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ON COVERS OF ABELIAN GROUPS BY COSETS

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Abstract. Let G be any abelian group and $\{a_sG_s\}_{s=1}^k$ be a finite system of cosets of subgroups $G_1, \ldots, G_k$. We show that if $\{a_sG_s\}_{s=1}^k$ covers all the elements of G at least m times with the coset a_tG_t irredundant then $[G : G_t] \leqslant 2^{k-m}$ and furthermore $k \geqslant m + f([G : G_t])$, where $f(\prod_{i=1}^r p_i^{\alpha_i}) = \sum_{i=1}^r \alpha_i(p_i - 1)$ if $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r$ are nonnegative integers. This extends Mycielski's conjecture in a new way and implies a conjecture of Gao and Geroldinger. Our new method involves algebraic number theory and characters of abelian groups.

1. Introduction

As in any textbook on group theory, for a subgroup H of a group G with the index $[G : H]$ finite, G can be partitioned into $k = [G : H]$ left cosets of H in G, i.e., all the k left cosets of H form a disjoint cover of G.

In 1954 B. H. Neumann [N1, N2] discovered the following basic result on covers of groups.

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Theorem 1.1 (Neumann)**.** *Let $\{a_sG_s\}_{s=1}^k$ be a cover of a group G by (finitely many) left cosets of subgroups $G_1, \dots, G_k$. Then G is the union of those a_sG_s with $[G : G_s] < \infty$. In other words, if $\{a_sG_s\}_{s\neq t}$ is not a cover of G then we have $[G : G_t] < \infty$.*

In 1966 J. Mycielski (cf. [MS]) posed an interesting conjecture on disjoint covers of abelian groups. Before stating the conjecture we give a definition first.

Definition 1.1. The *Mycielski function* $f : \mathbb{Z}^+ = \{1, 2, \dots\} \to \{0, 1, \dots\}$ is given by

$$f(n) = \sum_{p\in P(n)} \mathrm{ord}_p(n)(p-1), \tag{1.1}$$

where $P(n)$ denotes the set of prime divisors of n and $\mathrm{ord}_p(n)$ represents the largest nonnegative integer α such that $p^\alpha \mid n$.

Remark 1.1. Since $p \leqslant 2^{p-1}$ for any prime p, (1.1) implies that $n \leqslant 2^{f(n)}$ (i.e., $f(n) \geqslant \log_2 n$).

Mycielski's Conjecture. *Let G be an abelian group, and $\{a_sG_s\}_{s=1}^k$ be a disjoint cover of G by left cosets of subgroups. Then $k \geqslant 1 + f([G : G_t])$ for each $t = 1, \dots, k$.*

When G is the additive group $\mathbb{Z}$ of integers, Mycielski's conjecture says that for any disjoint cover $\{a_s(n_s)\}_{s=1}^k$ of $\mathbb{Z}$ by residue classes (where $a_s \in \mathbb{Z}$, $n_s \in \mathbb{Z}^+$ and $a_s(n_s) = a_s + n_s\mathbb{Z}$) we have $k \geqslant 1 + f(n_t)$ for every $t = 1, \dots, k$. This was first confirmed by Š. Znám [Z66]. For problems and results on covers of $\mathbb{Z}$, the reader is referred to [G04], [PS], [S03] and [S05].

Definition 1.2. For a subnormal subgroup H of a group G with finite index, we define

$$d(G, H) = \sum_{i=1}^{n}([H_i : H_{i-1}] - 1), \tag{1.2}$$

where $H_0 = H \subset H_1 \subset \cdots \subset H_n = G$ is any composition series from H to G.

By [S90, Theorem 6] and [S01, Theorem 3.1], for any subnormal subgroup H of a group G with $[G : H] < \infty$, we have $d(G, H) \geqslant f([G : H])$, and equality holds if and only if G/H_G is solvable, where $H_G = \bigcap_{g\in G} gHg^{-1}$ is the core of H in G (i.e., the largest normal subgroup of G contained in H).

The following result is stronger than Mycielski's conjecture.

