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ZERO-SUM PROBLEMS FOR ABELIAN p-GROUPS AND COVERS OF THE INTEGERS BY RESIDUE CLASSES

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Abstract. Zero-sum problems for abelian groups and covers of the integers by residue classes, are two different active topics initiated by P. Erdős more than 40 years ago and investigated by many researchers separately since then. In an earlier announcement [Electron. Res. Announc. Amer. Math. Soc. **9**(2003), 51-60], the author claimed some surprising connections among these seemingly unrelated fascinating areas. In this paper we establish further connections between zero-sum problems for abelian p-groups and covers of the integers. For example, we extend the famous Erdős-Ginzburg-Ziv theorem in the following way: If $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ covers each integer either exactly $2q-1$ times or exactly $2q$ times where q is a prime power, then for any $c_1,\ldots,c_k\in\mathbb{Z}/q\mathbb{Z}$ there exists an $I\subseteq\{1,\ldots,k\}$ such that $\sum_{s\in I}1/n_s=q$ and $\sum_{s\in I}c_s=0$. Our main theorem in this paper unifies many results in the two realms and also implies an extension of the Alon-Friedland-Kalai result on regular subgraphs.

1. Introduction

Let G be an abelian group (written additively). By $\mathcal{F}(G)$ we mean the set of all finite sequences of elements of G (with repetition allowed). A sequence $\{c_s\}_{s=1}^k\in\mathcal{F}(G)$ (which is often written as $\prod_{s=1}^k c_s$ by A. Geroldinger and his followers) is called a *zero-sum sequence* if $\sum_{s=1}^k c_s=0$.

In 1961 Erdős, Ginzburg and Ziv [EGZ] established the following celebrated theorem which initiated the study of zero-sum sequences.

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Theorem 1.1 (EGZ Theorem). *Let $n \in \mathbb{Z}^+ = \{1,2,3,\dots\}$. For any $c_1,\dots,c_{2n-1} \in \mathbb{Z}$, there is an $I \subseteq [1,2n-1] = \{1,\dots,2n-1\}$ with $|I| = n$ such that $\sum_{s\in I} c_s \equiv 0 \pmod{n}$. In other words, any sequence in $\mathcal{F}(\mathbb{Z}_n)$ of length $2n-1$ contains a zero-sum subsequence of length n, where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the additive group of residue classes modulo n.*

The EGZ theorem can be easily reduced to the case where n is a prime (and hence $\mathbb{Z}_n$ is a field), and then deduced from the well-known Cauchy-Davenport theorem or the Chevalley-Warning theorem. (See, e.g., Nathanson [N, pp. 48–51], and Geroldinger and Halter-Koch [GH, p. 349].) It remains valid if we replace the cyclic group $\mathbb{Z}_n$ by an arbitrary abelian group of order n. (Cf. T. Tao and V. Vu [TV, pp. 350–351].)

Let G be a finite abelian group. When $|G| > 1$, there is a unique sequence $d_1,\dots,d_r$ of positive integers with $d_1 > 1$ and $d_i \mid d_{i+1}$ for $1 \leqslant i < r$ such that G is isomorphic to the direct sum

$$\mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r};$$

in this case, r is $\operatorname{rank}(G)$ (the rank of G) and d_r is $\exp(G)$ (the exponent of G), and we define

$$d^*(G) = \sum_{i=1}^{r}(d_i - 1). \tag{1.1}$$

If $|G| = 1$, then $\operatorname{rank}(G) = \exp(G) = 1$ and we set $d^*(G) = 0$. Clearly $d^*(G) + 1 \leqslant |G|$.

Let G be a finite abelian group written additively. By $s(G)$ we denote the smallest positive integer k such that any sequence in $\mathcal{F}(G)$ of length k has a zero-sum subsequence of length $\exp(G)$. For any $n \in \mathbb{Z}^+$, we have $s(\mathbb{Z}_n) = 2n-1$ by the EGZ theorem, and $s(\mathbb{Z}_n \oplus \mathbb{Z}_n) = 4n-3$ by the Kemnitz conjecture proved by C. Reiher [Re] (see also [SC]), and $s(\mathbb{Z}_d \oplus \mathbb{Z}_n) = 2(d+n)-3$ by Theorem 5.8.3 of Geroldinger and Halter-Koch [GH, p. 362] where d is any positive divisor of n. The reader is referred to the survey [GG06], and the recent papers [E] and [EEGKR] for various problems and results on $s(G)$.

Let G be an abelian group of order n. For any $\{c_s\}_{s=1}^n \in \mathcal{F}(G)$, as the following elements

$$0,\ c_1,\ c_1+c_2,\ \dots,\ c_1+c_2+\cdots+c_n$$

cannot be distinct by the pigeon-hole principle, we have $\sum_{s\in I} c_s = 0$ for some $\emptyset \neq I \subseteq [1,n]$; furthermore, $\{c_s\}_{s=1}^n$ has a zero-sum subsequence $\{c_s\}_{s\in I}$ with $\emptyset \neq I \subseteq [1,n]$ and $\sum_{s\in I} 1/\operatorname{ord}(c_s) \leqslant 1$, by a celebrated theorem of Geroldinger [G93] (which was re-proved later by Elledge and Hurlbert [EH] via graph pebbling). The *Davenport constant* $D(G)$ of G is defined as the smallest positive integer k such that any sequence $\{c_s\}_{s=1}^k \in$

$\mathcal{F}(G)$ has a nonemepty zero-sum subsequence. (Note that we essentially impose no restriction on the length of the required zero-sum subsequence.) By the above, $D(G) \leqslant n = |G|$. In 1966 Davenport showed that if K is an algebraic number field with ideal class group G, then $D(G)$ is the maximal number of prime ideals (counting multiplicity) in the decomposition of an irreducible algebraic integer in K. The reader may consult Theorem 5.1.5 of [GH, pp. 305–306] for further results in this direction.

