

**NEW CONGRUENCES FOR
CENTRAL BINOMIAL COEFFICIENTS**

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ABSTRACT. Let p be a prime and let a be a positive integer. In this paper we determine $\sum_{k=0}^{p^a-1} \binom{2k}{k+d}/m^k$ and $\sum_{k=1}^{p-1} \binom{2k}{k+d}/(km^{k-1})$ modulo p for all $d = 0, \dots, p^a$, where m is any integer not divisible by p . For example, we show that if $p \neq 2, 5$ then

$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k} \equiv -5 \frac{F_{p-\left(\frac{p}{5}\right)}}{p} \pmod{p},$$

where F_n is the n th Fibonacci number and $(-)$ is the Jacobi symbol. We also prove that if $p > 3$ then

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv \frac{8}{9} p^2 B_{p-3} \pmod{p^3},$$

where B_n denotes the n th Bernoulli number.

Key words and phrases. Central binomial coefficients, congruences modulo primes, Fibonacci numbers, Bernoulli numbers.

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1. INTRODUCTION

A central binomial coefficient has the form $\binom{2n}{n}$ with $n \in \mathbb{N} = \{0, 1, \dots\}$. A well-known theorem of Wolstenholme (see, e.g., [5]) states that

$$\frac{1}{2} \binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3} \quad \text{for any prime } p > 3.$$

In 2006 H. Pan and Z. W. Sun [9] used a sophisticated combinatorial identity to deduce that if p is a prime then

$$\sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \left(\frac{p-d}{3}\right) \pmod{p} \quad \text{for } d = 0, \dots, p, \quad (1.1)$$

where the Jacobi symbol $\left(\frac{a}{3}\right)$ coincides with the unique integer $\varepsilon \in \{0, \pm 1\}$ satisfying $a \equiv \varepsilon \pmod{3}$. In a recent paper [16] the authors determined $\sum_{k=0}^{p^a-1} \binom{2k}{k+d} \pmod{p^2}$ for any prime p and $d \in \{0, 1, \dots, p^a\}$ with $a \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$.

In this paper we extend the congruence (1.1) in a new way and derive various congruences related to recurrences. Throughout this paper, for an assertion A we set

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

We also define two recurrences $\{u_n(x)\}_{n \in \mathbb{N}}$ and $\{v_n(x)\}_{n \in \mathbb{N}}$ of polynomials as follows:

$$u_0(x) = 0, \quad u_1(x) = 1, \quad \text{and } u_{n+1}(x) = xu_n(x) - u_{n-1}(x) \quad (n = 1, 2, \dots),$$

and

$$v_0(x) = 2, \quad v_1(x) = x, \quad \text{and } v_{n+1}(x) = xv_n(x) - v_{n-1}(x) \quad (n = 1, 2, \dots).$$

For a fixed integer x , the sequences $\{u_n(x)\}_{n \in \mathbb{N}}$ and $\{v_n(x)\}_{n \in \mathbb{N}}$ are linear recurrences of integers. By induction, for any $n \in \mathbb{N}$ we have

$$u_n(-x) = (-1)^{n-1} u_n(x) \quad \text{and} \quad v_n(-x) = (-1)^n v_n(x). \quad (1.2)$$

Now we state our first theorem.

Theorem 1.1. *Let p be a prime and let $d \in \{0, \dots, p^a\}$ with $a \in \mathbb{Z}^+$. Let $m \in \mathbb{Z}$ with $p \nmid m$. Then we have*

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k+d}}{m^k} \equiv u_{p^a-d}(m-2) \pmod{p} \quad (1.3)$$

and

$$d \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k+d}}{km^{k-1}} \equiv 2(-1)^d + v_{p^a-d}(m-2) \pmod{p} \quad \text{provided } d > 0. \quad (1.4)$$

If $p \neq 2$, then

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k+d}}{m^k} \equiv -u_{d-\left(\frac{m(m-4)}{p^a}\right)}(m-2) \pmod{p} \quad (1.5)$$

and also

$$d \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k+d}}{km^{k-1}} \equiv 2(-1)^d + v_{d-\left(\frac{m(m-4)}{p^a}\right)}(m-2) \pmod{p} \quad \text{provided } d > 0, \quad (1.6)$$

where $u_{-1}(x) = xu_0(x) - u_1(x) = -1$ and $v_{-1}(x) = xv_0(x) - v_1(x) = x$.

Remark 1.1. Let p be any prime and let $a \in \mathbb{Z}^+$. As $u_n(-1) = \binom{n}{3}$ for $n = 0, 1, 2, \dots$, (1.3) in the case $m = 1$ yields that

$$\sum_{k=0}^{p^a-1} \binom{2k}{k+d} \equiv \left(\frac{p^a - d}{3} \right) \pmod{p} \quad \text{for every } d = 0, 1, \dots, p^a.$$

Since $v_n(-1) = 3[3 \mid n] - 1$ for all $n \in \mathbb{N}$, by (1.4) in the case $m = 1$, for $d \in \{1, \dots, p^a\}$ we have

$$d \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k+d}}{k} \equiv \begin{cases} 2(-1)^d + 2 \pmod{p} & \text{if } p^a \equiv d \pmod{3}, \\ 2(-1)^d - 1 \pmod{p} & \text{otherwise.} \end{cases}$$

The well-known Fibonacci sequence $\{F_n\}_{n \in \mathbb{N}}$ is defined by

$$F_0 = 0, \quad F_1 = 1, \quad \text{and } F_{n+1} = F_n + F_{n-1} \text{ for } n = 1, 2, 3, \dots$$

Its companion $\{L_n\}_{n \in \mathbb{N}}$, the Lucas sequence, is given by

$$L_0 = 2, \quad L_1 = 1, \quad \text{and } L_{n+1} = L_n + L_{n-1} \text{ for } n = 1, 2, 3, \dots$$

Define

$$\begin{aligned} F_{-1} &= F_1 - F_0 = 1, \quad F_{-2} = F_0 - F_{-1} = -1, \\ L_{-1} &= L_1 - L_0 = -1, \quad L_{-2} = L_0 - L_{-1} = 3. \end{aligned}$$

By induction, $F_{2n} = u_n(3)$ and $L_{2n} = v_n(3)$ for $n = -1, 0, 1, \dots$. Note also that $u_{2n}(0) = v_{2n+1}(0) = 0$ and $v_{2n}(0)/2 = u_{2n+1}(0) = (-1)^n$ for all $n \in \mathbb{N}$. Thus, with the help of (1.2), Theorem 1.1 in the cases $m = -1, 2$ gives the following consequence.

Corollary 1.1. *Let p be an odd prime and let $d \in \{0, 1, \dots, p^a\}$ with $a \in \mathbb{Z}^+$. Then*

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k+d} \equiv (-1)^{d-[p \neq 5]} F_{2(d-\frac{p^a}{5})} \pmod{p}, \quad (1.7)$$

and

$$d \sum_{k=1}^{p^a-1} (-1)^k \frac{\binom{2k}{k+d}}{k} \equiv (-1)^{d-[p=5]} L_{2(d-\frac{p^a}{5})} - 2(-1)^d \pmod{p} \quad (1.8)$$

provided $d > 0$. Also,

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k+d}}{2^k} \equiv \begin{cases} 0 \pmod{p} & \text{if } p^a \equiv d \pmod{2}, \\ 1 \pmod{p} & \text{if } p^a \equiv d+1 \pmod{4}, \\ -1 \pmod{p} & \text{if } p^a \equiv d-1 \pmod{4}, \end{cases} \quad (1.9)$$

and for $d > 0$ we have

$$d \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k+d}}{k2^k} - (-1)^d \equiv \begin{cases} 0 \pmod{p} & \text{if } p^a \not\equiv d \pmod{2}, \\ 1 \pmod{p} & \text{if } p^a \equiv d \pmod{4}, \\ -1 \pmod{p} & \text{if } p^a \equiv d+2 \pmod{4}. \end{cases} \quad (1.10)$$

Our following result can be viewed as a complement to Theorem 1.1.

