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ON m-COVERS AND m-SYSTEMS

Zhi-Wei Sun

Abstract. Let $\mathcal{A}=\{a_s(\mathrm{mod}\ n_s)\}_{s=0}^k$ be a system of residue classes. With the help of cyclotomic fields we obtain a theorem which unifies several previously known results related to the covering multiplicity of $\mathcal{A}$. In particular, we show that if every integer lies in more than $m_0=\lfloor\sum_{s=1}^k 1/n_s\rfloor$ members of $\mathcal{A}$, then for any $a=0,1,2,\ldots$ there are at least $\binom{m_0}{\lfloor a/n_0\rfloor}$ subsets I of $\{1,\ldots,k\}$ with $\sum_{s\in I}1/n_s=a/n_0$. We also characterize when any integer lies in at most m members of $\mathcal{A}$, where m is a fixed positive integer.

1. The main results

For $a\in\mathbb{Z}$ and $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, we simply denote the residue class $\{x\in\mathbb{Z}:\ x\equiv a\ (\mathrm{mod}\ n)\}$ by $a(n)$. For a finite system

$$A=\{a_s(n_s)\}_{s=1}^k \tag{1.1}$$

of residue classes, the function $w_A:\mathbb{Z}\to\mathbb{N}=\{0,1,2,\ldots\}$ given by

$$w_A(x)=|\{1\leqslant s\leqslant k:\ x\in a_s(n_s)\}| \tag{1.2}$$

is called the *covering function* of A. Obviously $w_A(x)$ is periodic modulo the least common multiple N of the moduli $n_1,\ldots,n_k$, and it is easy to see that the average $\sum_{x=0}^{N-1}w_A(x)/N$ equals $\sum_{s=1}^k 1/n_s$. As in [S97] we call $m(A)=\min_{x\in\mathbb{Z}}w_A(x)$ the *covering multiplicity* of system (1.1).

Let m be any positive integer. If $w_A(x)\geqslant m$ for all $x\in\mathbb{Z}$ (i.e., $m(A)\geqslant m$), then (1.1) is said to be an *m-cover* of $\mathbb{Z}$ as in [S95, S96], and in this case $\sum_{s=1}^k 1/n_s\geqslant m$. Covers (i.e. 1-covers) of $\mathbb{Z}$ were first introduced by P. Erdős [E50] and they are also called covering systems. If $w_A(x)=m$ for all $x\in\mathbb{Z}$, then we call (1.1) an *exact m-cover* of $\mathbb{Z}$ as in [S96, S97] (and in this case $\sum_{s=1}^k 1/n_s=m$). By [PZ, Theorem 1.3], when $m\geqslant 2$ there are exact m-covers of $\mathbb{Z}$ that cannot split into two covers of $\mathbb{Z}$. If

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$w_A(x) \leqslant m$ for all $x \in \mathbb{Z}$, then we call (1.1) an *m-system*, and in this case $\sum_{s=1}^k 1/n_s \leqslant m$; any 1-system is said to be *disjoint*.

The reader may consult Guy [G04, pp. 383-390] and Simpson [Sim] for some problems and results in covering theory. Covers of $\mathbb{Z}$ have many surprising applications, see, e.g., [C71], [G04, sections A19 and B21], [S00], [SY] and [WS]. Sun [S09] showed that m-covers of $\mathbb{Z}$ are related to zero-sum problems for abelian groups. Also, the topic of covering systems stimulated the birth of some new algebraic results (cf. [S01] and [S05]).

Throughout this paper, for $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} : a \leqslant x \leqslant b\}$ and define $[a, b)$ and $(a, b]$ similarly. As usual, the integral part and the fractional part of a real number α are denoted by $\lfloor \alpha \rfloor$ and $\{\alpha\}$ respectively.

For system (1.1) we define its *dual system* A^* by

$$A^* = \{a_s + r(n_s) : \ 1 \leqslant r < n_s, \ 1 \leqslant s \leqslant k\}. \tag{1.3}$$

As $\{a_s + r(n_s)\}_{r=0}^{n_s-1}$ is a partition of $\mathbb{Z}$ for any $s \in [1, k]$, we have $w_A(x) + w_{A^*}(x) = k$ for all $x \in \mathbb{Z}$. Thus $w_A(x) \leqslant m$ for all $x \in \mathbb{Z}$ if and only if $w_{A^*}(x) \geqslant k - m$ for all $x \in \mathbb{Z}$. This simple and new observation shows that we can study m-systems via covers of $\mathbb{Z}$, and construct covers of $\mathbb{Z}$ via m-systems.

By a result in [S96], if (1.1) is an m-cover of $\mathbb{Z}$ then for any $m_1, \ldots, m_k \in \mathbb{Z}^+$ there are at least m positive integers in the form $\sum_{s \in I} m_s/n_s$ with $I \subseteq [1, k]$. Applying this result to the dual A^* of an m-system (1.1), we obtain that there are more than $k - m$ integers in the form $\sum_{s=1}^k x_s/n_s$ with $x_s \in [0, n_s)$; equivalently, at most $m - 1$ of the numbers in $[1, k]$ cannot be written in the form $\sum_{s=1}^k m_s/n_s = k - \sum_{s=1}^k (n_s - m_s)/n_s$ with $m_s \in [1, n_s]$. This implies the following result stated in [S03, Remark 1.3]: If (1.1) is an m-system, then there are $m_1, \ldots, m_k \in \mathbb{Z}^+$ such that $\sum_{s=1}^k m_s/n_s = m$.