Theorem 1.2 (I. Korec, Z. W. Sun)**.** *Let $a_1G_1,\dots,a_kG_k$ be left cosets of subnormal subgroups $G_1,\dots,G_k$ of a group G. If $\mathcal{A}=\{a_sG_s\}_{s=1}^k$ forms an exact m-cover of G, i.e., $\mathcal{A}$ covers each element of G exactly m times, then $[G:\bigcap_{s=1}^k G_s]<\infty$ and*

$$k\geqslant m+d\Big(G,\bigcap_{s=1}^k G_s\Big)\geqslant m+f\Big(\Big[G:\bigcap_{s=1}^k G_s\Big]\Big),$$

where the lower bound $m+d(G,\bigcap_{s=1}^k G_s)$ is best possible.

In the case $m=1$ and $G=\mathbb{Z}$, Theorem 1.2 was first conjectured by Znám [Z69]. When $m=1$ and $G_1,\dots,G_k$ are normal in G, Theorem 1.2 was obtained by Korec [K74] in 1974. In 1990 Sun [S90] deduced Theorem 1.2 in the case $m=1$ by a method different from that of Korec. The current version of Theorem 1.2 was established by Sun [S01] in 2001, the proof of which depends heavily on the condition that $\mathcal{A}$ covers all the elements of G the same number of times. Under the conditions of Theorem 1.2, Sun [S04] also showed that the indices $[G:G_s]$ $(1\leqslant s\leqslant k)$ cannot be distinct providing $k>1$.

Call a coset in an abelian group not containing the identity element a *proper coset*. In 2003 W. D. Gao and A. Geroldinger [GG] proved the following conjecture for any elementary abelian p-group G (they did not explicitly state this conjecture in [GG]).

Gao-Geroldinger Conjecture. *Let G be a finite abelian group with identity e. If $G\setminus\{e\}$ is a union of k proper cosets $a_1G_1,\dots,a_kG_k$ then we have $k\geqslant f(|G|)$.*

With the notations of the Gao-Geroldinger conjecture, if we set $a_0=e$ and $G_0=\{e\}$ then $\{a_sG_s\}_{s=0}^k$ forms a cover of G with $a_0G_0\cap a_sG_s=\emptyset$ for all $s=1,\dots,k$. Thus, by the result of [Z69], the Gao-Geroldinger conjecture holds when G is cyclic.

In this paper we aim to generalize Mycielski's conjecture in a new direction and prove an extended version of the Gao-Geroldinger conjecture.

Definition 1.3. Let G be a group and let $\mathcal{A}=\{a_sG_s\}_{s=1}^k$ be a finite system of left cosets of subgroups $G_1,\dots,G_k$. The *covering function* of $\mathcal{A}$ is given by

$$w_{\mathcal{A}}(x)=|\{1\leqslant s\leqslant k:\ x\in a_sG_s\}|\quad (x\in G).\tag{1.3}$$

Let m be a positive integer. We call $\mathcal{A}$ an *m-cover* of G if $w_{\mathcal{A}}(x)\geqslant m$ for all $x\in G$. If $\mathcal{A}$ forms an m-cover of G but none of its proper subsystems does, then $\mathcal{A}$ is said to be a *minimal m-cover* of G.

Now we state our main result in this paper, which (in the special case $m=1$) implies the Gao-Geroldinger conjecture for arbitrary finite abelian groups.

Theorem 1.3. *Let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be an m-cover of an abelian group G by left cosets. Then, for any $a \in G$ with $w_{\mathcal{A}}(a) = m$, we have*

$$N_a = \Big[G : \bigcap_{\substack{1\leqslant s\leqslant k\\ a\in a_sG_s}} G_s\Big] \leqslant 2^{k-m} \quad \text{and furthermore} \quad k \geqslant m + f(N_a). \tag{1.4}$$

In particular, if $\{a_sG_s\}_{s\neq t}$ fails to be an m-cover of G, then we have the inequalities

$$[G : G_t] \leqslant 2^{k-m} \quad \text{and} \quad k \geqslant m + f([G : G_t]), \tag{1.5}$$

the bounds of which are best possible.

Remark 1.2. When $G = \mathbb{Z}$, Theorem 1.3 was proved by Znám [Z75] in the case $m = 1$, and we can say something stronger in Section 2. Also, in the second inequality of (1.4), N_a cannot be replaced by $[G : \bigcap_{s=1}^k G_s]$ as illustrated by the following example.

