J. Number Theory 129(2009), no. 4, 964-969.

MIXED SUMS OF SQUARES AND TRIANGULAR NUMBERS (III)

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Abstract. In this paper we confirm a conjecture of Sun which states that each positive integer is a sum of a square, an odd square and a triangular number. Given any positive integer m, we show that $p = 2m + 1$ is a prime congruent to 3 modulo 4 if and only if $T_m = m(m + 1)/2$ cannot be expressed as a sum of two odd squares and a triangular number, i.e., $p^2 = x^2+8(y^2+z^2)$ for no odd integers x, y, z. We also show that a positive integer cannot be written as a sum of an odd square and two triangular numbers if and only if it is of the form $2T_m$ $(m > 0)$ with $2m + 1$ having no prime divisor congruent to 3 modulo 4.

1. Introduction

The study of expressing natural numbers as sums of squares has long history. Here are some well-known classical results in number theory.

(a) (Fermat-Euler theorem) Any prime $p \equiv 1 \pmod 4$ is a sum of two squares of integers.

(b) (Gauss-Legendre theorem, cf. [G, pp. 38–49] or [N, pp. 17–23]) $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as a sum of three squares of integers if and only if n is not of the form $4^k(8l + 7)$ with $k, l \in \mathbb{N}$.

Key words and phrases. Squares, triangular numbers, mixed sums, ternary quadratic forms, representations of natural numbers.

2000 *Mathematics Subject Classification.* Primary 11E25; Secondary 05A05, 11D85, 11P99, 11Y11.

The first author is supported by the Korea Research Foundation Grant (KRF-2008-314-C00004) funded by the Korean Government.

The second author is responsible for communications, and supported by the National Natural Science Foundation (grant 10871087) of People's Republic of China.

(c) (Lagrange's theorem) Every $n \in \mathbb{N}$ is a sum of four squares of integers.

Those integers $T_x = x(x+1)/2$ with $x \in \mathbb{Z}$ are called triangular numbers. Note that $T_x = T_{-x-1}$ and $8T_x + 1 = (2x+1)^2$. In 1638 P. Fermat asserted that each $n \in \mathbb{N}$ can be written as a sum of three triangular numbers (equivalently, $8n+3$ is a sum of three squares of odd integers); this follows from the Gauss-Legendre theorem.

Let $n \in \mathbb{N}$. As observed by L. Euler (cf. [D, p. 11]), the fact that $8n+1$ is a sum of three squares (of integers) implies that n can be expressed as a sum of two squares and a triangular number. This is remarkable since there are infinitely many natural numbers which cannot be written as a sum of three squares. According to [D, p. 24], E. Lionnet stated, and V. A. Lebesgue [L] and M. S. Réalis [R] showed that n is also a sum of two triangular numbers and a square. In 2006 these two results were re-proved by H. M. Farkas [F] via the theory of theta functions. Further refinements of these results are summarized in the following theorem.

Theorem 1.0. (i) (B. W. Jones and G. Pall [JP]) *For every $n \in \mathbb{N}$, we can write $8n+1$ in the form $8x^2 + 32y^2 + z^2$ with $x, y, z \in \mathbb{Z}$, i.e., n is a sum of a square, an even square and a triangular number.*

(ii) (Z. W. Sun [S07]) *Any natural number is a sum of an even square and two triangular numbers. If $n \in \mathbb{N}$ and $n \neq 2T_m$ for any $m \in \mathbb{N}$, then n is also a sum of an odd square and two triangular numbers.*

(iii) (Z. W. Sun [S07]) *A positive integer is a sum of an odd square, an even square and a triangular number unless it is a triangular number T_m $(m > 0)$ for which all prime divisors of $2m+1$ are congruent to 1 mod 4.*

We mention that Jones and Pall [JP] used the theory of ternary quadratic forms and Sun [S07] employed some identities on q-series. Motivated by Theorem 1.0(iii) and the fact that every prime $p \equiv 1 \pmod 4$ is a sum of an odd square and an even square, the second author [S09] conjectured that each natural number $n \neq 216$ can be written in the form $p + T_x$ with $x \in \mathbb{Z}$, where p is a prime or zero. Sun [S09] also made a general conjecture which states that for any $a, b \in \mathbb{N}$ and $r = 1, 3, 5, \ldots$ all sufficiently large integers can be written in the form $2^a p + T_x$ with $x \in \mathbb{Z}$, where p is either zero or a prime congruent to r mod 2^b.