Our following theorem unifies and generalizes several known results.

Theorem 1.1. *Let $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ be a finite system of residue classes with $m(\mathcal{A}) > m = \lfloor \sum_{s=1}^k m_s/n_s \rfloor$, where $m_1, \ldots, m_k \in \mathbb{Z}^+$. Then, for any $0 \leqslant \alpha < 1$, either*

$$\sum_{\substack{I \subseteq [1,k] \\ \sum_{s \in I} m_s/n_s = (\alpha + a)/n_0}} (-1)^{|I|} e^{2\pi i \sum_{s \in I} a_s m_s/n_s} = 0 \tag{1.4}$$

for any $a \in \mathbb{N}$, or

$$\left| \left\{ I \subseteq [1, k] : \sum_{s \in I} \frac{m_s}{n_s} = \frac{\alpha + a}{n_0} \right\} \right| \geqslant \binom{m}{\lfloor a/n_0 \rfloor} \tag{1.5}$$

for all $a = 0, 1, 2, \ldots$.

Example 1.1. Erdős observed that $\{0(2), 0(3), 1(4), 5(6), 7(12)\}$ is a cover of $\mathbb{Z}$ with the moduli

$$n_0 = 2,\ n_1 = 3,\ n_2 = 4,\ n_3 = 6,\ n_4 = 12$$

distinct. As $\lfloor \sum_{s=1}^{4} 1/n_s \rfloor = 0$, by Theorem 1.1 in the case $\alpha = 0$ we have $\sum_{s\in I} 1/n_s = 1/n_0 = 1/2$ for some $I \subseteq [1,4]$; we can actually take $I = \{1,3\}$. Since $\sum_{s=1}^{4} 1/n_s < (5/6+1)/n_0 = 11/12$, by Theorem 1.1 in the case $\alpha = 5/6$ the set $\mathcal{I} = \{I \subseteq [1,4] : \sum_{s\in I} 1/n_s = 5/12\}$ cannot have a single element; in fact, $\mathcal{I} = \{\{1,4\},\{2,3\}\}$ and

$$(-1)^{|\{1,4\}|} e^{2\pi i(0/n_1+7/n_4)} + (-1)^{|\{2,3\}|} e^{2\pi i(1/n_2+5/n_3)} = -e^{\pi i/6} + e^{\pi i/6} = 0.$$

Corollary 1.1. *If $\mathcal{A} = \{a_s(n_s)\}_{s=0}^{k}$ is a finite system of residue classes with $w_{\mathcal{A}}(x) > m = \lfloor \sum_{s=1}^{k} 1/n_s \rfloor$ for all $x \in \mathbb{Z}$, then*

$$\left|\left\{ I \subseteq [1,k] :\ \sum_{s\in I} \frac{1}{n_s} = \frac{a}{n_0} \right\}\right| \geqslant \binom{m}{\lfloor a/n_0 \rfloor} \quad \text{for all } a \in \mathbb{N}. \tag{1.6}$$

In particular, if (1.1) *has covering multiplicity $m(A) = \lfloor \sum_{s=1}^{k} 1/n_s \rfloor$, then*

$$\left|\left\{ I \subseteq [1,k] :\ \sum_{s\in I} \frac{1}{n_s} = n \right\}\right| \geqslant \binom{m(A)}{n} \quad \text{for each } n \in \mathbb{N}. \tag{1.7}$$

Proof. Observe that the left hand side of (1.4) is nonzero in the case $\alpha = a = 0$. So (1.6) follows from Theorem 1.1 immediately. In the case $n_0 = 1$ this yields the latter result in Corollary 1.1. □

Remark 1.1. Let (1.1) be an exact m-cover of $\mathbb{Z}$. Then $\sum_{s=1}^{k} 1/n_s = m$ and $\lfloor \sum_{s\in[1,k]\setminus\{t\}} 1/n_s \rfloor = m-1$ for any $t = 1, \ldots, k$. So Corollary 1.1 implies the following result in [S97]: For any $t \in [1,k]$ and $a \in \mathbb{N}$, we have

$$\left|\left\{ I \subseteq [1,k] \setminus \{t\} :\ \sum_{s\in I} \frac{1}{n_s} = \frac{a}{n_t} \right\}\right| \geqslant \binom{m-1}{\lfloor a/n_t \rfloor}.$$

As $m(A) = \sum_{s=1}^{k} 1/n_s$, we also have $|\{I \subseteq [1,k] : \sum_{s\in I} 1/n_s = n\}| \geqslant \binom{m}{n}$ for all $n = 0, 1, \ldots, m$, which was first established in [S92] by means of the Riemann zeta function.

Corollary 1.2. *Let* (1.1) *be an m-system with $m = \lceil \sum_{s=1}^{k} 1/n_s \rceil$, where $\lceil \alpha \rceil$ denotes the least integer not smaller than a real number α. Then*

$$\left|\left\{ \langle m_1, \ldots, m_k \rangle \in \mathbb{Z}^k :\ m_s \in [1, n_s],\ \sum_{s=1}^{k} \frac{m_s}{n_s} = n \right\}\right| \geqslant \binom{k-m}{n-m} \tag{1.8}$$

for every $n = m, \dots, k$.