Example 1.1. Let G be the abelian group $C_p \times C_p$ where p is a prime and C_p is the cyclic group of order p. Then any element $a \neq e$ of G has order p. Let $G_1, \ldots, G_k$ be all the distinct subgroups of G with order p. If $1 \leqslant i < j \leqslant k$, then $G_i \cap G_j = \{e\}$. Thus $\{G_s\}_{s=1}^k$ forms a minimal 1-cover of G with $\bigcap_{s=1}^k G_s = \{e\}$. Since $1 + k(p-1) = |\bigcup_{s=1}^k G_s| = |G| = p^2$, we have

$$k = p + 1 \geqslant 1 + f([G : G_s]) = 1 + f(p) = p.$$

However,

$$k = p + 1 \leqslant 2p - 1 = 1 + f([G : \{e\}]) = 1 + d\Big(G, \bigcap_{s=1}^k G_s\Big),$$

and the last inequality becomes strict when $p > 2$.

Example 1.1 also shows that we don't have an analogy of [S01, Theorem 2.1] for minimal m-covers of the abelian group $C_p \times C_p$ (where p is a prime), thus we cannot prove our Theorem 1.3 by the method in [S01]. To obtain Theorem 1.3 we employ some tools from algebraic number theory as well as characters of abelian groups.

Corollary 1.1. *Let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be an m-cover of a group G by left cosets. Provided that $a \in G$ and $w_{\mathcal{A}}(a) = m$, for any abelian subgroup K of G we have*

$$\begin{aligned} k - m \geqslant & |\{1 \leqslant s \leqslant k : \ a \notin a_sG_s \text{ and } K \not\subseteq G_s\}| \\ \geqslant & f\Big(\Big[K : K \cap \bigcap_{\substack{s=1\\ a\in a_sG_s}}^{k} G_s\Big]\Big). \end{aligned} \tag{1.6}$$

In particular, if $\{a_sG_s\}_{s\neq t}$ *fails to be an* m*-cover of* G*, then for any abelian subgroup* K *of* G *not contained in* G_t *we have*

$$|\{1\leqslant s\leqslant k:\ K\not\subseteq G_s\}|\geqslant 1+f([K:G_t\cap K]). \tag{1.7}$$

Proof. Let $J=\{1\leqslant s\leqslant k:\ a_sG_s\cap aK\neq\emptyset\}$. For each $s\in J$, $a^{-1}a_sG_s\cap K$ is a coset of $G_s\cap K$ in K. Observe that $\{a^{-1}a_sG_s\cap K\}_{s\in J}$ is an m-cover of K with $|\{s\in J:\ e\in a^{-1}a_sG_s\cap K\}|=|I_a|=m$ where $I_a=\{1\leqslant s\leqslant k:\ a\in a_sG_s\}$. Applying Theorem 1.3 to the abelian group K we get the inequality $|J|-m\geqslant f([K:\bigcap_{s\in I_a}G_s\cap K])$. If $s\in J$ and $K\subseteq G_s$, then $a^{-1}a_sG_s\cap K=K$ and hence $s\in I_a$. Thus

$$\begin{aligned}|J|-m=&|\{s\in J:\ e\notin a^{-1}a_sG_s\cap K\}|\\ \leqslant&|\{1\leqslant s\leqslant k:\ a\notin a_sG_s \text{ and } K\not\subseteq G_s\}|\leqslant k-m\end{aligned}$$

and hence (1.6) follows.

Now suppose that $\{a_sG_s\}_{s\neq t}$ is not an m-cover of G and K is an abelian subgroup of G with $K\not\subseteq G_t$. Then $w_{\mathcal{A}}(x)=m$ for some $x\in a_tG_t$. In light of the above,

$$\begin{aligned}&|\{1\leqslant s\leqslant k:\ s\neq t \text{ and } K\not\subseteq G_s\}|\\ \geqslant&|\{1\leqslant s\leqslant k:\ x\notin a_sG_s \text{ and } K\not\subseteq G_s\}|\geqslant f([K:K\cap G_t]).\end{aligned}$$

This proves (1.7) and we are done. □

Corollary 1.2. *Let* R *be any ring. Let* $a_1,\dots,a_k$ *be elements of* R *and* $I_1,\dots,I_k$ *ideals of* R*. If* $\{a_s+I_s\}_{s=1}^k$ *is an* m*-cover of* R *with the coset* a_t+I_t *irredundant, then for the quotient ring* R/I_t *we have* $|R/I_t|\leqslant 2^{k-m}$ *and furthermore* $k\geqslant m+f(|R/I_t|)$*.*

Proof. Since R is an additive abelian group, this follows from Theorem 1.3 immediately. □

In the next section we will present a new approach to Mycielski's problem on covers of $\mathbb{Z}$. In Section 3 we are going to work with covers of abelian groups and extend some ideas in Section 2, this will lead to our proof of Theorem 1.3.