Theorem 1.2. *Let p be a prime and let m be an integer not divisible by p . Then we have*

$$\frac{1}{2} \sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{km^{k-1}} \equiv \frac{m^p - V_p(m)}{p} \pmod{p}, \quad (1.11)$$

where the polynomial sequence $\{V_n(x)\}_{n \in \mathbb{N}}$ is defined as follows:

$$V_0(x) = 2, \quad V_1(x) = x, \quad \text{and } V_{n+1}(x) = x(V_n(x) + V_{n-1}(x)) \quad (n \in \mathbb{Z}^+).$$

Given a prime p and an integer a not divisible by p , we use $q_p(a)$ to denote the integer $(a^{p-1} - 1)/p$ and call $q_p(a)$ a *Fermat quotient* with base a . See E. Lehmer [7] for connections between Fermat quotients and Fermat's last theorem.

Corollary 1.2. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k2^{k-1}} \equiv \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k4^k} \equiv 2q_p(2) \pmod{p}. \quad (1.12)$$

If $p \neq 3$ then

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k3^{k-1}} \equiv 3q_p(3) \pmod{p}. \quad (1.13)$$

Corollary 1.3. *Let p be an odd prime.*

(i) *If $p \neq 5$, then we have*

$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k} \equiv -5 \frac{F_{p-(\frac{p}{5})}}{p} \pmod{p}, \quad (1.14)$$

$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k5^k} \equiv q_p(5) - 6 \frac{F_{p-(\frac{p}{5})}}{p} \pmod{p}, \quad (1.15)$$

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k5^k} \equiv q_p(5) - \frac{F_{p-(\frac{p}{5})}}{p} \pmod{p}. \quad (1.16)$$

(ii) *Define the Pell sequence $\{P_n\}_{n \in \mathbb{N}}$ by*

$$P_0 = 0, \quad P_1 = 1, \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad (n = 1, 2, 3, \dots).$$

Then

$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k4^k} \equiv 2q_p(2) - 4 \frac{P_{p-(\frac{p}{2})}}{p} \equiv 2 \sum_{0 < k < 3p/4} \frac{(-1)^{k-1}}{k} \pmod{p}. \quad (1.17)$$

(iii) *Let $\{S_n\}_{n \in \mathbb{N}}$ be the sequence defined by*

$$S_0 = 0, \quad S_1 = 1, \quad \text{and} \quad S_{n+1} = 4S_n - S_{n-1} \quad (n = 1, 2, 3, \dots).$$

If $p > 3$, then

$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k2^k} \equiv q_p(2) - 6 \left(\frac{2}{p}\right) \frac{S_{(p-(\frac{3}{p}))/2}}{p} \equiv \sum_{0 < k < 5p/6} \frac{(-1)^{k-1}}{k} \pmod{p} \quad (1.18)$$

and

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k6^k} \equiv q_p(2) + q_p(3) - 2 \left(\frac{2}{p}\right) \frac{S_{(p-(\frac{3}{p}))/2}}{p} \pmod{p}. \quad (1.19)$$

Remark 1.2. (a) A prime $p \neq 2, 5$ is called a Wall-Sun-Sun prime if $F_{p-(\frac{p}{5})} \equiv 0 \pmod{p^2}$ (cf. [1]). In 1992 Z. H. Sun and Z. W. Sun [13] showed that Fermat's equation $x^p + y^p = z^p$ has no integer solutions satisfying $p \nmid xyz$ unless p is a Wall-Sun-Sun prime. There are no Wall-Sun-Sun primes below 2×10^{14} (cf. [8]). In 1982 H. C. Williams [10] showed that

$$\frac{F_{p-(\frac{p}{5})}}{p} \equiv \frac{2}{5} \sum_{0 < k < 4p/5} \frac{(-1)^k}{k} \pmod{p}.$$

(b) The second congruences in (1.17) and (1.18) are essentially due to Z. W. Sun [14, 15]. For other information about the sequence $\{S_n\}_{n \in \mathbb{N}}$ the reader may consult [11].

In 2006 Pan and Sun [9] proved that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv 0 \pmod{p}$$

for any prime $p > 3$. Here we determine the sum modulo p^3 .

Theorem 1.3. *Let p be any prime and let $a \in \mathbb{Z}^+$. Then we have*

$$p^{a-1} \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k}}{k} \equiv \begin{cases} 2 \pmod{p^3} & \text{if } p = 2, \\ 5 \pmod{p^3} & \text{if } p = 3, \\ \frac{8}{9}p^2 B_{p-3} \pmod{p^3} & \text{otherwise,} \end{cases} \quad (1.20)$$

where B_0, B_1, B_2, \dots are the well-known Bernoulli numbers.

The following conjecture, which is related to (1.7) in the case $d = 0$, seems very challenging.

Conjecture 1.1. *Let $p \neq 2, 5$ be a prime and let $a \in \mathbb{Z}^+$. Then*

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left(\frac{p^a}{5}\right) \left(1 - 2F_{p^a-(\frac{p^a}{5})}\right) \pmod{p^3}.$$

In the next section we are going to present two auxiliary identities. Theorem 1.1, Theorem 1.2 and Corollaries 1.2-1.3, and Theorem 1.3 will be proved in Sections 3, 4 and 5 respectively.

2. AN AUXILIARY THEOREM

Theorem 2.1. *For any $n \in \mathbb{Z}^+$ and $d \in \mathbb{Z}$, we have*

$$\begin{aligned} & \sum_{0 \leq k < n} \binom{2k}{k+d} x^{n-1-k} + [d > 0] x^n u_d(x-2) \\ &= \sum_{0 \leq k < n+d} \binom{2n}{k} u_{n+d-k}(x-2) \end{aligned} \quad (2.1)$$

and

$$\begin{aligned}
 & d \sum_{0 < k < n} \frac{\binom{2k}{k+d}}{k} x^{n-k} - [d \geq 0] x^n v_d(x-2) + [d = 0] x^n \\
 &= - \sum_{0 \leq k < n+d} \binom{2n}{k} v_{n+d-k}(x-2) - 2 \binom{2n-1}{n+d-1}.
 \end{aligned} \tag{2.2}$$

Proof. (i) We use induction on $n \in \mathbb{Z}^+$ to prove (2.1).

Since $(x-2)u_d(x-2) = u_{d+1}(x-2) + u_{d-1}(x-2)$ for $d = 1, 2, 3, \dots$, we can easily see that (2.1) with $n = 1$ holds for all $d \in \mathbb{Z}$.