It is easy to see that $D(\mathbb{Z}_n) = n$ for any $n \in \mathbb{Z}^+$. For an abelian p-group G with p a prime, $D(G)$ is greater than $d^*(G)$ by a constructive example; on the other hand, in 1969 Olson [O] used the knowledge of group rings to show that $D(G) \leqslant d^*(G) + 1$ (and hence $D(G) = d^*(G)+1$). Olson's original method has been further refined and explored by many researchers, see, e.g., [GGH].

Theorem 1.2 (Olson's Theorem)**.** *Let p be a prime and let G be an additive abelian p-group. Given $c, c_1, \ldots, c_{d^*(G)+1} \in G$ we have*

$$\sum_{\substack{I\subseteq[1,d^*(G)+1]\\ \sum_{s\in I} c_s=c}} (-1)^{|I|} \equiv 0 \pmod p,$$

and in particular $\{c_s\}_{s=1}^{d^*(G)+1}$ has a nonempty zero-sum subsequence.

Let q be a power of a prime p, and let $c \in \mathbb{Z}_q$ and $\{c_s\}_{s=1}^{2q-1} \in \mathcal{F}(\mathbb{Z}_q)$. By Olson's theorem in the case $G = \mathbb{Z}_q^2 = \mathbb{Z}_q \oplus \mathbb{Z}_q$, we have

$$\sum_{\substack{I\subseteq[1,d^*(\mathbb{Z}_q^2)+1]\\ q\mid |I|,\ \sum_{s\in I} c_s=c}} (-1)^{|I|} \equiv 0 \pmod p.$$

In other words,

$$\left|\left\{I \subseteq [1, 2q-1] :\ |I| = q \text{ and } \sum_{s\in I} c_s = c\right\}\right| \equiv [\![c = 0]\!] \pmod p,$$

where for a predicate P we let $[\![P]\!]$ be 1 or 0 according as P holds or not. Thus, Olson's theorem implies the EGZ theorem.

Let G be a finite abelian group. A zero-sum sequence $\{c_s\}_{s=1}^{k}$ is called a *minimal zero-sum sequence* if $\sum_{s\in I} c_s = 0$ for no $\emptyset \neq I \subset [1, k]$. Though we don't study minimal zero-sum sequences in this paper, the reader is still recommended to see [GG99], [GGS] and [LS] for some results on minimal zero-sum sequences.

Now we turn to covers of the integers by residue classes.

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we call

$$a(n) = a + n\mathbb{Z} = \{a + nx :\ x \in \mathbb{Z}\}$$

a residue class with modulus n. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \tag{1.2}$$

of residue classes, its *covering function*

$$w_A(x) = |\{1 \leqslant s \leqslant k : x \in a_s(n_s)\}|$$

is periodic modulo the least common multiple $N_A = [n_1, \ldots, n_k]$ of the moduli $n_1, \ldots, n_k$. Sun [S97, S99] called $m(A) = \min_{x\in\mathbb{Z}} w_A(x)$ the *covering multiplicity* of (1.2). One can easily verify the following basic property:

$$\sum_{s=1}^k \frac{1}{n_s} = \frac{1}{N_A} \sum_{x=0}^{N_A-1} w_A(x) \geqslant m(A). \tag{1.3}$$

Further properties of the covering function $w_A(x)$ can be found in [S03a, S04].

If $\bigcup_{s=1}^k a_s(n_s) = \mathbb{Z}$ (i.e., $m(A) \geqslant 1$), then we call (1.2) a *cover* (or *covering system*) of $\mathbb{Z}$. This concept was first introduced by Erdős in the early 1930's (cf. [E50]), and many surprising applications have been found (cf. [Cr], [Gra], [S00] and [S01]). Erdős was very proud of this invention; in [E97] he said: *"Perhaps my favorite problem of all concerns covering systems."*

For $m \in \mathbb{Z}^+$, if $m(A) \geqslant m$ then A is said to be an *m-cover* of $\mathbb{Z}$; general m-covers were first studied by the author in [S95]. It is easy to construct an m-cover of $\mathbb{Z}$ which cannot be split into two covers of $\mathbb{Z}$ (cf. [PS, Example 1.1]).

If $w_A(x) = m$ for all $x \in \mathbb{Z}$, then we call (1.2) an *exact m-cover* of $\mathbb{Z}$. (Note that in this case we have $\sum_{s=1}^k 1/n_s = m$ by (1.3).) Clearly m copies of $0(1)$ form a trivial exact m-cover of $\mathbb{Z}$. Using a graph-theoretic argument Zhang [Z91] proved that for each $m = 2, 3, \ldots$ there are infinitely many exact m-covers of $\mathbb{Z}$ which cannot split into an exact n-cover and an exact $(m-n)$-cover with $0 < n < m$; such an exact m-cover is said to be *irreducible*. In 1973 Choi supplied the following example of an irreducible exact 2-cover of $\mathbb{Z}$:

$$\{1(2); 0(3); 2(6); 0, 4, 6, 8(10); 1, 2, 4, 7, 10, 13(15); 5, 11, 12, 22, 23, 29(30)\}.$$

Zhang [Z91] showed that the residue classes

$$1(2), 0(6), 0(10), 0(14), 1(15), 1(15), 8(21), 8(21), 36(105), 36(105),$$

together with some residue classes modulo 210, form an irreducible exact 3-cover of $\mathbb{Z}$. In 1992 Sun [S92] proved that if $\{a_s(n_s)\}_{s=1}^k$ forms an exact

m-cover of $\mathbb{Z}$ then for each $n = 0, 1, \ldots, m$ there are at least $\binom{m}{n}$ subsets I of $[1, k]$ with $\sum_{s\in I} 1/n_s = n$.

There are many problems and results on covers of $\mathbb{Z}$; the reader may consult sections F13 and F14 of the book [Gu, pp. 383–390], the survey [P-S], and the recent papers [S05a] and [FFKPY].