2. A new approach to Mycielski's problem

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of the rational field $\mathbb{Q}$ and $\overline{\mathbb{Z}}$ the ring of all algebraic integers in $\overline{\mathbb{Q}}$.

Lemma 2.1. *For* $s=1,\dots,k$ *let* $\zeta_s\in\overline{\mathbb{Z}}$ *be a root of unity with order* $n_s>1$*. Then* $n\in\mathbb{Z}^+$ *divides* $\prod_{s=1}^k(1-\zeta_s)$ *in* $\overline{\mathbb{Z}}$*, if and only if we have*

$$\sum_{\substack{s=1\\ P(n_s)=\{p\}}}^{k}\frac{1}{\varphi(n_s)}\geqslant \mathrm{ord}_p(n)\quad \textit{for any prime } p, \tag{2.1}$$

where φ is the well-known Euler function.

Proof. For each prime p, let $\mathbf{v}_p : \overline{\mathbb{Q}} \to \mathbb{Q}$ denote any extension of the p-adic valuation $\mathrm{ord}_p(\cdot)$ to $\overline{\mathbb{Q}}$, normed by $\mathbf{v}_p(p) = 1$. It is well known (cf. [W, Chap. 2]) that

$$\mathbf{v}_p(1-\zeta_s) = \begin{cases} 1/\varphi(n_s) & \text{if } n_s \text{ is a power of } p, \\ 0 & \text{otherwise.} \end{cases}$$

Now n divides $\prod_{s=1}^k(1-\zeta_s)$ in $\overline{\mathbb{Z}}$, if and only if for each valuation $\mathbf{v} : \overline{\mathbb{Q}} \to \mathbb{Q}$ one has $\mathbf{v}(n) \leqslant \sum_{s=1}^k \mathbf{v}(1-\zeta_s)$. Since any valuation $\mathbf{v}$ of $\overline{\mathbb{Q}}$ is (equivalent to) an extension of $\mathrm{ord}_p(\cdot)$ for some prime p, we immediately obtain the desired result. $\square$

Corollary 2.1. *Let $n > 1$ be an integer. Then $f(n)$ is the smallest positive integer k such that there are roots of unity $\zeta_1, \dots, \zeta_k$ different from 1 for which $\prod_{s=1}^k(1-\zeta_s) \in n\overline{\mathbb{Z}}$. Furthermore, this holds with $k = f(n)$ if and only if for any prime divisor p of n there are exactly $\mathrm{ord}_p(n)(p-1)$ of $\zeta_1, \dots, \zeta_k$ having order p.*

Proof. For $s = 1, \dots, k$ let ζ_s be a root of unity with order $n_s > 1$. By Lemma 2.1, n divides $\prod_{s=1}^k(1-\zeta_s)$ in $\overline{\mathbb{Z}}$ if and only if (2.1) holds. Clearly

$$\sum_{\substack{s=1 \\ P(n_s)=\{p\}}}^{k} \frac{1}{\varphi(n_s)} \leqslant \frac{|\{1 \leqslant s \leqslant k :\ P(n_s) = \{p\}\}|}{p-1} \quad \text{for every prime } p.$$

If (2.1) is valid, then

$$k \geqslant \sum_{p \in P(n)} |\{1 \leqslant s \leqslant k :\ P(n_s) = \{p\}\}| \geqslant \sum_{p\in P(n)} \mathrm{ord}_p(n)(p-1) = f(n).$$

Now assume that $k = f(n)$. When (2.1) is valid, equality holds in the last three inequalities and hence

$$|\{1 \leqslant s \leqslant k :\ n_s = p\}| = |\{1 \leqslant s \leqslant k :\ P(n_s) = \{p\}\}| = \mathrm{ord}_p(n)(p-1)$$

for any prime p. Conversely, (2.1) holds if $|\{1 \leqslant s \leqslant k : n_s = p\}| = \mathrm{ord}_p(n)(p-1)$ for all $p \in P(n)$.