Now fix $n \in \mathbb{Z}^+$ and assume (2.1) for all $d \in \mathbb{Z}$. Let d be any integer. For $k \in \mathbb{N}$, it is easy to see that

$$\binom{2n+2}{k} = \binom{2n}{k} + 2 \binom{2n}{k-1} + \binom{2n}{k-2}.$$

Thus,

$$\begin{aligned}
 & \sum_{0 \leq k < (n+1)+d} \binom{2n+2}{k} u_{n+1+d-k}(x-2) \\
 &= \sum_{0 \leq k < n+(d+1)} \binom{2n}{k} u_{n+(d+1)-k}(x-2) \\
 & \quad + 2 \sum_{0 \leq j < n+d} \binom{2n}{j} u_{n+d-j}(x-2) \\
 & \quad + \sum_{0 \leq i < n+(d-1)} \binom{2n}{i} u_{n+(d-1)-i}(x-2).
 \end{aligned}$$

By the induction hypothesis, for any $r \in \mathbb{Z}$ we have

$$\sum_{0 \leq k < n+r} \binom{2n}{k} u_{n+r-k}(x-2) = \sum_{0 \leq k < n} \binom{2k}{k+r} x^{n-1-k} + [r > 0] x^n u_r(x-2).$$

So, from the above we get

$$\begin{aligned}
& \sum_{0 \leq k < (n+1)+d} \binom{2n+2}{k} u_{n+1+d-k}(x-2) \\
&= \sum_{0 \leq k < n} \left(\binom{2k}{k+d+1} + 2 \binom{2k}{k+d} + \binom{2k}{k+d-1} \right) x^{n-1-k} \\
&\quad + [d \geq 0] x^n u_{d+1}(x-2) + 2[d \geq 0] x^n u_d(x-2) + [d > 0] x^n u_{d-1}(x-2) \\
&= \sum_{0 \leq k < n} \left(\binom{2k+1}{k+d+1} + \binom{2k+1}{k+d} \right) x^{n-1-k} - [d=0] x^n u_{-1}(x-2) \\
&\quad + [d \geq 0] x^n (u_{d+1}(x-2) + 2u_d(x-2) + u_{d-1}(x-2)) \\
&= \sum_{0 \leq k < n} \binom{2(k+1)}{(k+1)+d} x^{n-1-k} + [d=0] x^n + [d \geq 0] x^n x u_d(x-2) \\
&= \sum_{0 \leq k < n+1} \binom{2k}{k+d} x^{(n+1)-1-k} + [d > 0] x^{n+1} u_d(x-2).
\end{aligned}$$

This concludes the induction step and hence (2.1) holds.

(ii) By induction, $v_k(x-2) = 2u_{k+1}(x-2) - (x-2)u_k(x-2)$ for all $k \in \mathbb{Z}$. Thus, with the help of (2.1), we have

$$\begin{aligned}
& \sum_{0 \leq k \leq n+d} \binom{2n}{k} v_{n+d-k}(x-2) \\
&= 2 \sum_{0 \leq k < n+d+1} \binom{2n}{k} u_{n+d+1-k}(x-2) \\
&\quad - (x-2) \sum_{0 \leq k < n+d} \binom{2n}{k} u_{n+d-k}(x-2) \\
&= 2 \sum_{0 \leq k < n} \binom{2k}{k+d+1} x^{n-1-k} + [d+1 > 0] x^n 2u_{d+1}(x-2) \\
&\quad - (x-2) \left(\sum_{0 \leq k < n} \binom{2k}{k+d} x^{n-1-k} + [d > 0] x^n u_d(x-2) \right) \\
&= \sum_{0 \leq k < n} \left(2 \binom{2k}{k+d+1} - (x-2) \binom{2k}{k+d} \right) x^{n-1-k} + [d \geq 0] x^n v_d(x-2).
\end{aligned}$$

For $k \in \mathbb{Z}^+$ we have

$$\begin{aligned}
& \binom{2k-2}{k+d} + \binom{2k-2}{k+d-1} = \binom{2k-1}{k+d} = \binom{2k-1}{k-d-1} \\
&= \frac{k-d}{2k} \binom{2k}{k-d} = \frac{k-d}{2k} \binom{2k}{k+d} = \frac{1}{2} \binom{2k}{k+d} - \frac{d}{2k} \binom{2k}{k+d}.
\end{aligned}$$

Thus

$$\begin{aligned}
 & \frac{1}{2} \sum_{0 < k < n} \binom{2k}{k+d} x^{n-k} - \frac{d}{2} \sum_{0 < k < n} \frac{\binom{2k}{k+d}}{k} x^{n-k} \\
 &= \sum_{0 < k \leq n} \left(\binom{2k-2}{k+d} + \binom{2k-2}{k+d-1} \right) x^{n-k} - \binom{(2n-2)+1}{n+d} \\
 &= \sum_{0 \leq k < n} \left(\binom{2k}{k+d+1} + \binom{2k}{k+d} \right) x^{n-1-k} - \binom{2n-1}{n+d}.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 & d \sum_{0 < k < n} \frac{\binom{2k}{k+d}}{k} x^{n-k} + [d=0]x^n - 2 \binom{2n-1}{n+d} \\
 &= \sum_{0 \leq k < n} \left((x-2) \binom{2k}{k+d} - 2 \binom{2k}{k+d+1} \right) x^{n-1-k}.
 \end{aligned}$$

Combining the above we obtain

$$\begin{aligned}
 & \sum_{0 \leq k \leq n+d} \binom{2n}{k} v_{n+d-k}(x-2) - [d \geq 0]x^n v_d(x-2) \\
 &= -d \sum_{0 < k < n} \frac{\binom{2k}{k+d}}{k} x^{n-k} - [d=0]x^n + 2 \binom{2n-1}{n+d},
 \end{aligned}$$

from which (2.2) follows. \square

Corollary 2.1. *Let $n \in \mathbb{Z}^+$ and $d \in \mathbb{N}$. Then*

$$\sum_{0 \leq k < n} \binom{2k}{k+d} + \binom{d}{3} = \sum_{0 \leq k < n+d} \binom{2n}{k} \binom{n+d-k}{3}, \quad (2.3)$$

$$\sum_{0 \leq k < n} (-1)^{k+d} \binom{2k}{k+d} + F_{2d} = \sum_{0 \leq k < n+d} (-1)^k \binom{2n}{k} F_{2(n+d-k)}, \quad (2.4)$$

and

$$\begin{aligned}
 & d \sum_{0 < k < n} \frac{(-1)^{k+d}}{k} \binom{2k}{k+d} + \sum_{0 \leq k < n+d} \binom{2n}{k} (-1)^k L_{2(n+d-k)} \\
 &= L_{2d} - (-1)^{n+d} 2 \binom{2n-1}{n+d-1} - [d=0].
 \end{aligned} \quad (2.5)$$

Proof. For $j \in \mathbb{N}$ we have $u_j(-1) = \binom{j}{3}$, $(-1)^{j-1}u_j(-3) = u_j(3) = F_{2j}$ and $(-1)^j v_j(-3) = v_j(3) = L_{2j}$. Thus, (2.1) in the case $x = 1$ yields (2.3), and (2.1) and (2.2) in the case $x = -1$ reduce to (2.4) and (2.5) respectively. This concludes the proof. \square

3. PROOF OF THEOREM 1.1

Given $A, B \in \mathbb{Z}$ we define the Lucas sequence $u_n = u_n(A, B)$ ($n \in \mathbb{N}$) and its companion $v_n = v_n(A, B)$ ($n \in \mathbb{N}$) as follows:

$$u_0 = 0, u_1 = 1, \text{ and } u_{n+1} = Au_n - Bu_{n-1} \text{ for } n = 1, 2, 3, \dots,$$

and

$$v_0 = 2, v_1 = A, \text{ and } v_{n+1} = Av_n - Bv_{n-1} \text{ for } n = 1, 2, 3, \dots.$$

It is well known that

$$u_n = \sum_{0 \leq k < n} \alpha^k \beta^{n-1-k} \quad \text{and} \quad v_n = \alpha^n + \beta^n \quad \text{for all } n \in \mathbb{N},$$

where α and β are the two roots of the equation $x^2 - Ax + B = 0$. It follows that if $n \in \mathbb{N}$ and $m \in \{n, n+1, \dots\}$ then

$$Au_n + v_n = 2u_{n+1} \quad \text{and} \quad u_m v_n - u_n v_m = 2B^n u_{m-n}.$$

