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SYMMETRIC IDENTITIES FOR EULER POLYNOMIALS

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Abstract. In this paper we establish two symmetric identities on sums of products of Euler polynomials.

1. Introduction

The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are rational numbers given by

$$B_0 = 1, \text{ and } \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \text{ for } n = 1, 2, 3, \ldots.$$

The Euler numbers $E_0, E_1, E_2, \ldots$ are integers determined by

$$E_0 = 1, \text{ and } \sum_{\substack{k=0 \\ 2 \mid n-k}}^{n} \binom{n}{k} E_k = 0 \text{ for } n = 1, 2, 3, \ldots.$$

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. The Bernoulli polynomials $B_n(x)$ $(n \in \mathbb{N})$ and the Euler polynomials $E_n(x)$ $(n \in \mathbb{N})$ are defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \text{ and } E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2} \right)^{n-k}.$$

It is well known that

$$\Delta(B_n(x)) = nx^{n-1} \text{ and } \Delta^*(E_n(x)) = 2x^n$$

for all $n \in \mathbb{N}$, where we set

$$\Delta(P(x)) = P(x+1) - P(x) \text{ and } \Delta^*(P(x)) = P(x+1) + P(x)$$

for any polynomial $P(x)$. Bernoulli and Euler numbers and polynomials play important roles in many fields including number theory and combinatorics.

In 2006 Z. W. Sun and H. Pan [6] established the following theorem which unifies many curious identities concerning Bernoulli and Euler numbers and polynomials.

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Theorem 1.1 (Sun and Pan, 2006). *Let n be a positive integer and let $x+y+z=1$.*

(i) *If r,s,t are complex numbers with $r+s+t=n$, then we have the symmetric relation*

$$r\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s\begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t\begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0$$

where

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n := \sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}B_{n-k}(x)B_k(y).$$

(ii) *If $r+s+t=n-1$, then*

$$\begin{aligned}
&\frac{r}{2}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}E_l(y)E_{n-1-l}(x)\\
=&\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}B_k(x)E_{n-k}(z)\\
&-(-1)^n\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{t}{n-k}B_k(y)E_{n-k}(z).
\end{aligned}$$

Recently, by a sophisticated application of the generating function method, A. M. Fu, H. Pan and F. Zhang [2] extended Theorem 1.1(i) of Sun and Pan to an identity on sums of products of $m \ge 2$ Bernoulli polynomials.

In this paper we obtain a general identity only involving Euler polynomials and also give an extension of Theorem 1.1(ii) which involves both Bernoulli and Euler polynomials.

Theorem 1.2. *Let m and n be positive integers, and let $r_0, r_1, \ldots, r_m$ be complex numbers with $r_0 + r_1 + \cdots + r_m = n-1$.*

(i) *If m is odd, then we have the symmetric relation*

$$\begin{aligned}
&\sum_{\substack{k_1,\cdots,k_m\ge 0\\ k_1+\cdots+k_m=n}}\prod_{j=1}^{m}\binom{r_j}{k_j}E_{k_j}(x_j)\\
=&-\sum_{i=1}^{m}(-1)^i\sum_{\substack{k_1,\ldots,k_m\ge 0\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_i}E_{k_i}(1-x_i)\prod_{\substack{1\le j\le m\\ j\ne i}}\binom{r_j}{k_j}E_{k_j}(x_j-x_i+\mathbf{1}_{j>i}),
\end{aligned}\tag{1.1}$$

where $\mathbf{1}_{j>i}$ takes 1 or 0 according as $j>i$ or not.

(ii) *If m is even, then*

$$\frac{r_0}{2}\sum_{\substack{k_1,\cdots,k_m\geq 0\\ k_1+\cdots+k_m=n-1}}\prod_{j=1}^{m}\binom{r_j}{k_j}E_{k_j}(x_j)$$
$$=\sum_{i=1}^{m}(-1)^i\sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_i}B_{k_i}(1-x_i)\prod_{\substack{1\leq j\leq m\\ j\neq i}}\binom{r_j}{k_j}E_{k_j}(x_j-x_i+\mathbf{1}_{j>i}). \tag{1.2}$$