Combining the above we have completed the proof. $\square$

Lemma 2.2. *Suppose that $A = \{a_s(n_s)\}_{s=1}^k$ is an m-cover of $\mathbb{Z}$ by residue classes and $a \in \mathbb{Z}$ is covered by A exactly m times. Let N_a be the least common multiple of those n_s with $a \in a_s(n_s)$, and let $m_s \in \mathbb{Z}$ for $s \in J$ where $J = \{1 \leqslant s \leqslant k :\ a \notin a_s(n_s)\}$. Then, for any $0 \leqslant \alpha < 1$ we have*

$$C_0(\alpha) = C_1(\alpha) = \cdots = C_{N_a-1}(\alpha), \tag{2.2}$$

where

$$C_r(\alpha)=\sum_{\substack{I\subseteq J\\ \{\sum_{s\in I}m_s/n_s\}=(\alpha+r)/N_a}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}(a_s-a)m_s/n_s} \tag{2.3}$$

for every $r=0,1,\ldots,N_a-1$, *and we use* $\{\theta\}$ *to denote the fractional part of a real number* θ.

Proof. This follows from [S99, Lemma 2]. □

Theorem 2.1. *Let* $A=\{a_s(n_s)\}_{s=1}^k$ *be an* m*-cover of* $\mathbb{Z}$, *and suppose that* a *is an integer with* $w_A(a)=m$. *Then* $k\geqslant m+f(N_a)$ *where* N_a *is the least common multiple of those* n_s *with* $a\in a_s(n_s)$. *Furthermore, for any prime* p *we have*

$$|I(p)|\geqslant\sum_{s\in I(p)}\frac{1}{p^{\operatorname{ord}_p(n_s)-\operatorname{ord}_p(a_s-a)-1}}\geqslant\operatorname{ord}_p(N_a)(p-1), \tag{2.4}$$

where

$$I(p)=\left\{1\leqslant s\leqslant k:\ \frac{n_s}{p^{\operatorname{ord}_p(n_s)}}\ \middle|\ a_s-a\ \text{but}\ n_s\nmid a_s-a\right\}. \tag{2.5}$$

Proof. Let $J=\{1\leqslant s\leqslant k:\ a\notin a_s(n_s)\}$. For each $s\in J$, let m_s be an integer not divisible by $n_s/(n_s,a_s-a)>1$, then $\zeta_s=e^{2\pi i(a_s-a)m_s/n_s}$ is a primitive d_sth root of unity where $d_s=n_s/(n_s,(a_s-a)m_s)>1$.

Set

$$S=\left\{\left\{N_a\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq J\right\}.$$

Then

$$\begin{aligned}\prod_{s\in J}(1-\zeta_s)&=\sum_{I\subseteq J}(-1)^{|I|}e^{2\pi i\sum_{s\in I}(a_s-a)m_s/n_s}\\&=\sum_{\alpha\in S}\sum_{\substack{I\subseteq J\\ \{N_a\sum_{s\in I}m_s/n_s\}=\alpha}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}(a_s-a)m_s/n_s}\\&=\sum_{\alpha\in S}\sum_{r=0}^{N_a-1}C_r(\alpha)=N_a\sum_{\alpha\in S}C_0(\alpha),\end{aligned}$$

where $C_r(\alpha)$ $(0\leqslant r<N_a)$ are given by (2.3). So N_a divides $\prod_{s\in J}(1-\zeta_s)$ in the ring $\overline{\mathbb{Z}}$. By Corollary 2.1, we have $k-m=|J|\geqslant f(N_a)$. In view of Lemma 2.1,

$$\sum_{\substack{s\in J\\ P(d_s)=\{p\}}}\frac{1}{\varphi(d_s)}\geqslant\operatorname{ord}_p(N_a)\quad\text{for each prime } p.$$

Now we simply let $m_s = 1$ for all $s \in J$. By the above, for any prime p we have

$$\sum_{s\in I(p)} \frac{1}{\varphi(n_s/(n_s, a_s - a))} \geqslant \mathrm{ord}_p(N_a)$$

which is equivalent to (2.4). This concludes the proof. □

3. Working with abelian groups

We first recall some well-known facts from the theory of characters of finite abelian groups (see, e.g. [W, pp. 22-23]).

