y_i a_i^{-1}$ because $k \geq |S_i| > i - 1$. Thus

$$\begin{aligned} & \{a_1 f_k(x_1) + \cdots + a_n f_k(x_n) : x_1 \in A_1, \dots, x_n \in A_n, \text{ and } x_i \neq x_j \text{ if } i \neq j\} \\ &= \{a_1 (y_1 a_1^{-1}) + \cdots + a_n (y_n a_n^{-1}) : y_i \in \{1, \dots, q_i\} \text{ for } i = 1, \dots, n\} \\ &= \{(y_1 + \cdots + y_n) e : y_i \in \{1, \dots, q_i\} \text{ for } i = 1, \dots, n\} \\ &= \{n e, (n+1)e, \dots, (q_1 + \cdots + q_n) e\}, \end{aligned}$$

where e is the multiplicative identity of F . Note that

$$q_1 + \cdots + q_n - n = \sum_{i=1}^n (q_i - 1) = \sum_{i=1}^n \left\lfloor \frac{|A_i| - i}{k} \right\rfloor.$$

So we have

$$\begin{aligned} & |\{a_1 f_k(x_1) + \cdots + a_n f_k(x_n) : x_1 \in A_1, \dots, x_n \in A_n, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \\ &= \min \left\{ p(F), \sum_{i=1}^n \left\lfloor \frac{|A_i| - i}{k} \right\rfloor + 1 \right\}. \end{aligned}$$

Proof of Corollary 1.4. The case $|A| < n$ is trivial; below we assume $|A| \geq n$.

We first handle the case $n \leq k$. If $1 \leq i \leq \min\{n, r\}$ then we let A_i be a subset of A with cardinality $kq + i + \max\{r - n, 0\} \leq kq + r = |A|$; if $r < j \leq n$ then we let A_j be a subset of A with cardinality $k(q - 1) + j \leq k(q - 1) + n \leq kq$. (Note that when $r < n$ we have $q \neq 0$ since $|A| \geq n$.) Obviously,

$$\sum_{i=1}^n \left\lfloor \frac{|A_i| - i}{k} \right\rfloor = \sum_{i=1}^{\min\{n, r\}} q + \sum_{r < j \leq n} (q - 1) = n(q - 1) + \min\{n, r\}.$$

Applying Theorem 1.2 we obtain that

$$\begin{aligned} & |\{f(x_1, \dots, x_n) : x_1, \dots, x_n \in A, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \\ & \geq |\{f(x_1, \dots, x_n) : x_1 \in A_1, \dots, x_n \in A_n, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \\ & \geq \min\{p(F), n(q - 1) + \min\{n, r\} + 1\}. \end{aligned}$$

In particular, if $k = n$ then

$$\begin{aligned} & |\{f(x_1, \dots, x_n) : x_1, \dots, x_n \in A, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \\ & \geq \min\{p(F), n(q - 1) + r + 1\} = \min\{p(F), |A| - n + 1\}. \end{aligned}$$

Now suppose that $n > k$. Let c_{k+1}, \dots, c_n be $n - k$ distinct elements of A . Then $A' = A \setminus \{c_{k+1}, \dots, c_n\}$ has cardinality $|A| - (n - k)$. By what we have proved,

$$\begin{aligned} & |\{f(x_1, \dots, x_k, c_{k+1}, \dots, c_n) : x_1, \dots, x_k \in A', \text{ and } x_i \neq x_j \text{ if } i < j\}| \\ & \geq \min\{p(F), |A'| - k + 1\} = \min\{p(F), |A| - n + 1\}. \end{aligned}$$

So the desired inequality follows.

4. Discussion of Conjecture 1.1

Conjecture 1.1 in the case $f(x_1, \dots, x_n) = x_1 + \dots + x_n$, essentially gives the Dias da Silva–Hamidoune result mentioned in Section 1.

In the case $f(x_1, \dots, x_n) = x_1^k + \dots + x_n^k$ with $k > 1$, we may explain the symmetry between n and $|A| - n$ as follows: If $|A| > n$ then

$$\begin{aligned} & |\{x_1^k + \dots + x_n^k : x_1, \dots, x_n \in A, \text{ and } x_i \neq x_j \text{ if } i \neq j\}| \\ & = \left| \left\{ \sum_{a \in A} a^k - y_1^k - \dots - y_{|A|-n}^k : y_1, \dots, y_{|A|-n} \in A, \text{ and } y_i \neq y_j \text{ if } i \neq j \right\} \right| \\ & = |\{y_1^k + \dots + y_{|A|-n}^k : y_1, \dots, y_{|A|-n} \in A, \text{ and } y_i \neq y_j \text{ if } i \neq j\}|. \end{aligned}$$