Proof. Let $n \in [m,k]$. Clearly the left hand side of (1.8) coincides with

$$L := \left|\left\{\langle x_1,\dots,x_k\rangle :\ x_s \in [0,n_s-1],\ \sum_{s=1}^k \frac{x_s}{n_s} = \sum_{s=1}^k \frac{n_s}{n_s} - n = k-n\right\}\right|.$$

Since $\sum_{s=1}^k 1/n_s > m-1$, $w_A(x) = m$ for some $x \in \mathbb{Z}$. As the dual A^* of (1.1) has covering multiplicity $m(A^*) = k-m$, applying Corollary 1.1 to A^* we find that $L \geqslant \binom{k-m}{k-n} = \binom{k-m}{n-m}$. This concludes the proof. $\square$

Remark 1.2. When (1.1) is an exact m-cover of $\mathbb{Z}$, it was proved in [S97] (by a different approach) that for each $n \in \mathbb{N}$ the equation $\sum_{s=1}^k x_s/n_s = n$ with $x_s \in [0,n_s)$ has at least $\binom{k-m}{n}$ solutions.

Corollary 1.3. *Let $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ be a finite system of residue classes with $m(\mathcal{A}) > m = \lfloor \sum_{s=1}^k m_s/n_s \rfloor$, where $m_1,\dots,m_k \in \mathbb{Z}^+$. Suppose that $J \subseteq [1,k]$ and $\sum_{s\in I} m_s/n_s = \sum_{s\in J} m_s/n_s$ for no $I \subseteq [1,k]$ with $I \neq J$. Then*

$$\left\{n_0 \sum_{s\in J} \frac{m_s}{n_s}\right\} + \left\{n_0 \sum_{s\in \bar{J}} \frac{m_s}{n_s}\right\} < 1, \tag{1.9}$$

where $\bar{J} = [1,k] \setminus J$. Also,

$$\sum_{s\in J} \frac{m_s}{n_s} \geqslant m \quad \textit{or} \quad \sum_{s\in \bar{J}} \frac{m_s}{n_s} \geqslant m \tag{1.10}$$

Proof. Let $v = \sum_{s\in J} m_s/n_s$, $\alpha = \{n_0 v\}$ and $b = \lfloor n_0 v \rfloor$. Then $(\alpha+b)/n_0 = v$ and

$$\sum_{\substack{I\subseteq[1,k] \\ \sum_{s\in I} m_s/n_s = v}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} a_s m_s/n_s} = (-1)^{|J|} e^{2\pi i \sum_{s\in J} a_s m_s/n_s} \neq 0.$$

By Theorem 1.1, (1.5) holds for any $a \in \mathbb{N}$. Applying (1.5) with $a = mn_0 + n_0 - 1$ we find that $\sum_{s\in I} m_s/n_s = (\alpha + mn_0 + n_0 - 1)/n_0$ for some $I \subseteq [1,k]$, therefore $\sum_{s=1}^k m_s/n_s \geqslant m + (\alpha + n_0 - 1)/n_0$. As $\lfloor \sum_{s=1}^k m_s/n_s \rfloor = m$, we must have

$$\left\{\sum_{s=1}^k \frac{m_s}{n_s}\right\} \geqslant \frac{\alpha + n_0 - 1}{n_0}, \text{ i.e., } n_0 - 1 + \alpha \leqslant n_0 \left\{\sum_{s=1}^k \frac{m_s}{n_s}\right\} < n_0.$$

Therefore $\alpha \leqslant \{n_0\{\sum_{s=1}^k m_s/n_s\}\} = \{n_0 \sum_{s=1}^k m_s/n_s\}$, which is equivalent to (1.9).

(1.5) in the case $a = b$ gives that $\binom{m}{\lfloor b/n_0 \rfloor} \leqslant 1$, thus $\lfloor v \rfloor \in \{0, m\}$. As $n_0\{v\} - \alpha = \lfloor n_0\{v\} \rfloor \leqslant n_0 - 1$, $\{v\} \leqslant (\alpha + n_0 - 1)/n_0 \leqslant \{\sum_{s=1}^k m_s/n_s\}$. If $\lfloor v \rfloor = 0$, then $m + v \leqslant m + \{\sum_{s=1}^k m_s/n_s\} = \sum_{s=1}^k m_s/n_s$ and hence $\sum_{s\in\bar{J}} m_s/n_s \geqslant m$. Therefore (1.10) is valid. We are done. □

Remark 1.3. Let (1.1) be an exact m-cover of $\mathbb{Z}$. Theorem 4(ii) in [S95] asserts that if $\emptyset \neq J \subset [1, k]$ then $\sum_{s\in I} 1/n_s = \sum_{s\in J} 1/n_s$ for some $I \subseteq [1, k]$ with $I \neq J$. This follows from Corollary 1.3, for, $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ (where $a_0 = 0$ and $n_0 = 1$) is an $(m + 1)$-cover of $\mathbb{Z}$ with $\sum_{s\in J\cup\bar{J}} 1/n_s = \sum_{s=1}^k 1/n_s = m$.