In [S07] Sun investigated what kind of mixed sums $ax^2 + by^2 + cT_z$ or $ax^2 + bT_y + cT_z$ (with $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$) represent all natural numbers, and left two conjectures in this direction. In [GPS] S. Guo, H. Pan and Sun proved Conjecture 2 of [S07]. Conjecture 1 of Sun [S07] states that any positive integer n is a sum of a square, an odd square and a triangular number, i.e., $n - 1 = x^2 + 8T_y + T_z$ for some $x, y, z \in \mathbb{Z}$.

In this paper we prove Conjecture 1 of Sun [S07] and some other results concerning mixed sums of squares and triangular numbers. Our main result is as follows.

Theorem 1.1. (i) *Each positive integer is a sum of a square, an odd square and a triangular number. A triangular number T_m with $m \in \mathbb{Z}^+$ is a sum of two odd squares and a triangular number if and only if $2m+1$ is not a prime congruent to* 3 *mod* 4.

(ii) *A positive integer cannot be written as a sum of an odd square and two triangular numbers if and only if it is of the form $2T_m$ ($m \in \mathbb{Z}^+$) with $2m+1$ having no prime divisor congruent to* 3 *mod* 4.

Remark 1.1. In [S09] the second author conjectured that if a positive integer is not a triangular number then it can be written as a sum of two odd squares and a triangular number unless it is among the following 25 exceptions:

$$4,\ 7,\ 9,\ 14,\ 22,\ 42,\ 43,\ 48,\ 52,\ 67,\ 69,\ 72,\ 87,\ 114,$$
$$144,\ 157,\ 159,\ 169,\ 357,\ 402,\ 489,\ 507,\ 939,\ 952,\ 1029.$$

Here is a consequence of Theorem 1.1.

Corollary 1.1. (i) *An odd integer $p > 1$ is a prime congruent to* 3 *mod* 4 *if and only if $p^2 = x^2 + 8(y^2 + z^2)$ for no odd integers x, y, z.*

(ii) *Let $n > 1$ be an odd integer. Then all prime divisors of n are congruent to* 1 *mod* 4, *if and only if $n^2 = x^2 + 4(y^2 + z^2)$ for no odd integers x, y, z.*

Remark 1.2. In number theory there are very few simple characterizations of primes such as Wilson's theorem. Corollary 1.1(i) provides a surprising new criterion for primes congruent to 3 mod 4.

In the next section we will prove an auxiliary theorem. Section 3 is devoted to our proofs of Theorem 1.1 and Corollary 1.1.

2. An auxiliary theorem

In this section we prove the following auxiliary result.

Theorem 2.1. *Let m be a positive integer.*

(i) *Assume that $p = 2m+1$ be a prime congruent to* 3 *mod* 4. *Then T_m cannot be written in the form $x^2 + y^2 + T_z$ with $x, y, z \in \mathbb{Z}$, $x^2 + y^2 > 0$ and $x \equiv y \pmod 2$. Also, $2T_m$ is not a sum of a positive even square and two triangular numbers.*

(ii) *Suppose that all prime divisors of $2m+1$ are congruent to* 1 *mod* 4. *Then T_m cannot be written as a sum of an odd square, an even square and a triangular number. Also, $2T_m$ is not a sum of an odd square and two triangular numbers.*

To prove Theorem 2.1 we need the following result due to Hurwitz.

Lemma 2.1 (cf. [D, p. 271] or [S07, Lemma 3]). *Let n be a positive odd integer, and let $p_1,\ldots,p_r$ be all the distinct prime divisors of n congruent to 3 mod 4. Write $n=n_0\prod_{0<i\leqslant r}p_i^{\alpha_i}$, where $n_0,\alpha_1,\ldots,\alpha_r\in\mathbb{Z}^+$ and n_0 has no prime divisors congruent to 3 mod 4. Then*

$$|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=n^2\}|=6n_0\prod_{0<i\leqslant r}\left(p_i^{\alpha_i}+2\frac{p_i^{\alpha_i}-1}{p_i-1}\right).$$

As in [S07], for $n\in\mathbb{N}$ we define

$$r_0(n)=|\{(x,y,z)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\ x^2+T_y+T_z=n\text{ and }2\mid x\}|$$

and

$$r_1(n)=|\{(x,y,z)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\ x^2+T_y+T_z=n\text{ and }2\nmid x\}|.$$

By p. 108 and Lemma 2 of Sun [S07], we have the following lemma.