Now we mention some properties of covers of $\mathbb{Z}$ related to Egyptian fractions. The first nontrivial result of this nature is the following one discovered by Zhang [Z89] with the help of the Riemann zeta function: If $\{a_s(n_s)\}_{s=1}^k$ forms a cover of $\mathbb{Z}$, then

$$\sum_{s\in I} \frac{1}{n_s} \in \mathbb{Z} \quad \text{for some nonempty } I \subseteq [1, k]. \tag{1.4}$$

The following theorem contains two different extensions of Zhang's result.

Theorem 1.3. *Let $\{a_s(n_s)\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$, and let $m_1, \ldots, m_k \in \mathbb{Z}^+$.*

(i) (Sun [S95, S96]) *There are at least m positive integers in the form $\sum_{s\in I} m_s/n_s$ with $I \subseteq [1, k]$.*

(ii) (Pan and Sun [PS]) *For any $J \subseteq [1, k]$ there are at least 2^m subsets I of $[1, k]$ with $\sum_{s\in I} m_s/n_s - \sum_{s\in J} m_s/n_s \in \mathbb{Z}$.*

Note that a residue class $a(n) = a + n\mathbb{Z}$ is a coset of the subgroup $n\mathbb{Z}$ of the additive group $\mathbb{Z}$. There are also some investigations on covers of a general group by left cosets of subgroups, see, e.g., [S06] and the references therein. Gao and Geroldinger [GG03] reduced some zero-sum problems to the study of covering a certain subset of an abelian group by few proper cosets. However, in this paper we are only interested in covers of the integers and their surprising connections with zero-sum problems.

The purpose of this paper is to show that some classical results of zero-sum nature, such as the EGZ theorem and Olson's theorem, are special cases of our general results on covers of $\mathbb{Z}$. The key point is to compare Davenport constants of abelian p-groups with covering multiplicities of covers of $\mathbb{Z}$.

In Section 2 we state our main results connecting zero-sum problems for abelian p-groups with covers of the integers. A more general theorem will be presented in the third section, together with some consequences; its proof will be given in Section 4.

2. Connections between Zero-sum Sequences and Covers of $\mathbb{Z}$

The following theorem reveals unexpected connections between zero-sum sequences and covers of $\mathbb{Z}$.

Theorem 2.1 (Main Theorem)**.** *Let G be an additive abelian p-group where p is a prime. Suppose that $A = \{a_s(n_s)\}_{s=1}^k$ is a $(d^*(G) + p^h)$-cover* *of* $\mathbb{Z}$ *with* $h \in \mathbb{N} = \{0, 1, \dots\}$. *Let* $c_1, \dots, c_k \in G$ *and* $m_1, \dots, m_k \in \mathbb{Z}$. *Then*

$$\left|\left\{I \subseteq [1,k] : \sum_{s\in I} c_s = c \text{ and } \sum_{s\in I} \frac{m_s}{n_s} \in \alpha + p^h\mathbb{Z}\right\}\right| \neq 1 \tag{2.1}$$

for any $c \in G$ *and rational number* α. *In particular,* $\{c_s\}_{s=1}^k$ *has a zero-sum subsequence* $\{c_s\}_{s\in I}$ *with* $\emptyset \neq I \subseteq [1,k]$ *satisfying the restriction* $\sum_{s\in I} m_s/n_s \in p^h\mathbb{Z}$.

Now we deduce various consequences from Theorem 2.1.

Corollary 2.1. *Let* $A = \{a_s(n_s)\}_{s=1}^k$ *be a* $(d^*(G)+1)$*-cover of* $\mathbb{Z}$ *where* G *is an additive abelian* p*-group. Let* $m_1, \dots, m_k \in \mathbb{Z}$. *Then any sequence* $\{c_i\}_{i=1}^k \in \mathcal{F}(G)$ *has a zero-sum subsequence* $\{c_s\}_{s\in I}$ *with* $\emptyset \neq I \subseteq [1,k]$ *such that* $\sum_{s\in I} m_s/n_s \in \mathbb{Z}$.

Proof. Simply apply Theorem 2.1 with $h = 0$. □

Remark 2.1. (a) For an abelian p-group G, if we apply Corollary 2.1 to the trivial $(d^*(G)+1)$-cover consisting of $d^*(G)+1$ copies of $0(1)$, we then obtain Olson's result $D(G) \leqslant d^*(G) + 1$. We conjecture that the desired result in Corollary 2.1 still holds when G is a finite abelian group and $A = \{a_s(n_s)\}_{s=1}^k$ is a $D(G)$-cover of $\mathbb{Z}$.

(b) That (1.4) holds for any cover (1.2), follows from Corollary 2.1 in the case $G = \{0\}$.

(c) If we apply Corollary 2.1 to the trivial group $G = \{0\}$ and the trivial cover $\{r(n)\}_{r=0}^{n-1}$ (where $n \in \mathbb{Z}^+$), then we get the basic result $D(\mathbb{Z}_n) = n$ in zero-sum theory.

Corollary 2.2. *Let* $\{a_s(n_s)\}_{s=1}^k$ *be an* m*-cover of* $\mathbb{Z}$ *and let* $m_1, \dots, m_k \in \mathbb{Z}$. *Assume that* m *is a prime power (i.e.,* $m = p^n$ *for some prime* p *and* $n \in \mathbb{N}$*). Then* $\sum_{s\in I} m_s/n_s \in m\mathbb{Z}$ *for some* $\emptyset \neq I \subseteq [1,k]$, *and in particular*

$$\sum_{s\in I} \frac{1}{n_s} \in m\mathbb{Z}^+ \quad \text{for some } I \subseteq [1,k]. \tag{2.2}$$

Moreover, for any $J \subseteq [1,k]$ *there is an* $I \subseteq [1,k]$ *with* $I \neq J$ *such that*

$$\sum_{s\in I} \frac{m_s}{n_s} - \sum_{s\in J} \frac{m_s}{n_s} \in m\mathbb{Z}.$$

Proof. Just apply Theorem 2.1 with $G = \{0\}$ and $\alpha = \sum_{s\in J} m_s/n_s$. □

Remark 2.2. Corollary 2.2 is a new extension of Zhang's result that (1.4) holds if (1.2) is a cover of $\mathbb{Z}$. We conjecture that the corollary remains valid if we remove the condition that m is a prime power. In the special case $n_1 = \dots = n_k = 1$, this conjecture yields the basic fact $D(\mathbb{Z}_m) = m$.