For a finite abelian group G, let $\widehat{G}$ denote the group of all complex-valued characters of G. One has $\widehat{G} \cong G$. For any subgroup H of G let $H^\perp$ denote the group of those characters $\chi \in \widehat{G}$ with $\ker(\chi) = \{x \in G : \chi(x) = 1\}$ containing H. Then there is a canonical isomorphism $H^\perp \cong \widehat{G/H}$ by putting $\chi(aH) = \chi(a)$ for any $a \in G$ and any $\chi \in H^\perp$. Furthermore, for each $a \in G \setminus H$ there exists some $\chi \in H^\perp$ with $\chi(a) \neq 1$.

`Proof of Theorem 1.3`. Choose a minimal $I_* \subseteq \{1, \dots, k\}$ such that the system $\{a_s G_s\}_{s\in I_*}$ forms an m-cover of G. As $I_a = \{1 \leqslant s \leqslant k : a \in a_s G_s\}$ has cardinality m, I_a is contained in I_*. So we can simply assume that $\mathcal{A}$ is a minimal m-cover of G (i.e., $I_* = \{1, \dots, k\}$). By [S90, Corollary 1], $H = \bigcap_{s=1}^k G_s$ is of finite index in G. Instead of the minimal m-cover $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ of G, we may consider the minimal m-cover $\bar{\mathcal{A}} = \{\bar{a}_s \bar{G}_s\}_{s=1}^k$ of the finite abelian group $\bar{G} = G/H$, where $\bar{a}_s = a_s H$ and $\bar{G}_s = G_s/H$ (hence $[\bar{G} : \bar{G}_s] = [G : G_s]$). Therefore, without any loss of generality, we can assume that G is finite.

Put $H_a = \bigcap_{s\in I_a} G_s$; then $|H_a^\perp| = [G : H_a] = N_a$.

Note that $J = \{1 \leqslant j \leqslant k : a \notin a_j G_j\}$ has cardinality $k - m$. For each $j \in J$ we may choose a $\chi_j \in G_j^\perp$ with $\zeta_j := \chi_j(a^{-1}a_j) \neq 1$. For any $x \in G \setminus H_a$ we have $ax \notin \bigcap_{s\in I_a} aG_s = \bigcap_{s\in I_a} a_s G_s$. Since $\mathcal{A}$ is an m-cover of G, there exists some $j \in J$ with $ax \in a_j G_j$, and therefore $\chi_j(x) = \zeta_j$ by the choice of χ_j and the definition of ζ_j.

For $x \in G$ we define

$$\Psi(x) = \prod_{j\in J} (\chi_j(x) - \zeta_j)\,.$$

If $\chi \in H_a^\perp$ and $\chi(x) \neq 1$, then $x \notin H_a$ and hence $\Psi(x) = 0$ by the above. Thus $\Psi\chi = \Psi$ for all $\chi \in H_a^\perp$.

Observe that

$$\Psi(x) = \sum_{I\subseteq J} \Big(\prod_{j\in I} \chi_j(x)\Big) \prod_{j\in J\setminus I} (-\zeta_j) = \sum_{\psi\in\widehat{G}} c(\psi)\psi(x),$$

where

$$c(\psi) = \sum_{\substack{I \subseteq J \\ \prod_{j\in I}\chi_j = \psi}} \prod_{j\in J\setminus I} (-\zeta_j) \in \overline{\mathbb{Z}}.$$

Let $\mathbb{C}$ be the complex field. As the set $\widehat{G}$ is a basis of the $\mathbb{C}$-vector space

$$\mathbb{C}^G = \{g:\ g \text{ is a function from } G \text{ to } \mathbb{C}\}$$

(cf. [J, p. 291]), for any $\chi \in H_a^\perp$ we have $c(\psi\chi) = c(\psi)$ for all $\psi \in \widehat{G}$ because $\Psi\chi^{-1} = \Psi$.