In the 1960s Erdős made the following conjecture: For any system (1.1) with $1 < n_1 < \cdots < n_k$, if it is a cover of $\mathbb{Z}$ then $\sum_{s=1}^k 1/n_s > 1$, in other words it cannot be a disjoint cover of $\mathbb{Z}$. This was later confirmed by H. Davenport, L. Mirsky, D. Newman and R. Radó who proved that if (1.1) is a disjoint cover of $\mathbb{Z}$ with $1 < n_1 \leqslant \cdots \leqslant n_{k-1} \leqslant n_k$ then $n_{k-1} = n_k$.

Corollary 1.4. *Let* (1.1) *be an m-cover of $\mathbb{Z}$ with*

$$n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k \quad (0 < l < k). \tag{1.11}$$

Then, for any $r \in [0, l]$ with $r < n_k/n_{k-l}$, either $\sum_{s=1}^{k-r} 1/n_s \geqslant m$ or

$$\binom{l}{r} \in D(n_k) = \left\{ \sum_{p\mid n_k} px_p :\ x_p \in \mathbb{N} \text{ for any prime divisor } p \text{ of } n_k \right\}.$$

Proof. Set $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ where $a_0 = 0$ and $n_0 = 1$. Suppose that $\sum_{s=1}^{k-r} 1/n_s < m$. Then $\sum_{s=1}^k 1/n_s < m + r/n_k < m + 1 \leqslant m(\mathcal{A})$. Since $|\{I \subseteq [1, k] : \sum_{s\in I} 1/n_s = m + r/n_k\}| = 0 < \binom{m}{m}$, by Theorem 1.1 we must have

$$\sum_{\substack{I\subseteq[1,k] \\ \sum_{s\in I} 1/n_s = r/n_k}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} a_s/n_s} = 0.$$

Observe that $r/n_k < 1/n_{k-l} = \min\{1/n_s : 1 \leqslant s \leqslant k - l\}$. Therefore

$$0 = \sum_{\substack{I\subseteq(k-l,k] \\ \sum_{s\in I} 1/n_s = r/n_k}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} a_s/n_s} = (-1)^r \Sigma_r,$$

where

$$\Sigma_r = \sum_{\substack{I\subseteq(k-l,k] \\ |I|=r}} e^{2\pi i \sum_{s\in I} a_s/n_k}.$$

By [S03, Lemma 3.1], $\Sigma_r = 0$ implies that

$$\binom{l}{r} = |\{I \subseteq (k-l,k] : |I| = r\}| \in D(n_k).$$

This concludes the proof. $\square$

Remark 1.4. Let (1.1) be an m-cover of $\mathbb{Z}$ with (1.11). By Corollary 1.4 in the case $r = l$, either $l \geqslant n_k/n_{k-l} > 1$ or $\sum_{s=1}^{k-l} 1/n_s \geqslant m$; this is one of the main results in [S96]. Corollary 1.4 in the case $r = 1$ yields that either $\sum_{s=1}^{k-1} 1/n_s \geqslant m$ or $l \in D(n_k)$; this implies the extended Newman-Znám result (cf. [N71]) which asserts that if (1.1) is an exact m-cover of $\mathbb{Z}$ (and hence $\sum_{s=1}^{k-1} 1/n_s < \sum_{s=1}^{k} 1/n_s = m$) then l is not smaller than the least prime divisor of n_k.

Let (1.1) is an m-system with (1.11), and let $r \in \mathbb{N}$ and $r < n_k/n_{k-l}$. With the help of the dual system of (1.1), we can also show that either $\sum_{s=1}^{k} 1/n_s \leqslant m - r/n_k$ or

$$\binom{l+r-1}{r} = |\{\langle x_{k-l+1}, \ldots, x_k\rangle \in \mathbb{N}^l : \ x_{k-l+1} + \cdots + x_k = r\}| \in D(n_k).$$

If (1.1) is disjoint with $1 < n_1 < \cdots < n_k$, then $\sum_{s=1}^{k} 1/n_s < 1$ since (1.1) is not a disjoint cover of $\mathbb{Z}$; Erdős [E62] showed further that $\sum_{s=1}^{k} 1/n_s \leqslant 1 - 1/2^k$. Now we give a generalization of this result.

Theorem 1.2. *Let* (1.1) *be an m-system with $k \geqslant m$, $\sum_{s=1}^{k} 1/n_s \neq m$ and $n_1 \leqslant \cdots \leqslant n_k$. Then we have*

$$\sum_{s=1}^{k} \frac{1}{n_s} \leqslant m - \frac{1}{2^{k-m+1}}, \tag{1.12}$$

and equality holds if and only if $n_s = 2^{\max\{s-m+1,0\}}$ for all $s = 1, \ldots, k$.

Remark 1.5. Let $k \geqslant m \geqslant 1$ be integers. Then $m-1$ copies of $0(1)$, together with the following $k-m+1$ residue classes

$$1(2),\ 2(2^2),\ \ldots,\ 2^{k-m}(2^{k-m+1}),$$

form an m-system with the moduli $2^{\max\{s-m+1,0\}}$ $(s = 1, \ldots, k)$.

We will prove Theorems 1.1 and 1.2 in the next section. Section 3 deals with two characterizations of m-systems one of which is as follows.