Lemma 3.1. *Let $A, B \in \mathbb{Z}$ with $B \neq 0$. Let $u_n = u_n(A, B)$ for $n \in \mathbb{N}$, and define $u_{-1} = (u_1 - Au_0)/(-B) = -1/B$. Let p be an odd prime, and let $a \in \mathbb{Z}^+$ and $d \in \{0, 1, \dots, p^a\}$. Then we have*

$$B^d u_{p^a-d} \equiv -c(A, B) u_{d - (\frac{\Delta}{p^a})} \pmod{p}, \quad (3.1)$$

where $\Delta = A^2 - 4B$ and

$$c(A, B) = \begin{cases} A/2 & \text{if } p \mid \Delta, \\ B & \text{if } (\frac{\Delta}{p^a}) = 1, \\ 1 & \text{if } (\frac{\Delta}{p^a}) = -1. \end{cases}$$

Proof. The two roots of the equation $x^2 - Ax + B = 0$ are algebraic integers $\alpha = (A + \sqrt{\Delta})/2$ and $\beta = (A - \sqrt{\Delta})/2$. Since

$$\binom{p^a}{k} = \frac{p^a}{k} \binom{p^a-1}{k-1} \equiv 0 \pmod{p} \text{ for } k = 1, \dots, p^a-1,$$

we have

$$v_{p^a} = \alpha^{p^a} + \beta^{p^a} \equiv (\alpha + \beta)^{p^a} = A^{p^a} \equiv A^{p^{a-1}} \equiv \dots \equiv A \pmod{p}$$

with the help of Fermat's little theorem. If $\Delta \neq 0$, then

$$\begin{aligned} u_{p^a} &= \frac{\alpha^{p^a} - \beta^{p^a}}{\alpha - \beta} = \frac{1}{\sqrt{\Delta}} \left(\left(\frac{A + \sqrt{\Delta}}{2} \right)^{p^a} - \left(\frac{A - \sqrt{\Delta}}{2} \right)^{p^a} \right) \\ &= \frac{1}{2^{p^a} \sqrt{\Delta}} \sum_{\substack{k=0 \\ 2 \nmid k}}^{p^a} \binom{p^a}{k} A^{p^a-k} \left((\sqrt{\Delta})^k - (-\sqrt{\Delta})^k \right) \\ &= \frac{1}{2^{p^a-1}} \sum_{\substack{k=1 \\ 2 \nmid k}}^{p^a} \binom{p^a}{k} A^{p^a-k} \Delta^{(k-1)/2}, \end{aligned}$$

if $\Delta = 0$ then $\alpha = \beta = A/2$ and hence $u_{p^a} = p^a(A/2)^{p^a-1}$. So we always have

$$u_{p^a} = \frac{1}{2^{p^a-1}} \sum_{\substack{k=1 \\ 2 \nmid k}}^{p^a} \binom{p^a}{k} A^{p^a-k} \Delta^{(k-1)/2}.$$

Note that $2^{p^a-1} \equiv 1 \pmod{p}$ by Fermat's little theorem. Thus, by Euler's criterion,

$$u_{p^a} \equiv \binom{p^a}{p^a} \Delta^{(p^a-1)/2} = (\Delta^{(p-1)/2})^{\sum_{k=0}^{a-1} p^k} \equiv \left(\frac{\Delta}{p} \right)^a = \left(\frac{\Delta}{p^a} \right) \pmod{p}.$$

Observe that

$$2B^d u_{p^a-d} = u_{p^a} v_d - u_d v_{p^a} \equiv \left(\frac{\Delta}{p^a} \right) v_d - u_d A \pmod{p}.$$

When $p \mid \Delta$, this yields

$$B^d u_{p^a-d} \equiv -\frac{A}{2} u_d \pmod{p}.$$

If $\left(\frac{\Delta}{p^a} \right) = 1$, then

$$2B^d u_{p^a-d} \equiv v_d - Au_d = 2(u_{d+1} - Au_d) = -2Bu_{d-1} \pmod{p}$$

and hence $B^d u_{p^a-d} \equiv -Bu_{d-1} \pmod{p}$. If $\left(\frac{\Delta}{p^a} \right) = -1$, then

$$2B^d u_{p^a-d} \equiv -v_d - Au_d = -2u_{d+1} \pmod{p}$$

and thus $B^d u_{p^a-d} \equiv -u_{d+1} \pmod{p}$. So (3.1) follows. \square

Proof of Theorem 1.1. For $n = -1, 0, 1, \dots$ let $u_n = u_n(m-2)$ and $v_n = v_n(m-2)$.

By Theorem 2.1,

$$\sum_{k=0}^{p^a-1} \binom{2k}{k-d} m^{p^a-1-k} = \sum_{0 \leq k < p^a-d} \binom{2p^a}{k} u_{p^a-d-k};$$

also, for $d > 0$ we have

$$-d \sum_{0 < k < p^a} \frac{\binom{2k}{k-d}}{k} m^{p^a-k} = - \sum_{0 \leq k < p^a-d} \binom{2p^a}{k} v_{p^a-d-k} - 2 \binom{2p^a-1}{p^a-d-1}.$$

By Fermat's little theorem, $m^{p^a} \equiv m \pmod{p}$. For $k \in \{1, \dots, p^a-1\}$ clearly

$$\binom{2p^a}{k} = \frac{2p^a}{k} \binom{2p^a-1}{k-1} \equiv 0 \pmod{p};$$

also, if $d < p^a$ then

$$\binom{2p^a-1}{p^a-d-1} = \prod_{0 < j < p^a-d} \left(\frac{2p^a}{j} - 1 \right) \equiv (-1)^{p^a-d-1} \equiv (-1)^d \pmod{p}.$$

Therefore

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k+d}}{m^k} \equiv [d \neq p^a] \binom{2p^a}{0} u_{p^a-d} = u_{p^a-d} \pmod{p};$$

if $d > 0$ then

$$\begin{aligned} d \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k+d}}{km^{k-1}} &\equiv [d \neq p^a] \binom{2p^a}{0} v_{p^a-d} + 2[d \neq p^a] (-1)^d \\ &\equiv v_{p^a-d} + 2(-1)^d \pmod{p}. \end{aligned}$$

So we have (1.3) and (1.4).

Now assume $p \neq 2$ and set $\Delta = (m-2)^2 - 4 \times 1 = m(m-4)$. As $p \nmid m$, if $p \mid \Delta$ then $m \equiv 4 \pmod{p}$ and hence $(m-2)/2 \equiv 1 \pmod{p}$. Thus, with the help of Lemma 3.1, we have

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k+d}}{m^k} \equiv u_{p^a-d} \equiv -u_{d-(\frac{\Delta}{p^a})} \pmod{p},$$

which proves (1.5). If $d > 0$, then

$$\begin{aligned} v_{d-(\frac{\Delta}{p^a})} &= 2u_{d-(\frac{\Delta}{p^a})+1} - (m-2)u_{d-(\frac{\Delta}{p^a})} \\ &= -2u_{d-1-(\frac{\Delta}{p^a})} + (m-2)u_{d-(\frac{\Delta}{p^a})} \\ &\equiv 2u_{p^a-d+1} - (m-2)u_{p^a-d} = v_{p^a-d} \pmod{p}. \end{aligned}$$

Thus (1.6) follows from (1.4). We are done. \square

4. PROOFS OF THEOREM 1.2 AND COROLLARIES 1.2-1.3

Lemma 4.1. *For any positive integer n , we have*

$$\frac{1}{2} \sum_{0 < k < n} \frac{\binom{2k}{k}}{kx^k} = \sum_{0 < d < n} (-1)^{d-1} \sum_{0 < k < n} \frac{\binom{2k}{k+d}}{kx^k}. \quad (4.1)$$