Clearly

$$\prod_{j\in J}(1-\zeta_j) = \Psi(e) = \sum_{\psi\in\widehat{G}} c(\psi)\psi(e) = \sum_{\psi\in\widehat{G}} c(\psi).$$

Let $\psi_1 H_a^\perp \cup \cdots \cup \psi_l H_a^\perp$ be a coset decomposition of $\widehat{G}$ where $l = [\widehat{G} : H_a^\perp]$. Then

$$\sum_{\psi\in\widehat{G}} c(\psi) = \sum_{r=1}^{l}\sum_{\chi\in H_a^\perp} c(\psi_r\chi) = \sum_{r=1}^{l} |H_a^\perp| c(\psi_r) = N_a \sum_{r=1}^{l} c(\psi_r).$$

(That $c(\psi_r\chi) = c(\psi_r)$ for all $\chi \in H_a^\perp$ is an analogy of Lemma 2.2.) Therefore N_a divides $\prod_{j\in J}(1-\zeta_j)$ in $\overline{\mathbb{Z}}$, and Corollary 2.1 gives $k-m = |J| \geqslant f(N_a)$, and consequently $N_a \leqslant 2^{k-m}$ by Remark 1.1.

If $\{a_sG_s\}_{s\neq t}$ is not an m-cover of G, then for some $x \in a_tG_t$ we have $w_{\mathcal{A}}(x) = m$, hence $k-m \geqslant f(N_x) \geqslant f([G:G_t])$ and $[G:G_t] \leqslant N_x \leqslant 2^{k-m}$ by the above.

By [S01, Example 1.2], for any subgroup H of G (with $[G:H] < \infty$) and an arbitrary element x of G, the coset xH and $m-1+d(G,H) = m-1+f([G:H])$ other cosets of subgroups containing H form an (exact) m-cover of G with xH irredundant. Also, $m-1$ copies of $0(1)$, together with the following $k-m+1$ residue classes

$$1(2),\ 2(2^2),\ \ldots,\ 2^{k-m-1}(2^{k-m}),\ 0(2^{k-m}),$$

clearly form an (exact) m-cover of $\mathbb{Z}$ with the residue class $0(2^{k-m})$ irredundant. So the inequalities in (1.5) are really best possible and we are done. □

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References

[GG] W. D. Gao and A. Geroldinger, *Zero-sum problems and coverings by proper cosets*, European J. Combin. **24**(2003), 531–549.

[G04] R. K. Guy, *Unsolved Problems in Number Theory*, 3rd ed., Springer, 2004, Sections F13 and F14.

[J] N. Jacobson, *Basic Algebra II*, 2nd ed., Freeman & Co., 1985.

[K74] I. Korec, *On a generalization of Mycielski's and Znám's conjectures about coset decomposition of Abelian groups*, Fund. Math. **85**(1974), 41–48.

[MS] J. Mycielski and W. Sierpiński, *Sur une propriété des ensembles linéaires*, Fund. Math. **58**(1966), 143–147.

[N1] B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. **29**(1954), 236–248.

[N2] B. H. Neumann, *Groups covered by finitely many cosets*, Publ. Math. Debrecen **3**(1954), 227–242.

[PS] Š. Porubský and J. Schönheim, *Covering systems of Paul Erdös: past, present and future*, in: Paul Erdös and his Mathematics. I (edited by G. Halász, L. Lovász, M. Simonvits, V. T. Sós), Bolyai Soc. Math. Studies 11, Budapest, 2002, pp. 581–627.

[S90] Z. W. Sun, *Finite coverings of groups*, Fund. Math. **134**(1990), 37–53.

[S99] Z. W. Sun, *On covering multiplicity*, Proc. Amer. Math. Soc. **127**(1999), 1293–1300.

[S01] Z. W. Sun, *Exact m-covers of groups by cosets*, European J. Combin. **22**(2001), 415–429.

[S03] Z. W. Sun, *Unification of zero-sum problems, subset sums and covers of $\mathbb{Z}$*, Electron. Res. Announc. Amer. Math. Soc. **9**(2003), 51–60.

[S04] Z. W. Sun, *On the Herzog-Schönheim conjecture for uniform covers of groups*, J. Algebra **273**(2004), 153–175.

[S05] Z. W. Sun, *On the range of a covering function*, J. Number Theory **111**(2005), 190–196.

[W] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer, New York, 1982.

[Z66] Š. Znám, *On Mycielski's problem on systems of arithmetical progressions*, Colloq. Math. **15**(1966), 201–204.

[Z69] Š. Znám, *A remark to a problem of J. Mycielski on arithmetic sequences*, Colloq. Math. **20**(1969), 69–70.

[Z75] Š. Znám, *On properties of systems of arithmetic sequences*, Acta Arith. **26**(1975), 279–283.