Theorem 1.3. (1.1) *is an m-system if and only if for any* $n \in [m,k)$ *we have*

$$S(n,\alpha) = \begin{cases} (-1)^k & \text{if } \alpha = 0, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases} \tag{1.13}$$

where $S(n,\alpha)$ *represents the sum*

$$\sum_{\substack{m_1,\dots,m_k\in\mathbb{Z}^+ \\ \{\sum_{s=1}^k m_s/n_s\}=\alpha}} (-1)^{\lfloor \sum_{s=1}^k m_s/n_s \rfloor} \binom{n}{\lfloor \sum_{s=1}^k m_s/n_s \rfloor} e^{2\pi i \sum_{s=1}^k a_s m_s/n_s}.$$

Theorem 1.3 in the case $m = 1$ yields the following result.

Corollary 1.5. *If* (1.1) *is disjoint, then we have*

$$\sum_{\substack{m_1,\dots,m_k\in\mathbb{Z}^+ \\ \sum_{s=1}^k m_s/n_s=1}} e^{2\pi i \sum_{s=1}^k a_s m_s/n_s} = (-1)^{k-1}. \tag{1.14}$$

A residue class $a(n) = a + n\mathbb{Z}$ is a coset of $n\mathbb{Z}$ in the additive group $\mathbb{Z}$ with $[\mathbb{Z} : n\mathbb{Z}] = n$. In [S06] the author conjectured that if $\{a_s G_s\}_{s=1}^k$ $(1 < k < \infty)$ is a disjoint system of left cosets in a group G with all the indices $n_s = [G : G_s]$ finite, then $\gcd(n_s, n_t) \geqslant k$ for some $1 \leqslant s < t \leqslant k$.

2. Proofs of Theorems 1.1 and 1.2

Lemma 2.1. *Let* $N \in \mathbb{Z}^+$ *be a common multiple of the moduli* $n_1, \dots, n_k$ *in* (1.1)*. And let* $m, m_1, \dots, m_k \in \mathbb{Z}^+$*. If* (1.1) *is an m-cover of* $\mathbb{Z}$*, then* $(1 - z^N)^m$ *divides the polynomial* $\prod_{s=1}^k (1 - z^{Nm_s/n_s} e^{2\pi i a_s m_s/n_s})$*. When* $m_1, \dots, m_k$ *are relatively prime to* $n_1, \dots, n_k$ *respectively, the converse also holds.*

Proof. For any $r = 0, 1, \dots, N-1$, clearly $e^{2\pi i r/N}$ is a zero of the polynomial $\prod_{s=1}^k (1 - z^{Nm_s/n_s} e^{2\pi i a_s m_s/n_s})$ with multiplicity $M_r = |\{s \in [1,k] : n_s \mid m_s(r + a_s)\}|$. Observe that $M_r \geqslant w_A(-r)$. If m_s is relatively prime to n_s for each $s \in [1,k]$, then $M_r = w_A(-r)$. As $(1 - z^N)^m = \prod_{r=0}^{N-1} (1 - z e^{-2\pi i r/N})^m$, the desired result follows from the above.

Proof of Theorem 1.1. Set $m_0 = 1$, and let N_0 be the least common multiple of $n_0, n_1, \dots, n_k$. In light of Lemma 2.1, we can write $P(z) = \prod_{s=0}^k (1 - z^{N_0 m_s/n_s} e^{2\pi i a_s m_s/n_s})$ in the form $(1 - z^{N_0})^{m+1} Q(z)$ where $Q(z) \in \mathbb{C}[z]$. Clearly

$$\deg Q = \deg P - (m+1)N_0 = N_0 \left(\sum_{s=0}^k \frac{m_s}{n_s} - m - 1 \right) < \frac{N_0}{n_0}.$$

Also,

$$\begin{aligned}\prod_{s=1}^{k}\left(1-z^{N_0m_s/n_s}e^{2\pi i a_s m_s/n_s}\right)\\ =\sum_{n=0}^{m}(-1)^n\binom{m}{n}z^{nN_0}\sum_{r=0}^{n_0-1}z^{rN_0/n_0}e^{2\pi i r a_0/n_0}Q(z)\end{aligned}\tag{2.1}$$

since

$$\frac{1-z^{N_0}}{1-z^{N_0/n_0}e^{2\pi i a_0/n_0}}=\sum_{r=0}^{n_0-1}z^{rN_0/n_0}e^{2\pi i r a_0/n_0}.$$

Let $a\in\mathbb{N}$ and

$$C_a=(-1)^{\lfloor a/n_0\rfloor}\sum_{\substack{I\subseteq[1,k]\\ \sum_{s\in I}m_s/n_s=(\alpha+a)/n_0}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}(a_s-a_0)m_s/n_s}.$$