Lemma 2.2 ([S07]). *For $n\in\mathbb{N}$ we have*

$$|\{(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\ x^2+y^2+T_z=n\ \textit{and}\ x\equiv y\pmod 2\}|=r_0(2n)$$

and

$$|\{(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\ x^2+y^2+T_z=n\ \textit{and}\ x\not\equiv y\pmod 2\}|=r_1(2n).$$

Also, $r_0(2T_m)-r_1(2T_m)=(-1)^m(2m+1)$ for every $m\in\mathbb{N}$.

Proof of Theorem 2.1. By Lemma 2.2,

$$\begin{aligned}&r_0(2T_m)+r_1(2T_m)\\ =&|\{(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\ x^2+y^2+T_z=T_m\}|\\ =&\frac12|\{(x,y,z)\in\mathbb{Z}^3:\ 8x^2+8y^2+(8T_z+1)=8T_m+1\}|\\ =&\frac12|\{(x,y,z)\in\mathbb{Z}^3:\ 4(x+y)^2+4(x-y)^2+(2z+1)^2=(2m+1)^2\}|\\ =&\frac12|\{(u,v,z)\in\mathbb{Z}^3:\ 4(u^2+v^2)+(2z+1)^2=(2m+1)^2\}|\\ =&\frac16|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=(2m+1)^2\}|.\end{aligned}$$

(i) As $p=2m+1$ is a prime congruent to 3 mod 4, by Lemma 2.1 and the above we have

$$r_0(2T_m)+r_1(2T_m)=p+2.$$

On the other hand,

$$r_0(2T_m) - r_1(2T_m) = (-1)^m(2m+1) = -p$$

by Lemma 2.2. So

$$2r_0(2T_m) = r_0(2T_m) + r_1(2T_m) + (r_0(2T_m) - r_1(2T_m)) = p + 2 - p = 2.$$

Therefore

$$|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + T_z = T_m \text{ and } 2 \mid x - y\}| = r_0(2T_m) = 1$$

and also

$$|\{(x,y,z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + T_y + T_z = 2T_m \text{ and } 2 \mid x\}| = r_0(2T_m) = 1.$$

Since $T_m = 0^2 + 0^2 + T_m$ and $2T_m = 0^2 + T_m + T_m$, the desired results follow immediately.

(ii) As all prime divisors of $2m+1$ are congruent to 1 mod 4, we have

$$r_0(2T_m)+r_1(2T_m) = \frac{1}{6}|\{(x,y,z) \in \mathbb{Z}^3 : x^2+y^2+z^2 = (2m+1)^2\}| = 2m+1$$

in view of Lemma 2.1. Note that m is even since $2m+1 \equiv 1 \pmod 4$. By Lemma 2.2, $r_0(2T_m) - r_1(2T_m) = (-1)^m(2m+1) = 2m+1$. Therefore

$$|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + T_z = T_m \text{ and } 2 \nmid x - y\}| = r_1(2T_m) = 0.$$

This proves part (ii) of Theorem 2.1. □

3. Proofs of Theorem 1.1 and Corollary 1.1

Lemma 3.1. *Let $m \in \mathbb{N}$ with $2m+1 = k(w^2 + x^2 + y^2 + z^2)$ where $k, w, x, y, z \in \mathbb{Z}$. Then*

$$2T_m = k^2(wy + xz)^2 + k^2(wz - xy)^2 + 2T_v \quad \text{for some } v \in \mathbb{Z}.$$

Proof. Write the odd integer $k(w^2 + x^2 - (y^2 + z^2))$ in the form $2v+1$. Then

$$\begin{aligned} 8T_m + 1 =& (2m+1)^2 = k^2(w^2 + x^2 + y^2 + z^2)^2 \\ =& (2v+1)^2 + 4k^2(w^2 + x^2)(y^2 + z^2) \\ =& 8T_v + 1 + 4k^2((wy + xz)^2 + (wz - xy)^2) \end{aligned}$$

and hence

$$2T_m = 2T_v + k^2(wy + xz)^2 + k^2(wz - xy)^2.$$

This concludes the proof. $\square$

Proof of Theorem 1.1. (i) In view of Theorem 1.0(iii), it suffices to show the second assertion in part (i).

Let m be any positive integer. By Theorem 2.1(i), if $2m + 1$ is a prime congruent to 3 mod 4 then T_m cannot be written as a sum of two odd squares and a triangular number.

Now assume that $2m + 1$ is not a prime congruent to 3 mod 4. Since the product of two integers congruent to 3 mod 4 is congruent to 1 mod 4, we can write $2m + 1$ in the form $k(4n + 1)$ with $k, n \in \mathbb{Z}^+$.