Proof. Observe that

$$\begin{aligned} & \sum_{d=0}^{n-1} (-1)^d \sum_{0 < k < n} \frac{\binom{2k}{k+d}}{kx^k} \\ &= \sum_{0 < k < n} \frac{1}{k(-x)^k} \sum_{d=0}^{n-1} (-1)^{k+d} \binom{2k}{k+d} \\ &= \sum_{0 < k < n} \frac{1}{2k(-x)^k} \sum_{j=k}^{2k} \left((-1)^j \binom{2k}{j} + (-1)^{2k-j} \binom{2k}{2k-j} \right) \\ &= \sum_{0 < k < n} \frac{1}{2k(-x)^k} \left((1-1)^{2k} + (-1)^k \binom{2k}{k} \right) = \frac{1}{2} \sum_{0 < k < n} \frac{\binom{2k}{k}}{kx^k}. \end{aligned}$$

So (4.1) follows. \square

Proof of Theorem 1.2. By Lemma 4.1,

$$\frac{1}{2} \sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{km^{k-1}} = \sum_{d=1}^{p-1} (-1)^d \sum_{k=1}^{p-1} \frac{\binom{2k}{k+d}}{k(-m)^{k-1}}.$$

In view of (1.4) and the basic fact

$$\frac{1}{p} \binom{p}{d} = \frac{1}{d} \prod_{0 < k < d} \frac{p-k}{k} \equiv \frac{(-1)^{d-1}}{d} \pmod{p} \quad (d = 1, \dots, p-1),$$

we have

$$\begin{aligned} & \sum_{d=1}^{p-1} (-1)^d \sum_{k=1}^{p-1} \frac{\binom{2k}{k+d}}{k(-m)^{k-1}} \\ &\equiv \sum_{d=1}^{p-1} \frac{(-1)^d}{d} (v_{p-d}(-m-2) + 2(-1)^d) \\ &\equiv \sum_{d=1}^{p-1} \frac{(-1)^d}{d} v_{p-d}(-m-2) + \sum_{d=1}^{p-1} \left(\frac{1}{d} + \frac{1}{p-d} \right) \\ &\equiv -\frac{1}{p} \sum_{d=1}^{p-1} \binom{p}{d} v_{p-d}(-m-2) = -\frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} v_k(-m-2) \pmod{p}. \end{aligned}$$

Let α and β be the two roots of the equation $x^2 - mx - m = 0$. Then $(-\alpha - 1) + (-\beta - 1) = -m - 2$ and $(-\alpha - 1)(-\beta - 1) = 1$, also

$$V_p(m) = \alpha^p + \beta^p \equiv (\alpha + \beta)^p = m^p \equiv m \pmod{p}.$$

In the case $p \neq 2$, we have

$$\begin{aligned} \sum_{k=1}^{p-1} \binom{p}{k} v_k(-m-2) &= \sum_{k=1}^{p-1} \binom{p}{k} ((-\alpha-1)^k + (-\beta-1)^k) \\ &= (-\alpha)^p + (-\beta)^p - 2 - (-\alpha-1)^p - (-\beta-1)^p \\ &= (-1)^p V_p(m) - 2 - (-1)^p \frac{\alpha^{2p} + \beta^{2p}}{m^p} \\ &= -V_p(m) + \frac{(\alpha^p + \beta^p)^2}{m^p} = \left(1 + \frac{V_p(m) - m^p}{m^p}\right) (V_p(m) - m^p) \\ &\equiv V_p(m) - m^p \pmod{p^2} \quad (\text{since } V_p(m) \equiv m^p \pmod{p}). \end{aligned}$$

Note also that

$$\sum_{k=1}^{2-1} \binom{2}{k} v_k(-m-2) = 2(-m-2) \equiv 2m = V_2(m) - m^2 \pmod{2^2}.$$

Therefore (1.11) follows from the above. \square

Proof of Corollary 1.2. By induction, whenever $n \in \mathbb{N}$ we have

$$\begin{aligned} V_{4n}(-2) &= (-1)^n 2^{2n+1}, \quad V_{4n+1}(-2) = (-1)^{n+1} 2^{2n+1}, \\ V_{4n+2}(-2) &= 0, \quad V_{4n+3}(-2) = (-1)^n 2^{2n+2}. \end{aligned}$$

It follows that

$$V_p(-2) = -\binom{2}{p} 2^{(p+1)/2}.$$

Combining this with (1.11) in the case $m = -2$, we get

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k2^k} &\equiv \frac{V_p(-2) - (-2)^p}{p} = 2^{(p+1)/2} \frac{2^{(p-1)/2} - \binom{2}{p}}{p} \\ &\equiv \left(2^{(p-1)/2} + \binom{2}{p}\right) \frac{2^{(p-1)/2} - \binom{2}{p}}{p} = q_p(2) \pmod{p}. \end{aligned}$$

By induction, $V_n(-4) = (-1)^n 2^{n+1}$ for all $n \in \mathbb{N}$. Thus, by (1.11) with $m = -4$, we have

$$\frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k4^{k-1}} \equiv \frac{V_p(-4) - (-4)^p}{p} = 2^p \frac{2^p - 2}{p} \equiv 4q_p(2) \pmod{p}.$$

Therefore (1.12) holds.

Now assume that $p \neq 3$. By induction, for $n \in \mathbb{N}$ we have

$$V_n(-3) = \begin{cases} (3[3 | n] - 1)(-3)^{n/2} & \text{if } 2 \mid n, \\ \binom{n}{3}(-3)^{(n+1)/2} & \text{if } 2 \nmid n. \end{cases}$$

Applying (1.11) with $m = -3$ we get

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k3^{k-1}} &\equiv \frac{V_p(-3) - (-3)^p}{p} = -(-3)^{(p+1)/2} \frac{(-3)^{(p-1)/2} - \left(\frac{-3}{p}\right)}{p} \\ &\equiv \frac{3}{2} \left((-3)^{(p-1)/2} + \left(\frac{-3}{p}\right) \right) \frac{(-3)^{(p-1)/2} - \left(\frac{-3}{p}\right)}{p} \\ &\equiv \frac{3}{2} \cdot \frac{(-3)^{p-1} - 1}{p} = \frac{3}{2} q_p(3) \pmod{p}. \end{aligned}$$

So (1.13) is valid. \square

Proof of Corollary 1.3. (i) Applying Theorem 1.2 with $m = 1$, we obtain that

$$\frac{1}{2} \sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k} \equiv \frac{1 - L_p}{p} \pmod{p}.$$

Let α and β be the two roots of the equation $x^2 - x - 1 = 0$. Suppose $p \neq 5$ and set $n = (p - \left(\frac{p}{5}\right))/2$. It is known that

$$L_n^2 - 5F_n^2 = (\alpha^n + \beta^n)^2 - (\alpha - \beta)^2 \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^2 = 4(\alpha\beta)^n = 4(-1)^n$$

and

$$L_{2n} = \alpha^{2n} + \beta^{2n} = (\alpha^n + \beta^n)^2 - 2(\alpha\beta)^n = L_n^2 - 2(-1)^n.$$

By [13, Corollary 1], $p \mid F_n$ if $p \equiv 1 \pmod{4}$, and $p \mid L_n$ if $p \equiv 3 \pmod{4}$. Thus

$$L_{p - \left(\frac{p}{5}\right)} = L_{2n} = 5F_n^2 + 2(-1)^n = L_n^2 - 2(-1)^n \equiv 2 \left(\frac{p}{5}\right) \pmod{p^2}.$$

By induction,

$$2L_k = 5F_{k-1} + L_{k-1} = 5F_{k+1} - L_{k+1} \text{ for } k = 1, 2, 3, \dots$$

Therefore

$$2L_p = 5F_{p - \left(\frac{p}{5}\right)} + \left(\frac{p}{5}\right) L_{p - \left(\frac{p}{5}\right)} \equiv 5F_{p - \left(\frac{p}{5}\right)} + 2 \pmod{p^2}$$

and hence

$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k} \equiv -2 \frac{L_p - 1}{p} \equiv -5 \frac{F_{p-(\frac{p}{5})}}{p} \pmod{p}.$$

This proves (1.14).