Remark 1.1. If $r+s+t=n-1$, then (1.2) in the case $m=2$ gives

$$\begin{aligned}
&\frac{r}{2}\sum_{k=0}^{n-1}\binom{s}{k}E_k(1-y)\binom{t}{n-1-k}E_{n-1-k}(x)\\
=&-\sum_{k=0}^{n}\binom{r}{k}B_k(1-(1-y))\binom{t}{n-k}E_{n-k}(x-(1-y)+1)\\
&+\sum_{k=0}^{n}\binom{r}{k}B_k(1-x)\binom{s}{n-k}E_{n-k}((1-y)-x)\\
=&-(-1)^n\sum_{k=0}^{n}(-1)^k\binom{t}{n-k}E_{n-k}(1-x-y)\binom{r}{k}B_k(y)\\
&+\sum_{k=0}^{n}(-1)^k\binom{r}{k}B_k(x)\binom{s}{n-k}E_{n-k}(1-x-y),
\end{aligned}$$

which is equivalent to the identity of Sun and Pan in Theorem 1.1(ii) since $E_k(1-x)=(-1)^kE_k(x)$.

Our proof of Theorem 1.2 given in the next section involves the difference operator Δ and its companion operator Δ^*. We can also show Theorem 1.2 via the generating function approach.

Let k be any nonnegative integer. It is well known that $B_k=0$ if k is odd and greater than one. By [1, pp. 804-808],

$$B_k\left(\frac{1}{2}\right)=(2^{1-k}-1)B_k \quad\text{and}\quad E_k(x)=\frac{2}{k+1}\left(B_{k+1}(x)-2^{k+1}B_{k+1}\left(\frac{x}{2}\right)\right).$$

Thus

$$(-1)^kE_k(1)=E_k(0)=2(1-2^{k+1})\frac{B_{k+1}}{k+1}.$$

In view of these, Theorem 1.2 in the case $x_1=\cdots=x_m=1/2$ yields the following consequence involving Euler numbers and Bernoulli numbers.

Corollary 1.1. *Let m and n be positive integers, and let $r_0,r_1,\ldots,r_m$ be complex numbers with $r_0+r_1+\cdots+r_m=n-1$.*

(i) *If m is odd, then*

$$\begin{aligned}&(-1)^n\sum_{\substack{k_1,\cdots,k_m\geq 0\\k_1+\cdots+k_m=n}}\prod_{j=1}^m\binom{r_j}{k_j}E_{k_j}\\=&\sum_{i=1}^m(-1)^i\sum_{\substack{k_1,\ldots,k_m\geq 0\\k_1+\cdots+k_m=n}}(-1)^{|\{i<j\leq m:\ k_j>0\}|}\binom{r_0}{k_i}E_{k_i}\prod_{\substack{1\leq j\leq m\\j\neq i}}\binom{r_j}{k_j}\tilde{B}_{k_j+1},\end{aligned}\tag{1.3}$$

where $\tilde{B}_k=2^k(2^k-1)B_k/k$ for $k=1,2,3,\ldots$.

(ii) *If m is even, then*

$$\begin{aligned}&(-1)^n r_0\sum_{\substack{k_1,\cdots,k_m\geq 0\\k_1+\cdots+k_m=n-1}}\prod_{j=1}^m\binom{r_j}{k_j}E_{k_j}\\=&\sum_{i=1}^m(-1)^i\sum_{\substack{k_1,\ldots,k_m\geq 0\\k_1+\cdots+k_m=n}}(-1)^{|\{i<j\leq m:\ k_j>0\}|}\binom{r_0}{k_i}(2^{k_i}-2)B_{k_i}\prod_{\substack{1\leq j\leq m\\j\neq i}}\binom{r_j}{k_j}\tilde{B}_{k_j+1}.\end{aligned}\tag{1.4}$$

2. Proof of Theorem 1.2

As usual we let $\mathbb{C}$ denote the field of complex numbers. By [4, Lemma 3.1], for $P(x),Q(x)\in\mathbb{C}[x]$, we have $P(x)=Q(x)$ if $\Delta^*(P(x))=\Delta^*(Q(x))$. This property will play a central role in our proof of Theorem 1.2.

Lemma 2.1. *Let $P_1(x),\cdots,P_m(x)\in\mathbb{C}[x]$. Then*

$$\begin{aligned}&P_1(x)\sum_{1<i\leq m}(-1)^i\Delta^*(P_i(x))\prod_{\substack{1<j\leq m\\j\neq i}}P_j(x+\mathbf{1}_{j<i})\\=&\begin{cases}\Delta^*(P_1(x)\cdots P_m(x))-\Delta^*(P_1(x))P_2(x+1)\cdots P_m(x+1) & \text{if } 2\nmid m,\\ \Delta^*(P_1(x)\cdots P_m(x))-\Delta(P_1(x))P_2(x+1)\cdots P_m(x+1) & \text{if } 2\mid m.\end{cases}\end{aligned}$$