Set $w = 1 + (-1)^n$. Observe that $4n + 1 - w^2$ is a positive integer congruent to 5 mod 8. By the Gauss-Legendre theorem on sums of three squares, there are integers x, y, z with x odd such that $4n + 1 - w^2 = x^2 + y^2 + z^2$. Clearly both y and z are even. As $y^2 + z^2 \equiv 4 \pmod 8$, we have $y_0 \not\equiv z_0 \pmod 2$ where $y_0 = y/2$ and $z_0 = z/2$.

Since $2m + 1 = k(w^2 + x^2 + y^2 + z^2)$, by Lemma 3.1 there is an integer v such that

$$2T_m = 2T_v + k^2(wy + xz)^2 + k^2(wz - xy)^2.$$

Thus

$$\begin{aligned} T_m =& T_v + 2(kwy_0 + kxz_0)^2 + 2(kwz_0 - kxy_0)^2 \\ =& T_v + (kwy_0 + kxz_0 + (kwz_0 - kxy_0))^2 \\ &+ (kwy_0 + kxz_0 - (kwz_0 - kxy_0))^2. \end{aligned}$$

As w is even and $kxy_0 \equiv y_0 \not\equiv z_0 \equiv kxz_0 \pmod 2$, we have

$$kwy_0 + kxz_0 \pm (kwz_0 - kxy_0) \equiv 1 \pmod 2.$$

Therefore $T_m - T_v$ is a sum of two odd squares.

(ii) In view of Theorem 1.0(ii) and Theorem 2.1(ii), it suffices to show that if $2m + 1$ $(m \in \mathbb{Z}^+)$ has a prime divisor congruent to 3 mod 4 then $2T_m$ is a sum of an odd square and two triangular numbers.

Suppose that $2m+1 = k(4n-1)$ with $k, n \in \mathbb{Z}^+$. Write $w = 1+(-1)^n$. Then $4n-1-w^2$ is a positive integer congruent to 3 mod 8. By the Gauss-Legendre theorem on sums of three squares, there are integers x, y, z such that $4n - 1 - w^2 = x^2 + y^2 + z^2$. Clearly $x \equiv y \equiv z \equiv 1 \pmod 2$ and $2m + 1 = k(w^2 + x^2 + y^2 + z^2)$. By Lemma 3.1, for some $v \in \mathbb{Z}$ we have

$$2T_m = k^2(wy + xz)^2 + k^2(wz - xy)^2 + 2T_v.$$

Let $u = kwz - kxy$. Then

$$T_{v+u} + T_{v-u} = \frac{(v+u)^2 + (v-u)^2 + (v+u) + (v-u)}{2} = u^2 + 2T_v.$$

Thus

$$2T_m = (kwy + kxz)^2 + T_{v+u} + T_{v-u}.$$

Note that $kwy + kxz$ is odd since w is even and k, x, z are odd.

Combining the above we have completed the proof of Theorem 1.1. □

Proof of Corollary 1.1. (i) Let $m = (p-1)/2$. Observe that

$$\begin{aligned} &T_m = T_x + (2y+1)^2 + (2z+1)^2 \\ \iff &p^2 = 8T_m + 1 = (2x+1)^2 + 8(2y+1)^2 + 8(2z+1)^2. \end{aligned}$$

So the desired result follows from Theorem 1.1(i).

(ii) Let $m = (n-1)/2$. Clearly

$$\begin{aligned} &2T_m = T_x + T_y + (2z+1)^2 \\ \iff &2n^2 = 16T_m + 2 = (2x+1)^2 + (2y+1)^2 + 8(2z+1)^2 \\ \iff &n^2 = (x+y+1)^2 + (x-y)^2 + 4(2z+1)^2. \end{aligned}$$

So $2T_m$ is a sum of an odd square and two triangular numbers if and only if $n^2 = x^2 + (2y)^2 + 4z^2$ for some odd integers x, y, z. (If x and z are odd but y is even, then $x^2 + (2y)^2 + 4z^2 \equiv 5 \not\equiv n^2 \pmod 8$.) Combining this with Theorem 1.1(ii) we obtain the desired result. □

Remark 3.1. We can deduce Corollary 1.1 in another way by using some known results (cf. [E], [EHH] and [SP]) in the theory of ternary quadratic forms, but this approach involves many sophisticated concepts.

Acknowledgment. The authors are grateful to the referee for his/her helpful comments.

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