Theorem 2.1 also implies the following central result of this paper.

Theorem 2.2. *Let G be an abelian p-group with p a prime, and let $q > d^*(G)$ be a power of p (e.g., $q = |G|$).*

(i) *Let $A = \{a_s(n_s)\}_{s=1}^k$ with $\{w_A(x) : x \in \mathbb{Z}\} \subseteq [d^*(G) + q, 2q]$. Then any $\{c_s\}_{s=1}^k \in \mathcal{F}(G)$ has a zero-sum subsequence $\{c_s\}_{s\in I}$ with $I \subseteq [1, k]$ and $\sum_{s\in I} 1/n_s = q$.*

(ii) *Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact $3q$-cover of $\mathbb{Z}$. Then any zero-sum sequence $\{c_s\}_{s=1}^k \in \mathcal{F}(G \oplus G)$ has a zero-sum subsequence $\{c_s\}_{s\in I}$ with $I \subseteq [1, k]$ and $\sum_{s\in I} 1/n_s = q$.*

Proof. (i) By Theorem 2.1, $\{c_s\}_{s=1}^k$ has a zero-sum subsequence $\{c_s\}_{s\in I}$ with $\emptyset \neq I \subseteq [1, k]$ satisfying $\sum_{s\in I} 1/n_s \in q\mathbb{Z}$. Observe that

$$\sum_{s\in I} \frac{1}{n_s} \leqslant \sum_{s=1}^{k} \frac{1}{n_s} = \frac{1}{N_A} \sum_{x=0}^{N_A-1} w_A(x) \leqslant 2q.$$

If $\sum_{s\in I} 1/n_s \neq q$, then $\sum_{s\in I} 1/n_s = 2q$ and $w_A(x) = 2q$ for all $x \in \mathbb{Z}$. When A is an exact $2q$-cover of $\mathbb{Z}$, for the system $A_* = \{a_s(n_s)\}_{s=1}^{k-1}$ we still have $\{w_{A_*}(x) : x \in \mathbb{Z}\} \subseteq [d^*(G)+q, 2q]$, hence $\{c_s\}_{s=1}^k$ has a zero-sum subsequence $\{c_s\}_{s\in I_*}$ with $\emptyset \neq I_* \subseteq [1, k-1]$ and $\sum_{s\in I_*} 1/n_s = q$. This completes the proof of part (i).

(ii) Note that $d^*(G \oplus G) = 2d^*(G)$. As $3q - 1 > d^*(G \oplus G) + q$, $A_* = \{a_s(n_s)\}_{s=1}^{k-1}$ is a $(d^*(G \oplus G) + q)$-cover of $\mathbb{Z}$. Applying Theorem 2.1 to the system A_* we find that $\{c_s\}_{s=1}^k$ has a zero-sum subsequence $\{c_s\}_{s\in I}$ with $\emptyset \neq I \subseteq [1, k-1]$ and $n = \sum_{s\in I} 1/n_s \in q\mathbb{Z}$. As $n < \sum_{s=1}^k 1/n_s = 3q$, n is q or $2q$. If $n = 2q$, then for $\bar{I} = [1, k] \setminus I$ we have

$$\sum_{s\in \bar{I}} c_s = \sum_{s=1}^{k} c_s - \sum_{s\in I} c_s = 0 \text{ and } \sum_{s\in \bar{I}} \frac{1}{n_s} = \sum_{s=1}^{k} \frac{1}{n_s} - n = 3q - 2q = q.$$

This concludes the proof. □

Remark 2.3. It is interesting to view $1/n_s$ in Theorem 2.2 as a weight of $s \in [1, k]$. In the case $n_1 = \cdots = n_k = 1$, part (i) yields the EGZ theorem for abelian p-groups, and part (ii) gives Lemma 3.2 of Alon and Dubiner [AD], which is an indispensable lemma in the study of the Kemnitz conjecture (cf. [Ro] and [Re]). Note that our Theorem 2.2(i) is quite different from the so-called weighted EGZ theorem proved by Grynkiewicz [Gry].

Theorem 2.2 tells that our following conjecture holds when n is a prime power.

Conjecture 2.1. *Let G be a finite abelian group of order n.*

(i) *If* $\{a_s(n_s)\}_{s=1}^k$ *covers each integer either exactly* $2n-1$ *times or exactly* $2n$ *times, then any* $\{c_s\}_{s=1}^k \in \mathcal{F}(G)$ *has a zero-sum subsequence* $\{c_s\}_{s\in I}$ *with* $I \subseteq [1,k]$ *and* $\sum_{s\in I} 1/n_s = n$.

(ii) *When* $\{a_s(n_s)\}_{s=1}^k$ *forms an exact* $3n$*-cover of* $\mathbb{Z}$*, any zero-sum sequence* $\{c_s\}_{s=1}^k \in \mathcal{F}(G\oplus G)$ *has a zero-sum subsequence* $\{c_s\}_{s\in I}$ *with* $I \subseteq [1,k]$ *and* $\sum_{s\in I} 1/n_s = n$.

An undirected graph is said to be q-*regular* if all the vertices have degree q. In 1984 Alon, Friedland and Kalai [AFK1, AFK2] proved that if q is a prime power then any loopless (undirected) graph with average degree bigger than $2q-2$ and maximum degree at most $2q-1$ must contain a q-regular subgraph. Now we apply Theorem 2.1 to strengthen this result.