By (1.11) in the case $m = 5$,

$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k5^k} \equiv \frac{2}{5} \cdot \frac{5^p - V_p(5)}{p} \pmod{p}.$$

Since $(5 + 3\sqrt{5})/2$ and $(5 - 3\sqrt{5})/2$ are the two roots of the equation $x^2 - 5x - 5 = 0$,

$$\begin{aligned} V_p(5) &= \left(\frac{5 + 3\sqrt{5}}{2} \right)^p + \left(\frac{5 - 3\sqrt{5}}{2} \right)^p \\ &= \sqrt{5}^p \left(\left(\frac{1 + \sqrt{5}}{2} \right)^{2p} - \left(\frac{1 - \sqrt{5}}{2} \right)^{2p} \right) \\ &= 5^{(p+1)/2} \frac{\alpha^p - \beta^p}{\alpha - \beta} (\alpha^p + \beta^p) = 5^{(p+1)/2} F_p L_p. \end{aligned}$$

As

$$L_p \equiv 1 + \frac{5}{2} F_{p-(\frac{p}{5})} \pmod{p^2}$$

and

$$L_p = F_p + 2F_{p-1} = 2F_{p+1} - F_p = 2F_{p-(\frac{p}{5})} + \binom{p}{5} F_p,$$

we have

$$\begin{aligned} \binom{p}{5} F_p L_p &= L_p (L_p - 2F_{p-(\frac{p}{5})}) \\ &\equiv \left(1 + \frac{5}{2} F_{p-(\frac{p}{5})} \right) \left(1 + \frac{1}{2} F_{p-(\frac{p}{5})} \right) \equiv 1 + 3F_{p-(\frac{p}{5})} \pmod{p^2} \end{aligned}$$

and hence

$$\begin{aligned} V_p(5) &= 5^{(p+1)/2} F_p L_p \\ &\equiv 5^{(p+1)/2} \binom{5}{p} (1 + 3F_{p-(\frac{p}{5})}) \equiv 5^{(p+1)/2} \binom{5}{p} + 15F_{p-(\frac{p}{5})} \pmod{p^2}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k5^k} &\equiv \frac{2}{5} \cdot \frac{5^p - 5^{(p+1)/2} \binom{5}{p} - 15F_{p-(\frac{p}{5})}}{p} \\ &\equiv \left(5^{(p-1)/2} + \binom{5}{p} \right) \frac{5^{(p-1)/2} - \binom{5}{p}}{p} - 6 \frac{F_{p-(\frac{p}{5})}}{p} \\ &\equiv q_p(5) - 6 \frac{F_{p-(\frac{p}{5})}}{p} \pmod{p}. \end{aligned}$$

So (1.15) also holds.

Applying (1.11) with $m = -5$ we get

$$\frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k5^{k-1}} \equiv \frac{V_p(-5) + 5^p}{p} \pmod{p}.$$

As the two roots of the equation $x^2 + 5x + 5 = 0$ are $(-5 \pm \sqrt{5})/2$, we have

$$\begin{aligned} V_p(-5) &= \left(\frac{-5 + \sqrt{5}}{2} \right)^p + \left(\frac{-5 - \sqrt{5}}{2} \right)^p \\ &= \sqrt{5}^p \left(\left(\frac{1 - \sqrt{5}}{2} \right)^p - \left(\frac{1 + \sqrt{5}}{2} \right)^p \right) = -\sqrt{5}^{p+1} F_p. \end{aligned}$$

Recall that

$$\binom{5}{p} F_p = L_p - 2F_{p-(\frac{p}{5})} \equiv 1 + \frac{1}{2} F_{p-(\frac{p}{5})} \pmod{p^2}.$$

Thus

$$\begin{aligned} 5^{(p-1)/2} F_p - 1 &\equiv 5^{(p-1)/2} \binom{5}{p} \left(1 + \frac{1}{2} F_{p-(\frac{p}{5})} \right) - 1 \\ &\equiv \binom{5}{p} \left(5^{(p-1)/2} - \binom{5}{p} \right) + \frac{1}{2} F_{p-(\frac{p}{5})} \\ &\equiv \frac{1}{2} \left(5^{(p-1)/2} + \binom{5}{p} \right) \left(5^{(p-1)/2} - \binom{5}{p} \right) + \frac{1}{2} F_{p-(\frac{p}{5})} \\ &\equiv \frac{5^{p-1} - 1}{2} + \frac{1}{2} F_{p-(\frac{p}{5})} \pmod{p^2} \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k5^{k-1}} &\equiv \frac{5^p - 5^{(p+1)/2} F_p}{p} = \frac{5^p - 5}{p} - 5 \frac{5^{(p-1)/2} F_p - 1}{p} \\ &\equiv 5 \left(q_p(5) - \frac{q_p(5)}{2} - \frac{F_{p-(\frac{p}{5})}}{2p} \right) \pmod{p}. \end{aligned}$$

This proves (1.16).

(ii) As $2+2\sqrt{2}$ and $2-2\sqrt{2}$ are the two roots of the equation $x^2-4x-4=0$, we have

$$V_p(4) = (2+2\sqrt{2})^p + (2-2\sqrt{2})^p = 2^p \left((1+\sqrt{2})^p + (1-\sqrt{2})^p \right) = 2^p Q_p,$$

where the sequence $\{Q_n\}_{n \in \mathbb{N}}$ is given by

$$Q_0 = Q_1 = 2 \text{ and } Q_{n+1} = 2Q_n + Q_{n-1} \text{ (} n = 1, 2, 3, \dots \text{)}.$$

By [15, Remark 3.1],

$$4 \binom{\frac{2}{p}}{p} P_p - Q_p = \binom{\frac{2}{p}}{p} Q_{p-\left(\frac{2}{p}\right)} \equiv 2 \pmod{p^2}$$

and

$$P_{p-\left(\frac{2}{p}\right)} \equiv \binom{\frac{2}{p}}{p} P_p - 1 \pmod{p^2}.$$

Thus

$$Q_p - 2 \equiv 4 \left(\binom{\frac{2}{p}}{p} P_p - 1 \right) \equiv 4P_{p-\left(\frac{2}{p}\right)} \pmod{p^2}$$

and hence

$$\begin{aligned} \sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k4^k} &\equiv \frac{4^p - V_p(4)}{2p} = 2^{p-1} \frac{2^p - Q_p}{p} \\ &\equiv 2q_p(2) - \frac{Q_p - 2}{p} \equiv 2q_p(2) - 4 \frac{P_{p-\left(\frac{2}{p}\right)}}{p} \pmod{p} \end{aligned}$$

with the help of (1.11) in the case $m = 4$.