Proof. Observe that

$$\begin{aligned}&\sum_{1<i\leq m}(-1)^i\Delta^*(P_i(x))\prod_{\substack{1<j\leq m\\j\neq i}}P_j(x+\mathbf{1}_{j<i})\\=&\sum_{1<i\leq m}\left((-1)^i\prod_{1<j\leq m}P_j(x+\mathbf{1}_{j<i})-(-1)^{i+1}\prod_{1<j\leq m}P_j(x+\mathbf{1}_{j<i+1})\right)\\=&(-1)^2\prod_{1<j\leq m}P_j(x)-(-1)^{m+1}\prod_{1<j\leq m}P_j(x+1).\end{aligned}$$

Therefore

$$
\begin{aligned}
&P_1(x) \sum_{1<i\le m} (-1)^i \Delta^*(P_i(x)) \prod_{\substack{1<j\le m\\ j\ne i}} P_j(x+\mathbf{1}_{j<i})\\
=&P_1(x)\cdots P_m(x) + (-1)^m P_1(x) \prod_{1<j\le m} P_j(x)\\
=&\Delta^*(P_1(x)\cdots P_m(x)) - (P_1(x+1) + (-1)^{m-1}P_1(x)) \prod_{1<j\le m} P_j(x).
\end{aligned}
$$

This proves the desired identity. □

Lemma 2.2. *Let $a_0, \bar{a}_0, a_1, \bar{a}_1 \dots, a_n, \bar{a}_n$ be complex numbers, and set*

$$
A_k(t) = \sum_{l=0}^{k} \binom{k}{l} (-1)^l a_l t^{k-l} \quad \textit{and} \quad \bar{A}_k(t) = \sum_{l=0}^{k} \binom{k}{l} (-1)^l \bar{a}_l t^{k-l}
$$

for $k = 0, \dots, n$. Let $r_0 + r_1 + \dots + r_m = n - 1$. Then

$$
\begin{aligned}
&\sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1+\dots+k_m=n}} \binom{r_0}{k_1} (-x_1)^{k_1} \prod_{j=2}^{m} \binom{r_j}{k_j} A_{k_j}(x_j - x_1)\\
=&\sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1+\dots+k_m=n}} \binom{r_1}{k_1} {x_1}^{k_1} \prod_{j=2}^{m} \binom{r_j}{k_j} A_{k_j}(x_j). \qquad (2.1)
\end{aligned}
$$

Also, for any $i = 2, \dots, m$ we have

$$
\begin{aligned}
&\sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1+\dots+k_m=n}} \binom{r_0}{k_1} A_{k_1}(-x_1) \binom{r_i}{k_i} (x_i - x_1)^{k_i} \prod_{\substack{2\le j\le m\\ j\ne i}} \binom{r_j}{k_j} \bar{A}_{k_j}(x_j - x_1)\\
=&\sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1+\dots+k_m=n}} \binom{r_1}{k_1} (x_1 - x_i)^{k_1} \binom{r_0}{k_i} A_{k_i}(-x_i) \prod_{\substack{2\le j\le m\\ j\ne i}} \binom{r_j}{k_j} \bar{A}_{k_j}(x_j - x_i). \qquad (2.2)
\end{aligned}
$$

Proof. By Remark 1.1 of Sun [5],

$$
A_k(x+y) = \sum_{l=0}^{k} \binom{k}{l} x^{k-l} A_l(y) \quad \text{and} \quad \bar{A}_k(x+y) = \sum_{l=0}^{k} \binom{k}{l} x^{k-l} \bar{A}_l(y)
$$

for every $k=0,\ldots,n$. Observe that

$$\begin{aligned}
&\sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_1}(-x_1)^{k_1}\prod_{j=2}^{m}\binom{r_j}{k_j}A_{k_j}(x_j-x_1)\\
=&\sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_1}(-x_1)^{k_1}\prod_{j=2}^{m}\binom{r_j}{k_j}\sum_{l_j=0}^{k_j}\binom{k_j}{l_j}(-x_1)^{k_j-l_j}A_{l_j}(x_j)\\
=&\sum_{\substack{l_1,\ldots,l_m\geq 0\\ l_1+\cdots+l_m=n}}(-x_1)^{l_1}\prod_{j=2}^{m}\binom{r_j}{l_j}A_{l_j}(x_j)\sum_{\substack{k_1\geq 0,\ k_j\geq l_j\ (1<j\leq m)\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_1}\prod_{j=2}^{m}\binom{r_j-l_j}{k_j-l_j}.
\end{aligned}$$

Given $l_1,\ldots,l_m\in\mathbb{N}$ with $l_1+\cdots+l_m=n$, by the Chu-Vandermonde convolution identity (cf. [3, (5.22)]), we have

$$\begin{aligned}
&\sum_{\substack{k_1\geq 0,\, k_j\geq l_j\ (1<j\leq m)\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_1}\prod_{j=2}^{m}\binom{r_j-l_j}{k_j-l_j}\\
=&\binom{r_0+(r_2-l_2)+\cdots+(r_m-l_m)}{n-l_2-\cdots-l_m}=\binom{l_1-1-r_1}{l_1}=(-1)^{l_1}\binom{r_1}{l_1}.
\end{aligned}$$

So (2.1) follows.