Theorem 2.3. *Let* G *be a loopless graph of* l *vertices with the edge set* $\{1,\ldots,k\}$*. Suppose that all the vertices of* G *have degree not exceeding* $2p^n-1$ *and that* $\{a_s(n_s)\}_{s=1}^k$ *forms an* $(l(p^n-1)+p^h)$*-cover of* $\mathbb{Z}$*, where* p *is a prime and* $n,h\in\mathbb{N}$*. Then, for any* $m_1,\ldots,m_k\in\mathbb{Z}$*, there exists a* p^n*-regular subgraph* H *of* G *with* $\sum_{s\in E(H)} m_s/n_s \in p^h\mathbb{Z}$*, where* $E(H)$ *denotes the edge set of* H.

Proof. Let $v_1,\ldots,v_l$ be all the vertices of graph G. For $s\in[1,k]$ and $t\in[1,l]$, set

$$\delta_{st} = [\![v_t \text{ is an endvertex of the edge } s]\!].$$

Note that $\sum_{s=1}^k \delta_{st}$ is just the degree $d_G(v_t)$.

By Theorem 2.1 in the case $G=\mathbb{Z}_{p^n}^l$, there is a nonempty $I\subseteq[1,k]$ such that $\sum_{s\in I} m_s/n_s \in p^h\mathbb{Z}$ and $p^n \mid \sum_{s\in I}\delta_{st}$ for all $t\in[1,l]$. Let V_I be the set of vertices incident with edges in I, and let H be the subgraph (V_I, I) of graph G. As $\sum_{s\in I}\delta_{st} \leqslant d_G(v_t) \leqslant 2p^n-1$, $p^n \mid \sum_{s\in I}\delta_{st}$ for all $t\in[1,l]$ if and only if $d_H(v)=p^n$ for all $v\in V_I$ (i.e., H is a p^n-regular subgraph of G). This concludes the proof. □

Remark 2.4. For the graph G in Theorem 2.3, clearly

$$k \geqslant l(p^n-1)+1 \Leftrightarrow 2k > l(2p^n-2) \Leftrightarrow \sum_{v\in V(G)} d_G(v) > (2p^n-2)|V(G)|$$

where $V(G)$ is the vertex set of graph G. So Theorem 2.3 in the case $h=0$ and $n_1=\cdots=n_k=1$ implies the Alon-Friedland-Kalai result.

3. A General Theorem and its Consequences

Let Ω be the ring of all algebraic integers. For $\omega_1,\omega_2,\gamma\in\Omega$, by $\omega_1\equiv\omega_2 \pmod{\gamma}$ we mean $\omega_1-\omega_2\in\gamma\Omega$. For $a,b,m\in\mathbb{Z}$ it is well known that $a-b\in m\Omega$ if and only if $a-b\in m\mathbb{Z}$ (see, e.g. [IR, p. 68]). For $m\in\mathbb{Z}$ and a root ζ of unity, if $\zeta\equiv 0 \pmod{m}$ then $1=\zeta\zeta^{-1}\equiv 0 \pmod{m}$ (since $\zeta^{-1}\in\Omega$) and hence m must be 1 or -1.

Theorem 1.2 of zero-sum nature, Theorem 1.3(i) on covers of $\mathbb{Z}$, and our useful Theorem 2.1 are special cases of the following general theorem (which is inevitably complicated since it unifies many results).

Theorem 3.1. *Let G be an additive abelian p-group where p is a prime. Suppose that $A = \{a_s(n_s)\}_{s=1}^k$ is a $(d^*(G) + p^h + \Delta)$-cover of $\mathbb{Z}$ with $h, \Delta \in \mathbb{N}$. Let $m_1, \ldots, m_k \in \mathbb{Z}$ and $c, c_1, \ldots, c_k \in G$. Let α belong to the rational field $\mathbb{Q}$, and set*

$$\mathcal{I} = \left\{ I \subseteq [1, k] : \sum_{s \in I} c_s = c \text{ and } \sum_{s \in I} \frac{m_s}{n_s} \in \alpha + p^h \mathbb{Z} \right\}. \tag{3.1}$$

Let $P(x_1, \ldots, x_k) \in \mathbb{Q}[x_1, \ldots, x_k]$ have degree not exceeding $d \in \mathbb{Z}^+$ and $P([\![1 \in I]\!], \ldots, [\![k \in I]\!]) \in \mathbb{Z}$ for all $I \subseteq [1, k]$ with $\sum_{s \in I} m_s/n_s - \alpha \in \mathbb{Z}$. Then, either we have the inequality

$$|\{P([\![1 \in I]\!], \ldots, [\![k \in I]\!]) \bmod p : I \in \mathcal{I}\}| > 1 + \frac{\Delta}{d}, \tag{3.2}$$

or $|\{I \in \mathcal{I} : P([\![1 \in I]\!], \ldots, [\![k \in I]\!]) \in p\mathbb{Z}\}| \neq 1$ and furthermore

$$\sum_{\substack{I \in \mathcal{I} \\ p \mid P([\![1 \in I]\!], \ldots, [\![k \in I]\!])}} (-1)^{|I|} e^{2\pi i \sum_{s \in I} a_s m_s / n_s} \equiv 0 \pmod{p}. \tag{3.3}$$

Remark 3.1. (a) By taking $P(x_1, \ldots, x_k) = 0$ in Theorem 3.1, we get the congruence

$$\sum_{I \in \mathcal{I}} (-1)^{|I|} e^{2\pi i \sum_{s \in I} a_s m_s / n_s} \equiv 0 \pmod{p}$$

under the conditions of Theorem 3.1. (Thus Theorem 2.1 follows from Theorem 3.1.) In the case $h = 0$ and $n_1 = \cdots = n_k = 1$, this yields Theorem 1.2 of Olson.

(b) When G is an elementary abelian p-group, and $h = 0$ and $d = 1$, Theorem 3.1 is equivalent to the first part of the Main Theorem in the announcement [S03b].