By comparing the coefficients of $z^{N_0(\alpha+a)/n_0}$ on both sides of (2.1) we obtain that

$$\begin{aligned}&\sum_{\substack{I\subseteq[1,k]\\ \sum_{s\in I}m_s/n_s=(\alpha+a)/n_0}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}a_s m_s/n_s}\\ =&(-1)^{\lfloor a/n_0\rfloor}\binom{m}{\lfloor a/n_0\rfloor}e^{2\pi i a_0\{a/n_0\}}[z^{\alpha N_0/n_0}]Q(z),\end{aligned}$$

where $[z^{\alpha N_0/n_0}]Q(z)$ denotes the coefficient of $z^{\alpha N_0/n_0}$ in $Q(z)$. Therefore

$$C_a=e^{-2\pi i\alpha a_0/n_0}\binom{m}{\lfloor a/n_0\rfloor}[z^{\alpha N_0/n_0}]Q(z)=\binom{m}{\lfloor a/n_0\rfloor}C_0.\tag{2.2}$$

For an algebraic integer ω in the field $K=\mathbb{Q}(e^{2\pi i/N_0})$, the norm $N(\omega)=\prod_{1\leqslant r\leqslant N_0,\ \gcd(r,N_0)=1}\sigma_r(\omega)$ (with respect to the field extension $K/\mathbb{Q}$) is a rational integer, where σ_r is the automorphism of K (in the Galois group $\mathrm{Gal}(K/\mathbb{Q})$) induced by $\sigma_r(e^{2\pi i/N_0})=e^{2\pi i r/N_0}$. (See, e.g., [K97, Chapter 1].) As $N((-1)^{\lfloor a/n_0\rfloor}C_a)$ equals

$$\prod_{\substack{1\leqslant r\leqslant N_0\\ \gcd(r,N_0)=1}}\sum_{\substack{I\subseteq[1,k]\\ \sum_{s\in I}m_s/n_s=(\alpha+a)/n_0}}(-1)^{|I|}e^{2\pi i r\sum_{s\in I}(a_s-a_0)m_s/n_s},$$

we have

$$\begin{aligned}|N(C_a)|=&\prod_{\substack{1\leqslant r\leqslant N_0\\ \gcd(r,N_0)=1}}\left|\sum_{\substack{I\subseteq[1,k]\\ \sum_{s\in I}m_s/n_s=(\alpha+a)/n_0}}(-1)^{|I|}e^{2\pi i r\sum_{s\in I}(a_s-a_0)m_s/n_s}\right|\\ \leqslant&\left|\left\{I\subseteq[1,k]:\ \sum_{s\in I}\frac{m_s}{n_s}=\frac{\alpha+a}{n_0}\right\}\right|^{\varphi(N_0)},\end{aligned}$$

where φ is Euler's totient function. Also,

$$|N(C_a)| = \left|N\left(\binom{m}{\lfloor a/n_0\rfloor}\right)\right| \times |N(C_0)| = \binom{m}{\lfloor a/n_0\rfloor}^{\varphi(N_0)}|N(C_0)|.$$

Suppose that $C_b \neq 0$ for some $b \in \mathbb{N}$. Then $N(C_b) \neq 0$ and hence $N(C_0) \in \mathbb{Z}$ is nonzero. For any $a \in \mathbb{N}$, we have

$$\left|\left\{I \subseteq [1,k] : \sum_{s\in I}\frac{m_s}{n_s} = \frac{\alpha + a}{n_0}\right\}\right|^{\varphi(N_0)} \geqslant |N(C_a)| \geqslant \binom{m}{\lfloor a/n_0\rfloor}^{\varphi(N_0)}$$

and hence (1.5) holds. This concludes the proof. □

Proof of Theorem 1.2. We use induction on k.

In the case $k = m$, we have $n_k > 1$ and hence

$$\sum_{s=1}^{k}\frac{1}{n_s} \leqslant k - 1 + \frac{1}{n_k} \leqslant m - \frac{1}{2} = m - \frac{1}{2^{k-m+1}},$$

also $\sum_{s=1}^{k} 1/n_s = m - 1/2$ if and only if $n_1 = \cdots = n_{k-1} = 1$ and $n_k = 2$.

Now let $k > m$. Clearly $\sum_{s=1}^{k-1} 1/n_s < \sum_{s=1}^{k} 1/n_s < m$. Assume that

$$\sum_{s=1}^{k-1}\frac{1}{n_s} \leqslant m - \frac{1}{2^{(k-1)-m+1}} = m - \frac{1}{2^{k-m}}$$

and that equality holds if and only if $n_s = 2^{\max\{s-m+1,0\}}$ for all $s \in [1, k-1]$. When $n_k > 2^{k-m+1}$, we have

$$\sum_{s=1}^{k}\frac{1}{n_s} = \sum_{s=1}^{k-1}\frac{1}{n_s} + \frac{1}{n_k} < \left(m - \frac{1}{2^{k-m}}\right) + \frac{1}{2^{k-m+1}} = m - \frac{1}{2^{k-m+1}}.$$

If $\sum_{s=1}^{k} 1/n_s > m - 1/n_k$, then $\lceil\sum_{s=1}^{k} 1/n_s\rceil = m$, thus $\sum_{s=1}^{k} m_s/n_s = m$ for some $m_1, \ldots, m_k \in \mathbb{Z}^+$ (by Corollary 1.2) and hence

$$m - \sum_{s=1}^{k}\frac{1}{n_s} \geqslant \min\left\{\frac{1}{n_s} : 1 \leqslant s \leqslant k\right\} = \frac{1}{n_k}.$$