Conjecture 1.1 holds when $n = 2$. In fact, if we set $A_1 = \{a_1x : x \in A\}$ and $A_2 = \{a_2x : x \in A\}$, then $|A_1| + |A_2| - 3 = 2(|A| - 2) + 1$ and

$$\begin{aligned} & |\{a_1x_1 + a_2x_2 : x_1, x_2 \in A \text{ and } x_1 \neq x_2\}| \\ & = |\{y_1 + y_2 : y_1 \in A_1, y_2 \in A_2 \text{ and } a_1^{-1}y_1 \neq a_2^{-1}y_2\}| \\ & = |\{y_1 + y_2 : y_1 \in A_1, y_2 \in A_2 \text{ and } y_1 - a_1a_2^{-1}y_2 \neq 0\}| \\ & \geq \min\{p(F) - \delta, |A_1| + |A_2| - 3\} \quad (\text{by Corollary 3 of [11]}). \end{aligned}$$

The following example illustrates how the lower bound in (1.7) comes out.

Example 4.1. Let $k, n \in \mathbb{Z}^+$ and $q \in \mathbb{N}$. Let

$$A = \{z \in \mathbb{C} : z^k \in \{1, \dots, q\}\} \cup R,$$

where \mathbb{C} is the field of complex numbers and R is a subset of $\{z \in \mathbb{C} : z^k = q + 1\}$ with cardinality $r < k$. Suppose that $n \geq k$ and $|A| = kq + r \geq n = k\lfloor n/k \rfloor + s$ with $s \in \{0, \dots, k-1\}$. Clearly,

$$V = \{x_1^k + \dots + x_n^k : x_1, \dots, x_n \in A, \text{ and } x_i \neq x_j \text{ if } i \neq j\}$$

just consists of those sums $\sum_{i=1}^n y_i$ with $y_i \in \{1, \dots, q+1\}$, $|\{i : y_i = q+1\}| \leq r$ and $|\{i : y_i = j\}| \leq k$ for all $j = 1, \dots, q$. Thus the smallest element of V is

$$m_V = k \sum_{i=1}^{\lfloor n/k \rfloor} i + s \left(\left\lfloor \frac{n}{k} \right\rfloor + 1 \right) = \left(\frac{k}{2} \left\lfloor \frac{n}{k} \right\rfloor + s \right) \left(\left\lfloor \frac{n}{k} \right\rfloor + 1 \right),$$

while the largest element of V is

$$M_V = r(q+1) + k \sum_{i=0}^{\lfloor n/k \rfloor - 1} (q-i) + (s-r) \left(q - \left\lfloor \frac{n}{k} \right\rfloor \right) - d(r, s),$$

where

$$d(r, s) = \begin{cases} r - s & \text{if } r \geq s, \\ 0 & \text{if } r < s. \end{cases}$$

It follows that

$$\begin{aligned} & d(r, s) + M_V - m_V \\ &= r(q+1) + kq \left\lfloor \frac{n}{k} \right\rfloor - k \left\lfloor \frac{n}{k} \right\rfloor^2 + (s-r)q + (r-s) \left\lfloor \frac{n}{k} \right\rfloor - s \left(\left\lfloor \frac{n}{k} \right\rfloor + 1 \right) \\ &= \left\lfloor \frac{n}{k} \right\rfloor \left(kq + r - s - k \left\lfloor \frac{n}{k} \right\rfloor - s \right) + r + sq - s \\ &= \left\lfloor \frac{n}{k} \right\rfloor (|A| - s - n) + s(q-1) + r \\ &= \left\lfloor \frac{n}{k} \right\rfloor (|A| - n) + s \left(q - \left\lfloor \frac{n}{k} \right\rfloor - 1 \right) + r \\ &= \left\lfloor \frac{n}{k} \right\rfloor (|A| - n) + s \left\lfloor \frac{|A| - n}{k} \right\rfloor + \begin{cases} r - s & \text{if } r \geq s, \\ r & \text{if } r < s. \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} |V| &= |\{m_V, \dots, M_V\}| = M_V - m_V + 1 \\ &= 1 + \left\lfloor \frac{n}{k} \right\rfloor (|A| - n) + s \left\lfloor \frac{|A| - n}{k} \right\rfloor + \begin{cases} r & \text{if } r < s, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, we note that

$$\begin{aligned}
& \left\lfloor \frac{n}{k} \right\rfloor (|A| - n) + s \left\lfloor \frac{|A| - n}{k} \right\rfloor \\
&= \left(\frac{n}{k} - \left\{ \frac{n}{k} \right\} \right) (|A| - n) + k \left\{ \frac{n}{k} \right\} \left\lfloor \frac{|A| - n}{k} \right\rfloor \\
&= \frac{n(|A| - n)}{k} - k \left\{ \frac{n}{k} \right\} \left\{ \frac{|A| - n}{k} \right\}.
\end{aligned}$$

Acknowledgments. The author thanks the referees for their helpful comments.

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