Corollary 3.1. *Let $\{a_s(n_s)\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$. Let $m_1, \ldots, m_k \in \mathbb{Z}$, and let $\mu_1, \ldots, \mu_k$ be rational numbers such that $\sum_{s \in I} \mu_s \in \mathbb{Z}$ for all those $I \subseteq [1, k]$ with $\sum_{s \in I} m_s/n_s \in \mathbb{Z}$. For any prime p, if there is no $\emptyset \neq I \subseteq [1, k]$ such that $\sum_{s \in I} m_s/n_s \in \mathbb{Z}$ and $\sum_{s \in I} \mu_s \in p\mathbb{Z}$, then*

$$\left| \left\{ \sum_{s \in I} \mu_s \bmod p : \emptyset \neq I \subseteq [1, k] \text{ and } \sum_{s \in I} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \geqslant m. \tag{3.4}$$

Proof. Apply Theorem 3.1 with $G=\{0\}$, $h=\alpha=0$ and $P(x_1,\dots,x_k)=\sum_{s=1}^k \mu_s x_s$. $\square$

Remark 3.2. (a) Under the conditions of Corollary 3.1, if $\mu_1,\dots,\mu_k$ are positive then by taking a prime $p>\mu_1+\cdots+\mu_k$ we get that

$$\left|\left\{\sum_{s\in I}\mu_s : \emptyset\neq I\subseteq[1,k] \text{ and } \sum_{s\in I}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\geqslant m;$$

in particular,

$$\left|\left\{|I| : \emptyset\neq I\subseteq[1,k] \text{ and } \sum_{s\in I}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\geqslant m. \tag{3.5}$$

Theorem 1.3(i) follows if we set $\mu_s=m_s/n_s$ for $s=1,\dots,k$.

(b) In the special case $n_1=\cdots=n_k=1$, Corollary 3.1 gives the following result: If $c_1,\dots,c_k\in\mathbb{Z}_p$ with p a prime, and $\sum_{s\in I}c_s=0$ for no $\emptyset\neq I\subseteq[1,k]$, then $|\{\sum_{s\in I}c_s : \emptyset\neq I\subseteq[1,k]\}|\geqslant k$.

Corollary 3.2. *Let $\{a_s(n_s)\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$. Given $m_1,\dots,m_k\in\mathbb{Z}$ and $J\subseteq[1,k]$, we have*

$$\left|\left\{\sum_{s\in I}\frac{a_sm_s}{n_s}-\frac{|I|}{2} : I\subseteq[1,k] \text{ and } \sum_{s\in I}\frac{m_s}{n_s}-\sum_{s\in J}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|>m. \tag{3.6}$$

Proof. Let N be the least common multiple of $2,n_1,\dots,n_k$. Set

$$\mathcal{I}=\left\{I\subseteq[1,k] : \sum_{s\in I}\frac{m_s}{n_s}-\sum_{s\in J}\frac{m_s}{n_s}\in\mathbb{Z}\right\} \text{ and } l=\max_{I\subseteq[1,k]}\left|N\sum_{s\in I}\mu_s\right|$$

where $\mu_s=a_sm_s/n_s-1/2$. Choose a prime $p>\max\{|\mathcal{I}|,2l\}$ and set

$$P(x_1,\dots,x_k)=N\left(\sum_{s=1}^k\mu_sx_s-\sum_{s\in J}\mu_s\right)\in\mathbb{Z}[x_1,\dots,x_k].$$

Since $l<p/2$, for $I_1,I_2\subseteq[1,k]$ we have

$$N\sum_{s\in I_1}\mu_s\equiv N\sum_{s\in I_2}\mu_s \pmod p \iff \sum_{s\in I_1}\mu_s=\sum_{s\in I_2}\mu_s.$$

Now assume that (3.6) fails. Then

$$|\{P([\![1\in I]\!],\dots,[\![k\in I]\!]) \bmod p : I\in\mathcal{I}\}|\leqslant m.$$

Applying Theorem 3.1 with $G=\{0\}$, $h=0$ and $\alpha=\sum_{s\in J}m_s/n_s$, we then obtain that

$$\sum_{\substack{I\in\mathcal{I}\\ p\mid P([\![1\in I]\!],\dots,[\![k\in I]\!])}} e^{2\pi i\sum_{s\in I}\mu_s}\equiv 0 \pmod p,$$

i.e., $|\{I\in\mathcal{I} : \sum_{s\in I}\mu_s=\sum_{s\in J}\mu_s\}|e^{2\pi i\sum_{s\in J}\mu_s}\equiv 0 \pmod p$, which is impossible since $p>|\mathcal{I}|$. This concludes our proof. $\square$

Remark 3.3. Clearly Corollary 3.2 implies [S99, Theorem 1(i)].

4. Proof of Theorem 3.1 and a Characterization of m-Covers of $\mathbb{Z}$

At first we introduce some notations. For a real number α, we let $\{\alpha\}$ denote the fractional part of α. For a polynomial $f(x_1,\ldots,x_k)$ over the field $\mathbb{C}$ of complex numbers, we use $[x_1^{j_1}\cdots x_k^{j_k}]f(x_1,\ldots,x_k)$ to represent the coefficient of the monomial $x_1^{j_1}\cdots x_k^{j_k}$ in $f(x_1,\ldots,x_k)$. Also, we fix a finite system (1.2) of residue classes, and set $I_z=\{1\leqslant s\leqslant k:\ z\in a_s(n_s)\}$ for $z\in\mathbb{Z}$. Note that $|I_z|=w_A(z)\geqslant m(A)$ for all $z\in\mathbb{Z}$.