This shows that we do have $\sum_{s=1}^{k} 1/n_s \leqslant m - 1/n_k$. Providing $n_k \leqslant 2^{k-m+1}$, (1.12) holds, and also

$$\begin{aligned}\sum_{s=1}^{k}\frac{1}{n_s} = m - \frac{1}{2^{k-m+1}} &\iff n_k = 2^{k-m+1} \text{ and } \sum_{s=1}^{k-1}\frac{1}{n_s} = m - \frac{1}{2^{k-m}}\\ &\iff n_s = 2^{\max\{s-m+1,0\}} \text{ for } s = 1, \ldots, k-1, k.\end{aligned}$$

This concludes the induction step and we are done. □

3. Characterizations of m-systems

Proof of Theorem 1.3. Like Lemma 2.1, (1.1) is an m-system if and only if $f(z)=(1-z^N)^m/\prod_{s=1}^k(1-z^{N/n_s}e^{2\pi i a_s/n_s})$ is a polynomial, where N is the least common multiple of $n_1,\ldots,n_k$.

Set $c=m-\sum_{s=1}^k 1/n_s$. If $f(z)$ is a polynomial, then $\deg f=cN$ and $[z^{cN}]f(z)=(-1)^{k-m}e^{-2\pi i\sum_{s=1}^k a_s/n_s}$.

For $|z|<1$ we have

$$f(z)=\sum_{n=0}^m\binom{m}{n}(-1)^n z^{nN}\prod_{s=1}^k\sum_{x_s=0}^\infty e^{2\pi i a_s x_s/n_s}z^{Nx_s/n_s}.$$

Let $\alpha\geqslant 0$. Then

$$\begin{aligned}[z^{(c+\alpha)N}]f(z)=&\sum_{n=0}^m(-1)^n\binom{m}{n}\sum_{\substack{x_1,\ldots,x_k\in\mathbb{N}\\ \sum_{s=1}^k x_s/n_s=c+\alpha-n}}e^{2\pi i\sum_{s=1}^k a_s x_s/n_s}\\ =&\sum_{n=0}^m(-1)^n\binom{m}{n}\sum_{\substack{m_1,\ldots,m_k\in\mathbb{Z}^+\\ \sum_{s=1}^k m_s/n_s=\alpha+m-n}}e^{2\pi i\sum_{s=1}^k a_s(m_s-1)/n_s}\\ =&(-1)^m e^{-2\pi i\sum_{s=1}^k a_s/n_s}S(m,\alpha),\end{aligned}$$

where $S(n,\alpha)$ $(n\in\mathbb{N})$ represents the sum

$$\sum_{\substack{m_1,\ldots,m_k\in\mathbb{Z}^+\\ \sum_{s=1}^k m_s/n_s-\alpha\in\mathbb{N}}}(-1)^{\sum_{s=1}^k m_s/n_s-\alpha}\binom{n}{\sum_{s=1}^k m_s/n_s-\alpha}e^{2\pi i\sum_{s=1}^k a_s m_s/n_s}$$

which agrees with its definition in the case $0\leqslant\alpha<1$ given in Theorem 1.3.

(i) Suppose that (1.1) is an m-system. Then $f(z)$ is a polynomial of degree cN and hence

$$S(m,\alpha)=(-1)^m e^{2\pi i\sum_{s=1}^k a_s/n_s}[z^{(c+\alpha)N}]f(z)=\begin{cases}(-1)^k & \text{if } \alpha=0,\\ 0 & \text{if } \alpha>0.\end{cases}$$

For any integer $n\geqslant m$, (1.1) is also an n-system and so we have (1.13).

(ii) Now assume that (1.13) holds for all $n\in[m,k)$. For any $n\geqslant k$ we also have (1.13) by (i) because (1.1) is a k-system.

If $0<\alpha<1$ then $S(n,\alpha)=0$ for any integer $n\geqslant m$. Fix $\alpha>0$. If $S(n,\alpha)=0$ for all integers $n\geqslant m$, then for any integer $n\geqslant m$ we have

$$S(n,\alpha+1)=S(n,\alpha)-S(n+1,\alpha)=0$$

because $\binom{n}{j-1} = \binom{n+1}{j} - \binom{n}{j}$ for $j = 1, 2, \ldots$. Thus, by induction, $S(n, \alpha) = 0$ for all $\alpha > 0$ and $n = m, m+1, \ldots$. It follows that $[z^{(c+\alpha)N}]f(z) = 0$ for any $\alpha > 0$. So $f(z)$ is a polynomial and (1.1) is an m-system.

The proof of Theorem 1.3 is now complete. □

The following characterization of m-covers plays important roles in [S95, S96].

Lemma 3.1 (Sun [S95])**.** *Let $m, m_1, \ldots, m_k \in \mathbb{Z}^+$. If* (1.1) *forms an m-cover of $\mathbb{Z}$, then*

$$\sum_{\substack{I \subseteq [1,k] \\ \{\sum_{s \in I} m_s/n_s\} = \theta}} (-1)^{|I|} \binom{\lfloor \sum_{s \in I} m_s/n_s \rfloor}{n} e^{2\pi i \sum_{s \in I} a_s m_s/n_s} = 0 \tag{3.1}$$

for all $0 \leqslant \theta < 1$ and $n = 0, 1, \ldots, m-1$. We also have the converse if $m_1, \ldots, m_k$ are relatively prime to $n_1, \ldots, n_k$ respectively.