(2.2) can be proved similarly. Let Σ denote the left-hand side of (2.2). Then

$$\begin{aligned}
\Sigma=&\sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_1}\sum_{l_i=0}^{k_1}\binom{k_1}{l_i}(x_i-x_1)^{k_1-l_i}A_{l_i}(-x_i)\binom{r_i}{k_i}(x_i-x_1)^{k_i}\\
&\times\prod_{\substack{1<j\leq m\\ j\neq i}}\binom{r_j}{k_j}\sum_{l_j=0}^{k_j}\binom{k_j}{l_j}(x_i-x_1)^{k_j-l_j}\bar{A}_{l_j}(x_j-x_i)\\
=&\sum_{\substack{l_1,\ldots,l_m\geq 0\\ l_1+\cdots+l_m=n}}(x_i-x_1)^{l_1}\binom{r_0}{l_i}A_{l_i}(-x_i)\prod_{\substack{1<j\leq m\\ j\neq i}}\binom{r_j}{l_j}\bar{A}_{l_j}(x_j-x_i)\\
&\times\sum_{\substack{k_j\geq l_j\ (1\leq j\leq m\ \&\ j\neq i)\\ k_i\geq 0,\ k_1+\cdots+k_m=n}}\binom{r_0-l_i}{k_1-l_i}\binom{r_i}{k_i}\prod_{\substack{1<j\leq m\\ j\neq i}}\binom{r_j-l_j}{k_j-l_j}\\
=&\sum_{\substack{l_1,\ldots,l_m\geq 0\\ l_1+\cdots+l_m=n}}(x_i-x_1)^{l_1}\binom{r_0}{l_i}A_{l_i}(-x_i)\prod_{\substack{1<j\leq m\\ j\neq i}}\binom{r_j}{l_j}\bar{A}_{l_j}(x_j-x_i)\times(-1)^{l_1}\binom{r_1}{l_1}.
\end{aligned}$$

This concludes the proof. $\square$

Remark 2.1. If we set $a_l = (-1)^l B_l$ and $\bar{a}_l = (-1)^l E_l(0)$ for $l = 0, \dots, n$ in Lemma 2.2, then $A_k(t) = B_k(t)$ and $\bar{A}_k(t) = E_k(t)$ for any $k = 0, \dots, n$.

Proof of Theorem 1.2. We fix $x_2, \dots, x_m$.

(i) Suppose that m is odd. Set

$$P(x_1) = \sum_{\substack{k_1,\dots,k_m \geqq 0 \\ k_1+\cdots+k_m=n}} \binom{r_0}{k_1} E_{k_1}(1-x_1) \prod_{j=2}^{m} \binom{r_j}{k_j} E_{k_j}(x_j - x_1 + 1).$$

Applying Lemma 2.1, we get

$$\begin{aligned}
&\Delta^*(P(x_1)) \\
=&\sum_{i=2}^{m}(-1)^i \sum_{\substack{k_1,\dots,k_m \geqq 0 \\ k_1+\cdots+k_m=n}} \binom{r_0}{k_1} E_{k_1}(1-x_1) \binom{r_i}{k_i} 2(x_i - x_1)^{k_i} \prod_{\substack{2\le j\le m \\ j\ne i}} \binom{r_j}{k_j} E_{k_j}(x_j - x_1 + \mathbf{1}_{j>i}) \\
&+ \sum_{\substack{k_1,\dots,k_m \geqq 0 \\ k_1+\cdots+k_m=n}} \binom{r_0}{k_1} 2(-x_1)^{k_1} \prod_{j=2}^{m} \binom{r_j}{k_j} E_{k_j}(x_j - x_1).
\end{aligned}$$