Lemma 4.1. *Let $A=\{a_s(n_s)\}_{s=1}^k$ and let $f(x_1,\ldots,x_k)\in\mathbb{C}[x_1,\ldots,x_k]$ with $\deg f\leqslant m(A)$. Let $m_1,\ldots,m_k\in\mathbb{Z}$. If $[\prod_{s\in I_z}x_s]f(x_1,\ldots,x_k)=0$ for all $z\in\mathbb{Z}$, then we have $\psi(\theta)=0$ for any $0\leqslant\theta<1$, where*

$$\psi(\theta)=\sum_{\substack{I\subseteq[1,k]\\ \{\sum_{s\in I}m_s/n_s\}=\theta}}(-1)^{|I|}f([\![1\in I]\!],\ldots,[\![k\in I]\!])e^{2\pi i\sum_{s\in I}a_sm_s/n_s}.$$

The converse holds when $m_1,\ldots,m_k$ are relatively prime to $n_1,\ldots,n_k$ respectively.

Proof. Let

$$S=\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq[1,k]\right\}. \tag{4.1}$$

By [S07, Lemma 1], for any $z\in\mathbb{Z}$ we have

$$\sum_{\theta\in S}e^{-2\pi iz\theta}\psi(\theta)=(-1)^kc(I_z)\prod_{\substack{s=1\\ s\notin I_z}}^{k}\left(e^{2\pi i(a_s-z)m_s/n_s}-1\right), \tag{4.2}$$

where $c(I_z)=[\prod_{s\in I_z}x_s]f(x_1,\ldots,x_k)$.

Observe that we must have $c(I_z)=0$ if $\psi(\theta)=0$ for all $0\leqslant\theta<1$ and each m_s is relatively prime to n_s.

Assume that $c(I_z)=0$ for all $z\in[a,\,a+|S|-1]$ where $a\in\mathbb{Z}$. Then $\sum_{\theta\in S}e^{-2\pi in\theta}(e^{-2\pi ia\theta}\psi(\theta))=0$ for every $n\in[0,|S|-1]$. Note that the Vandermonde determinant $|(e^{-2\pi i\theta})^n|_{n\in[0,|S|-1],\ \theta\in S}$ does not vanish. So $\psi(\theta)=0$ for each $\theta\in S$. If $0\leqslant\theta<1$ and $\theta\notin S$, then we obviously have $\psi(\theta)=0$.

The proof is now complete. □

Remark 4.1. Let $A=\{a_s(n_s)\}_{s=1}^k$ and let S be the set given by (4.1) where $m_1,\ldots,m_k\in\mathbb{Z}$ are relatively prime to $n_1,\ldots,n_k$ respectively. Let y be any integer with $|I_y|=w_A(y)=m(A)$. By Lemma 4.1 and its proof, for any $a\in\mathbb{Z}$ there is a $z\in[a,a+|S|-1]$ such that $[\prod_{s\in I_z}x_s]\prod_{s\in I_y}x_s\neq0$, hence $I_y=I_z$ and $z\in y([n_s]_{s\in I_y})$, where $[n_s]_{s\in I_y}$ is the least common multiple of those n_s with $s \in I_y$. Therefore $|S| \geqslant [n_s]_{s\in I_y}$, and we also get the following local-global result of Sun [S95, S96, S04]: $\{a_s(n_s)\}_{s=1}^k$ *forms an m-cover of $\mathbb{Z}$ if it covers $|S|$ consecutive integers at least m times.* In the case $m = 1$, this local-global principle was conjectured by Erdős with $|S|$ replaced by 2^k (cf. [CV]). The reference [S05b] contains a local-global theorem of another type.

Now we use Lemma 4.1 to characterize m-covers of $\mathbb{Z}$.

Theorem 4.1. *Let $m \in \mathbb{Z}^+$, and let $P_0(x), \ldots, P_{m-1}(x) \in \mathbb{C}[x]$ have degrees $0, \ldots, m-1$ respectively. Let $m_1, \ldots, m_k \in \mathbb{Z}$ and $\mu_1, \ldots, \mu_k \in \mathbb{C}$. If $A = \{a_s(n_s)\}_{s=1}^k$ forms an m-cover of $\mathbb{Z}$, then we have*

$$\sum_{\substack{I\subseteq[1,k]\\ \{\sum_{s\in I} m_s/n_s\}=\theta}} (-1)^{|I|} P_n\Big(\sum_{s\in I}\mu_s\Big) e^{2\pi i \sum_{s\in I} a_s m_s/n_s} = 0 \tag{4.3}$$

for all $0 \leqslant \theta < 1$ and $n \in [0, m-1]$. The converse holds provided that $m_1, \ldots, m_k$ are relatively prime to $n_1, \ldots, n_k$ respectively, and $\mu_1, \ldots, \mu_k$ are all nonzero.

Proof. For $n \in [0, m-1]$ set $f_n(x_1, \ldots, x_k) = P_n(\sum_{s=1}^k \mu_s x_s)$. If $A = \{a_s(n_s)\}_{s=1}^k$ is an m-cover of $\mathbb{Z}$, then $\deg f_n < m \leqslant m(A)$ and hence $[\prod_{s\in I_z} x_s] f_n(x_1, \ldots, x_k) = 0$ for all $z \in \mathbb{Z}$ since $|I_z| = w_A(z) \geqslant m$, therefore (4.3) holds for any $0 \leqslant \theta < 1$ in view of Lemma 4.1.