By [14],

$$-2^{(p+1)/2} \frac{P_p - 2^{(p-1)/2}}{p} \equiv \sum_{k=1}^{(p-1)/2} \frac{1}{k2^k} \equiv \sum_{k=1}^{\lfloor 3p/4 \rfloor} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

(The last congruence was first conjectured by Z. H. Sun in 1988.) Observe that

$$\begin{aligned} -2^{(p+1)/2} \frac{P_p - 2^{(p-1)/2}}{p} &\equiv -2^{(p+1)/2} \frac{\binom{\frac{2}{p}}{p} (1 + P_{p-\left(\frac{2}{p}\right)}) - 2^{(p-1)/2}}{p} \\ &\equiv -2 \frac{P_{p-\left(\frac{2}{p}\right)}}{p} + 2^{(p+1)/2} \frac{2^{(p-1)/2} - \binom{\frac{2}{p}}{p}}{p} \\ &\equiv -2 \frac{P_{p-\left(\frac{2}{p}\right)}}{p} + q_p(2) \pmod{p}. \end{aligned}$$

So we also have

$$2q_p(2) - 4 \frac{P_{p-\binom{2}{p}}}{p} \equiv 2 \sum_{k=1}^{\lfloor 3p/4 \rfloor} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

(iii) Now suppose $p > 3$. By Theorem 1.2 in the case $m = 2$, we have

$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k2^k} \equiv \frac{2^p - V_p(2)}{p} \pmod{p}.$$

Observe that the two roots of the equation $x^2 - 2x - 2 = 0$ are $1 \pm \sqrt{3}$. Thus

$$\begin{aligned} V_p(2) &= (1 + \sqrt{3})^p + (1 - \sqrt{3})^p = 2 \sum_{k=0}^{(p-1)/2} \binom{p}{2k} (\sqrt{3})^{2k} \\ &= 2 + \sum_{k=1}^{(p-1)/2} \frac{2p}{2k} \binom{p-1}{2k-1} 3^k \\ &\equiv 2 - p \sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \pmod{p^2}. \end{aligned}$$

As observed by Eisenstein [2],

$$2q_p(2) = \frac{2^p - 2}{p} = \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} = \sum_{k=1}^{p-1} \frac{\binom{p-1}{k-1}}{k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

By a congruence of Z. W. Sun [15],

$$\sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv \sum_{0 < k < p/6} \frac{(-1)^k}{k} \equiv \sum_{0 < k < p/6} \frac{(-1)^{p-k}}{p-k} \pmod{p}.$$

Thus

$$\begin{aligned} &\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k2^k} \\ &\equiv \frac{2^p - 2}{p} - \frac{V_p(2) - 2}{p} \equiv 2q_p(2) + \sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \\ &\equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} + \sum_{5p/6 < k < p} \frac{(-1)^k}{k} = \sum_{0 < k < 5p/6} \frac{(-1)^{k-1}}{k} \pmod{p}. \end{aligned}$$

In light of [15],

$$\sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv -q_p(2) - 6 \binom{2}{p} \frac{S_{(p-(\frac{3}{p}))}/2}}{p} \pmod{p}.$$

So we also have

$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k2^k} \equiv q_p(2) - 6 \binom{2}{p} \frac{S_{(p-(\frac{3}{p}))}/2}}{p} \pmod{p}.$$

Therefore (1.18) follows.

Let $u_n = u_n(2, -2)$ and $v_n = v_n(2, -2)$ for $n \in \mathbb{N}$. By induction,

$$v_n = 2u_{n+1} - 2u_n = 2u_n + 4u_{n-1} \quad \text{for } n = 1, 2, 3, \dots$$

Thus

$$v_p = 2 \binom{3}{p} u_p + \left(3 + \binom{3}{p} \right) u_{p-(\frac{3}{p})}.$$

Clearly

$$\begin{aligned} 2\sqrt{3}u_{p-(\frac{3}{p})} &= (1 + \sqrt{3})^{p-(\frac{3}{p})} - (1 - \sqrt{3})^{p-(\frac{3}{p})} \\ &= 2^{(p-(\frac{3}{p}))/2} \left((2 + \sqrt{3})^{(p-(\frac{3}{p}))/2} - (2 - \sqrt{3})^{(p-(\frac{3}{p}))/2} \right) \end{aligned}$$

and hence

$$u_{p-(\frac{3}{p})} = 2^{(p-(\frac{3}{p}))/2} S_{(p-(\frac{3}{p}))/2} \equiv \binom{2}{p} 2^{(1-(\frac{3}{p}))/2} S_{(p-(\frac{3}{p}))/2} \pmod{p^2}.$$

Recall that

$$v_p = V_p(2) \equiv 2 - p \sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv 2 + (2^{p-1} - 1) + 6 \binom{2}{p} S_{(p-(\frac{3}{p}))/2} \pmod{p^2}.$$

Therefore

$$\begin{aligned} 2 \binom{3}{p} u_p - 2 &= v_p - 2 - \left(3 + \binom{3}{p} \right) u_{p-(\frac{3}{p})} \\ &\equiv 2^{p-1} - 1 + 6 \binom{2}{p} S_{(p-(\frac{3}{p}))/2} - \left(3 + \binom{3}{p} \right) \binom{2}{p} 2^{(1-(\frac{3}{p}))/2} S_{(p-(\frac{3}{p}))/2} \\ &\equiv 2^{p-1} - 1 + 2 \binom{2}{p} S_{(p-(\frac{3}{p}))/2} \pmod{p^2}. \end{aligned}$$

Applying (1.11) with $m = -6$, we get

$$\frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k6^{k-1}} \equiv \frac{V_p(-6) + 6^p}{p} \equiv \frac{V_p(-6) + 6}{p} + 6(q_p(2) + q_p(3)) \pmod{p}.$$

Observe that

$$\begin{aligned} V_p(-6) &= (-3 + \sqrt{3})^p + (-3 - \sqrt{3})^p \\ &= -\sqrt{3}^p \left((1 + \sqrt{3})^p - (1 - \sqrt{3})^p \right) = -2 \times 3^{(p+1)/2} u_p \end{aligned}$$

and hence

$$\begin{aligned} V_p(-6) + 6 &\equiv -6 \left(3^{(p-1)/2} - \left(\frac{3}{p} \right) \right) u_p - 6 \left(\frac{3}{p} \right) u_p + 6 \\ &\equiv -6 \left(3^{(p-1)/2} - \left(\frac{3}{p} \right) \right) \left(\frac{3}{p} \right) \\ &\quad - 3 \left(2^{p-1} - 1 + 2 \left(\frac{2}{p} \right) S_{(p-(\frac{3}{p}))/2} \right) \\ &\equiv -3 \left((3^{p-1} - 1) + 2^{p-1} - 1 + 2 \left(\frac{2}{p} \right) S_{(p-(\frac{3}{p}))/2} \right) \pmod{p^2}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k6^{k-1}} &\equiv -3 \left(q_p(3) + q_p(2) + 2 \left(\frac{2}{p} \right) \frac{S_{(p-(\frac{3}{p}))/2}}{p} \right) \\ &\quad + 6(q_p(2) + q_p(3)) \\ &\equiv 3 \left(q_p(2) + q_p(3) - 2 \left(\frac{2}{p} \right) \frac{S_{(p-(\frac{3}{p}))/2}}{p} \right) \pmod{p}. \end{aligned}$$

So (1.19) is valid.