With the help of Lemma 2.2, we have

$$\begin{aligned}
&\Delta^*(P(x_1)) \\
=&2\sum_{i=2}^{m}(-1)^i \sum_{\substack{k_1,\dots,k_m \geqq 0 \\ k_1+\cdots+k_m=n}} \binom{r_1}{k_1} (x_1 - x_i)^{k_1} \binom{r_0}{k_i} E_{k_i}(1-x_i) \prod_{\substack{2\le j\le m \\ j\ne i}} \binom{r_j}{k_j} E_{k_j}(x_j - x_i + \mathbf{1}_{j>i}) \\
&+ 2 \sum_{\substack{k_1,\dots,k_m \geqq 0 \\ k_1+\cdots+k_m=n}} \binom{r_1}{k_1} x_1^{k_1} \prod_{j=2}^{m} \binom{r_j}{k_j} E_{k_j}(x_j).
\end{aligned}$$

It follows that $\Delta^*(P(x_1)) = \Delta^*(Q(x_1))$, where

$$\begin{aligned}
Q(x_1) = &\sum_{1<i\le m} (-1)^i \sum_{\substack{k_1,\dots,k_m \ge 0 \\ k_1+\cdots+k_m=n}} \binom{r_0}{k_i} E_{k_i}(1-x_i) \prod_{\substack{1\le j\le m \\ j\ne i}} \binom{r_j}{k_j} E_{k_j}(x_j - x_i + \mathbf{1}_{j>i}) \\
&+ \sum_{\substack{k_1,\cdots,k_m \ge 0 \\ k_1+\cdots+k_m=n}} \prod_{j=1}^{m} \binom{r_j}{k_j} E_{k_j}(x_j).
\end{aligned}$$

Therefore $P(x_1) = Q(x_1)$ by [4, Lemma 3.1]. This proves (1.1).

(ii) Now assume that m is even. Define

$$P(x_1)=\sum_{\substack{k_1,\ldots,k_m\geqq 0\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_1}B_{k_1}(1-x_1)\prod_{j=2}^m\binom{r_j}{k_j}E_{k_j}(x_j-x_1+1).$$

For $k_1=0,1,2,\ldots$, clearly

$$\binom{r_0}{k_1}(B_{k_1}(1-(x_1+1))-B_{k_1}(1-x_1))=-\binom{r_0}{k_1}k_1(-x_1)^{k_1-1}=-r_0\binom{r_0-1}{k_1-1}(-x_1)^{k_1-1}.$$

(As usual $\binom{x}{-1}$ is regarded as 0.) Thus, by Lemma 2.1 we have

$$\begin{aligned}&\Delta^*(P(x_1))\\ =&2\sum_{i=2}^m(-1)^i\sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_1}B_{k_1}(1-x_1)\binom{r_i}{k_i}(x_i-x_1)^{k_i}\prod_{\substack{1<j\leq m\\ j\neq i}}\binom{r_j}{k_j}E_{k_j}(x_j-x_1+\mathbf{1}_{j>i})\\ &-r_0\sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n-1}}\binom{r_0-1}{k_1}(-x_1)^{k_1}\prod_{1<j\leq m}\binom{r_j}{k_j}E_{k_j}(x_j-x_1).\end{aligned}$$

With the help of Lemma 2.2,

$$\begin{aligned}&\Delta^*(P(x_1))\\ =&2\sum_{i=2}^m(-1)^i\sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n}}\binom{r_1}{k_1}(x_1-x_i)^{k_1}\binom{r_0}{k_i}B_{k_i}(1-x_i)\prod_{\substack{1<j\leq m\\ j\neq i}}\binom{r_j}{k_j}E_{k_j}(x_j-x_i+\mathbf{1}_{j>i})\\ &-r_0\sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n-1}}\binom{r_1}{k_1}x_1^{k_1}\prod_{1<j\leq m}\binom{r_j}{k_j}E_{k_j}(x_j).\end{aligned}$$

So we have $\Delta^*(P(x_1))=\Delta^*(Q(x_1))$, where

$$\begin{aligned}Q(x_1)=&\sum_{i=2}^m(-1)^i\sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n}}\binom{r_0}{k_i}B_{k_i}(1-x_i)\prod_{\substack{1\leq j\leq m\\ j\neq i}}\binom{r_j}{k_j}E_{k_j}(x_j-x_i+\mathbf{1}_{j>i})\\ &-\frac{r_0}{2}\sum_{\substack{k_1,\cdots,k_m\geq 0\\ k_1+\cdots+k_m=n-1}}\prod_{j=1}^m\binom{r_j}{k_j}E_{k_j}(x_j).\end{aligned}$$

Therefore, $P(x_1)$ coincides with $Q(x_1)$ by [4, Lemma 3.1]. So (1.2) holds. This concludes the proof. □

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