Now assume that $m_1, \ldots, m_k$ are relatively prime to $n_1, \ldots, n_k$ respectively and that $\mu_1 \cdots \mu_k \neq 0$. Suppose that $m(A) = n \in [0, m-1]$. Then there is a $z \in \mathbb{Z}$ such that $|I_z| = w_A(z) = n$. As (4.3) holds for all $0 \leqslant \theta < 1$, by Lemma 4.1 the coefficient $c := [\prod_{s\in I_z} x_s] f_n(x_1, \ldots, x_k)$ vanishes. On the other hand,

$$\begin{aligned} c =& \Big[\prod_{s\in I_z} x_s\Big] P_n\Big(\sum_{s=1}^k \mu_s x_s\Big) = [x^n]P_n(x) \times \Big[\prod_{s\in I_z} x_s\Big]\Big(\sum_{s=1}^k \mu_s x_s\Big)^n \\ =& [x^n]P_n(x) \times \frac{n!}{\prod_{s=1}^k [\![s \in I_z]\!]!} \prod_{s\in I_z} \mu_s \neq 0. \end{aligned}$$

This contradiction concludes our proof. □

Remark 4.2. In the case $\mu_s = m_s/n_s$ $(1 \leqslant s \leqslant k)$ and $P_n(x) = \binom{x}{n}$ $(n \in [0, m-1])$, Theorem 4.1 is equivalent to a characterization of m-covers of $\mathbb{Z}$ obtained by the author [S95, S96] via an analytic method.

Let a be an integer and let p be a prime. Fermat's little theorem tells that we can characterize whether p divides a as follows:

$$[\![p \mid a]\!] \equiv 1 - a^{p-1} \pmod p.$$

To handle general abelian p-groups in a similar way, we need to characterize whether a given power of p divides a. Thus, our following lemma is of technical importance. (It appeared even in the first version of this paper posted as `arXiv:math/NT/0305369` on May 24, 2003.)

Lemma 4.2. *Let p be a prime, and let $h \in \mathbb{N}$ and $a \in \mathbb{Z}$. Then we have the following congruence*

$$\binom{a-1}{p^h-1} \equiv [\![p^h \mid a]\!] \pmod p. \tag{4.4}$$

Proof. (4.4) is trivial if $h = 0$. Below we let $h > 0$.

Write $a = p^h q + r$ where $q, r \in \mathbb{N}$ and $r < p^h$. For $j \in [1, p^h - 1]$, if we write $j = p^{h_j} q_j$ with $h_j \in \mathbb{N}$, $q_j \in \mathbb{Z}^+$ and $p \nmid q_j$, then $0 \leqslant h_j < h$ and

$$\frac{p^h q \pm j}{p^h \pm j} = \frac{p^{h-h_j} q \pm q_j}{p^{h-h_j} \pm q_j} \equiv 1 \pmod p.$$

Thus, when $r = 0$ we have

$$\binom{a-1}{p^h-1} = \prod_{j=1}^{p^h-1} \frac{p^h q - j}{p^h - j} \equiv 1 \pmod p.$$

In the case $0 < r \leqslant p^h - 1$,

$$\binom{a-1}{p^h-1} = \prod_{j=1}^{r-1} \frac{p^h q + j}{j} \times \frac{p^h q}{r} \times \prod_{j=1}^{p^h-1-r} \frac{p^h q - j}{p^h - j} \equiv \frac{p^h q}{r} \equiv 0 \pmod p.$$

Therefore (4.4) holds. $\square$

`Proof of Theorem 3.1`. Write

$$\{P([\![1 \in I]\!], \ldots, [\![k \in I]\!]) \bmod p : I \in \mathcal{I}\} = \{r \bmod p : r \in R\}$$

with $R \subseteq [0, p-1]$. If $0 \notin R$, then (3.3) holds trivially. So we assume $0 \in R$ from now on.

Suppose that $G \cong \mathbb{Z}_{p^{h_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{h_l}}$ where $h_1, \ldots, h_l \in \mathbb{N}$. (When $|G| = 1$ we have $G \cong \mathbb{Z}_{p^0}$.) We can identify $c \in G$ with a vector

$$(c^{(1)} \bmod p^{h_1}, \ldots, c^{(l)} \bmod p^{h_l}),$$

and identify c_s ($s \in [1, k]$) with a vector

$$(c_s^{(1)} \bmod p^{h_1}, \ldots, c_s^{(l)} \bmod p^{h_l}),$$

where $c^{(t)}$ and $c_s^{(t)}$ are integers for $t = 1, \ldots, l$.

Let $\theta = \{\alpha\}$ and

$$f(x_1, \ldots, x_k) = \prod_{t=1}^{l} \binom{\sum_{s=1}^{k} c_s^{(t)} x_s - c^{(t)} - 1}{p^{h_t} - 1} \times \binom{\sum_{s=1}^{k} m_s x_s / n_s - \alpha - 1}{p^h - 1} \prod_{r \in R \setminus \{0\}} (P(x_1, \ldots, x_k) - r).$$

For any $I \subseteq [1, k]$ with $\{\sum_{s \in I} m_s/n_s\} = \theta$, by Lemma 4.2 we have

$$\begin{aligned} f([\![1 \in I]\!], \ldots, [\![k \in I]\!]) &\equiv [\![I \in \mathcal{I}]\!] \prod_{r \in R \setminus \{0\}} (P([\![1 \in I]\!], \ldots, [\![k \in I]\!]) - r) \\ &\equiv [\![I \in \mathcal{I} \ \&\ p \mid P([\![1 \in I]\!], \ldots, [\![k \in I]\!])]\!] C \pmod{p}, \end{aligned}$$

where $C = \prod_{r \in R \setminus \{0\}} (-r) \not\equiv 0 \pmod{p}$. Thus

$$\sigma := \sum_{\substack{I \subseteq [1,k] \\ \{\sum_{s \in I} m_s/n_s\} = \theta}} (-1)^{|I|} f([\![1 \in I]\!], \ldots, [\![k \in I]\!]) e^{2\pi i \sum_{s \in I} a_s m_s / n_s}$$

is congruent to the left-hand side of (3.3) times $C \in \mathbb{Z} \setminus p\mathbb{Z}$ modulo p.

Suppose that (3.2) fails. Then

$$\deg f \leqslant \sum_{t=1}^{l} (p^{h_t} - 1) + p^h - 1 + (|R| - 1)d \leqslant d^*(G) + p^h - 1 + \Delta < m(A),$$

and hence $\sigma = 0$ in light of Lemma 4.1. Therefore (3.3) holds. We are done. □

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