We can provide a new proof of Lemma 3.1 in a way similar to the proof of Theorem 1.3.

Lemma 3.2. *Let $n \in \mathbb{Z}^+$ and $l \in [0, n-1]$. Then*

$$\sum_{\substack{J \subseteq [1,n) \\ |J| = l}} e^{2\pi i \sum_{j \in J} j/n} = (-1)^l. \tag{3.2}$$

Proof. Clearly we have the identity

$$\prod_{0<j<n} \left(1 - z e^{2\pi i j/n}\right) = \frac{1 - z^n}{1 - z} = 1 + z + \cdots + z^{n-1}.$$

Comparing the coefficients of z^l we then obtain (3.2). □

Using Lemmas 3.1 and 3.2 we can deduce another characterization of m-systems.

Theorem 3.1. (1.1) *is an m-system if and only if we have*

$$\sum_{\substack{x_s \in [0, n_s) \text{ for } s \in [1,k] \\ \{\sum_{s=1}^k x_s/n_s\} = \theta}} \binom{\lfloor \sum_{s=1}^k x_s/n_s \rfloor}{n} e^{2\pi i \sum_{s=1}^k a_s x_s/n_s} = 0 \tag{3.3}$$

for all $0 \leqslant \theta < 1$ and $n \in [0, k-m)$.

Proof. The case $k \leqslant m$ is trivial, so we just let $k > m$. Recall that (1.1) is an m-system if and only if its dual A^* is a $(k-m)$-cover of $\mathbb{Z}$.

By Lemma 3.1 in the case $m_1 = \cdots = m_k = 1$, A^* forms an $(k-m)$-cover of $\mathbb{Z}$ if and only if for any $0 \leqslant \theta < 1$ and $n \in [0, k-m)$ the sum

$$\sum_{\substack{x_s \in [0,n_s) \text{ for } s \in [1,k] \\ \{\sum_{s=1}^k x_s/n_s\} = \theta}} (-1)^{\sum_{s=1}^k x_s} \binom{\lfloor \sum_{s=1}^k x_s/n_s \rfloor}{n} e^{2\pi i \sum_{s=1}^k a_s x_s/n_s} \prod_{s=1}^k f_s(x_s)$$

vanishes, where

$$f_s(x_s) = \sum_{\substack{J \subseteq [1,n_s) \\ |J| = x_s}} e^{2\pi i \sum_{j \in J} j/n_s} = (-1)^{x_s}$$

by Lemma 3.2. This concludes the proof. □

The following consequence extends Corollary 1.5.

Corollary 3.1. *Let* (1.1) *be an m-system. Then we have*

$$\sum_{\substack{m_s \in [1,n_s] \text{ for } s \in [1,k] \\ m - \sum_{s=1}^k m_s/n_s \in \mathbb{N}}} \binom{k-1-\sum_{s=1}^k m_s/n_s}{m - \sum_{s=1}^k m_s/n_s} e^{2\pi i \sum_{s=1}^k a_s m_s/n_s} = (-1)^{k-m}.$$

Proof. If $k \leqslant m$, then the left hand side of the last equality coincides with

$$\binom{k-1-\sum_{s=1}^k n_s/n_s}{m - \sum_{s=1}^k n_s/n_s} e^{2\pi i \sum_{s=1}^k a_s n_s/n_s} = \binom{-1}{m-k} = (-1)^{m-k}.$$

Now let $k > m$. As $\{-a_s(n_s)\}_{s=1}^k$ is an m-system, by Theorem 4.1 and the identity

$$(-1)^{k-m-1} \binom{x-1}{k-m-1} = \sum_{n=0}^{k-m-1} (-1)^n \binom{x}{n}$$

(cf. [GKP, (5.16)]) we have

$$\begin{aligned} 0 &= \sum_{\substack{x_s \in [0,n_s) \text{ for } s \in [1,k] \\ \{\sum_{s=1}^k x_s/n_s\} = 0}} \binom{\lfloor \sum_{s=1}^k x_s/n_s \rfloor - 1}{k-m-1} e^{2\pi i \sum_{s=1}^k (-a_s) x_s/n_s} \\ &= \sum_{\substack{m_s \in [1,n_s] \text{ for } s \in [1,k] \\ \sum_{s=1}^k (n_s - m_s)/n_s \in \mathbb{N}}} \binom{\sum_{s=1}^k (n_s - m_s)/n_s - 1}{k-m-1} e^{-2\pi i \sum_{s=1}^k a_s (n_s - m_s)/n_s} \\ &= \sum_{\substack{m_s \in [1,n_s] \text{ for } s \in [1,k] \\ \sum_{s=1}^k m_s/n_s \in [0,k-1]}} \binom{k-1-\sum_{s=1}^k m_s/n_s}{k-1-m} e^{2\pi i \sum_{s=1}^k a_s m_s/n_s} \\ &\qquad + \binom{k-1-\sum_{s=1}^k n_s/n_s}{k-1-m} e^{2\pi i \sum_{s=1}^k a_s n_s/n_s}. \end{aligned}$$

So the desired equality follows. □

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Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

E-mail address: zwsun@nju.edu.cn *Homepage*: http://math.nju.edu.cn/∼zwsun