The proof of Corollary 1.3 is now complete. \square

5. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. By an identity of T. B. Staver [12],

$$\sum_{k=1}^n \frac{1}{k} \binom{2k}{k} = \frac{2n+1}{3n^2} \binom{2n}{n} \sum_{k=1}^n \frac{1}{\binom{n-1}{k-1}^2} = \frac{n+1}{3} \binom{2n+1}{n} \sum_{k=1}^n \frac{1}{k^2 \binom{n}{k}^2}$$

for all $n = 1, 2, 3, \dots$. Taking $n = p^a - 1$ in the identity, we get

$$\sum_{k=1}^{p^a-1} \frac{1}{k} \binom{2k}{k} = \frac{p^a}{3} \binom{2p^a-1}{p^a-1} \sum_{k=1}^{p^a-1} \frac{1}{k^2 \binom{p^a-1}{k}^2}. \quad (5.1)$$

Recall that

$$\binom{2p^a - 1}{p^a - 1} \equiv 1 + p[p = 2] + p^2[p = 3] \pmod{p^3}$$

by [16, Lemma 2.2]. For $k = 1, \dots, p^a - 1$, we set $H_k = \sum_{0 < j \leq k} 1/j$ and note that

$$\begin{aligned} \frac{1}{\binom{p^a - 1}{k}^2} &= \prod_{0 < j \leq k} \frac{1}{(1 - p^a/j)^2} \\ &\equiv \prod_{0 < j \leq k} \frac{(1 - p^{3a}/j^3)^2}{(1 - p^a/j)^2} = \prod_{0 < j \leq k} \left(1 + \frac{p^a}{j} + \frac{p^{2a}}{j^2}\right)^2 \\ &\equiv \prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j} + \frac{p^{2a}}{j^2} + 2\frac{p^{2a}}{j^2}\right) \pmod{p^3} \\ &\equiv \prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j}\right) \pmod{p^{2+[p=3]}} \end{aligned}$$

Therefore (5.1) implies that

$$\begin{aligned} p^{a-1} \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k}}{k} &= \frac{p}{3} \binom{2p^a - 1}{p^a - 1} \sum_{k=1}^{p^a-1} \frac{p^{2(a-1)}}{k^2 \binom{p^a - 1}{k}^2} \\ &\equiv \frac{p}{3} (1 + p[p = 2] + p^2[p = 3]) \sum_{k=1}^{p^a-1} \frac{p^{2(a-1)}}{k^2} \prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j}\right) \pmod{p^3}. \end{aligned}$$

So we have

$$p^{a-1} \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k}}{k} \equiv \left(\frac{p}{3} + p^2[p \leq 3]\right) \sum_{k=1}^{p^a-1} \frac{p^{2(a-1)}}{k^2} \prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j}\right) \pmod{p^3}. \quad (5.2)$$

For $k = 1, \dots, p^a - 1$, clearly

$$\begin{aligned} \prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j}\right) &\equiv 1 + 2p^a H_k + 4p^{2a} \sum_{0 < i < j \leq k} \frac{1}{ij} \\ &\equiv 1 + 2p^a H_k + 2p^{2a} \left(H_k^2 - \sum_{j=1}^k \frac{1}{j^2}\right) \pmod{p^3}. \end{aligned}$$

In the case $a \geq 2$, if $1 \leq k \leq p^a - 1$ and $p^{a-2} \nmid k$ then $p^{2(a-1)}/k^2 \equiv$

0 (mod p^4). When $a \geq 2$ and $k \in \{1, \dots, p^2 - 1\}$, we have

$$\begin{aligned} \prod_{j=1}^{p^{a-2}k} \left(1 + 2\frac{p^a}{j}\right) &\equiv 1 + 2 \sum_{j=1}^{p^{a-2}k} \frac{p^a}{j} + 2 \left(\sum_{j=1}^{p^{a-2}k} \frac{p^a}{j} \right)^2 - 2 \sum_{j=1}^{p^{a-2}k} \frac{p^{2a}}{j^2} \\ &\equiv 1 + 2 \sum_{i=1}^k \frac{p^a}{p^{a-2}i} + 2 \left(\sum_{i=1}^k \frac{p^a}{p^{a-2}i} \right)^2 - 2 \sum_{i=1}^k \frac{p^{2a}}{(p^{a-2}i)^2} \\ &\equiv 1 + 2p^2 H_k + 2(p^2 H_k)^2 - 2 \sum_{i=1}^k \frac{p^4}{i^2} \pmod{p^3}. \end{aligned}$$

Therefore, if $a \geq 2$ then (5.2) implies that

$$\begin{aligned} p^{a-1} \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k}}{k} &\equiv \left(\frac{p}{3} + p^2[p \leq 3]\right) \sum_{k=1}^{p^2-1} \frac{p^{2(a-1)}}{(p^{a-2}k)^2} \prod_{j=1}^{p^{a-2}k} \left(1 + 2\frac{p^a}{j}\right) \\ &\equiv \left(\frac{p}{3} + p^2[p \leq 3]\right) \sum_{k=1}^{p^2-1} \frac{p^2}{k^2} \prod_{j=1}^k \left(1 + 2\frac{p^2}{j}\right) \pmod{p^3}. \end{aligned}$$

In the case $p = 3$, this yields (1.20) for $a \geq 2$. (1.20) in the case $p = 3$ and $a = 1$ can be verified directly.

Below we assume that $p \neq 3$. For $k = 1, \dots, p^a - 1$, if $p^{a-1} \nmid k$ then $p^{2(a-1)}/k^2 \equiv 0 \pmod{p^2}$. Also,

$$p^a H_{p^{a-1}k} = \sum_{j=1}^{p^{a-1}k} \frac{p^a}{j} \equiv \sum_{i=1}^k \frac{p^a}{p^{a-1}i} = p H_k \pmod{p^2}$$

for every $k = 1, \dots, p - 1$. Thus (5.2) implies that

$$\begin{aligned} p^{a-1} \sum_{k=1}^{p^a-1} \frac{\binom{2k}{k}}{k} &\equiv \left(\frac{p}{3} + p^2[p = 2]\right) \sum_{k=1}^{p-1} \frac{p^{2(a-1)}}{(p^{a-1}k)^2} (1 + 2p^a H_{p^{a-1}k}) \\ &\equiv \left(\frac{p}{3} + p^2[p = 2]\right) \sum_{k=1}^{p-1} \frac{1 + 2p H_k}{k^2} \pmod{p^3}. \end{aligned}$$

This yields (1.20) in the case $p = 2$.

Now we handle the remaining case $p > 3$. By the above, it suffices to show that

$$\sum_{k=1}^{p-1} \frac{1 + 2p H_k}{k^2} \equiv \frac{8}{3} p B_{p-3} \pmod{p^2}. \quad (5.3)$$

Let $n \in \mathbb{N}$. It is well known that

$$\sum_{j=0}^{k-1} j^n = \frac{1}{n+1} \sum_{i=0}^n \binom{n+1}{i} B_i k^{n+1-i} \quad \text{for } k \in \mathbb{Z}^+,$$

and that

$$\sum_{k=1}^{p-1} k^n \equiv pB_n \equiv 0 \pmod{p} \quad \text{if } n \not\equiv 0 \pmod{p-1}.$$

(See, e.g., [6, p. 235].) Therefore

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{j=0}^k j^{p-2} &= \sum_{k=1}^{p-1} \left(k^{p-4} + \frac{1}{k^2(p-1)} \sum_{i=0}^{p-2} \binom{p-1}{i} B_i k^{p-1-i} \right) \\ &= \sum_{k=1}^{p-1} k^{p-4} + \frac{1}{p-1} \sum_{i=0}^{p-2} \binom{p-1}{i} B_i \sum_{k=1}^{p-1} k^{p-3-i} \\ &\equiv \binom{p-1}{p-3} B_{p-3} + \frac{B_{p-2}}{2} \sum_{k=1}^{p-1} \left(\frac{1}{k} + \frac{1}{p-k} \right) \pmod{p} \end{aligned}$$

and hence

$$\sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv B_{p-3} \pmod{p}. \quad (5.4)$$

By a result of J. W. L. Glaisher [3, 4],

$$\binom{2p-1}{p-1} \equiv 1 - p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 1 - \frac{2}{3} p^3 B_{p-3} \pmod{p^4}$$

and thus

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \frac{2}{3} p B_{p-3} \pmod{p^2}. \quad (5.5)$$

Note that (5.3) follows from (5.4) and (5.5). We are done. \square

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