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ON 2-ADIC ORDERS OF SOME BINOMIAL SUMS

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Abstract. We prove that for any nonnegative integers n and r the binomial sum

$$\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2r}$$

is divisible by $2^{2n-\min\{\alpha(n),\alpha(r)\}}$, where $\alpha(n)$ denotes the number of 1s in the binary expansion of n. This confirms a recent conjecture of Guo and Zeng [J. Number Theory, **130**(2010), 172–186].

In 1976 Shapiro [3] introduced the Catalan triangle $(\frac{k}{n}\binom{2n}{n-k})_{n\geqslant k\geqslant 1}$ and determined the sum of entries in the nth row; namely, he showed that

$$\sum_{k=1}^{n}k\binom{2n}{n-k}=\frac{n}{2}\binom{2n}{n}.$$

Let $n,r\in\mathbb{N}=\{0,1,2,\dots\}$. Recently, Guo and Zeng [1] proved that

$$\frac{2}{n^2\binom{2n}{n}}\sum_{k=1}^{n}\binom{2n}{n-k}k^{2r+1}$$

is an odd integer if $n,r\in\mathbb{Z}^+=\{1,2,3,\dots\}$. They also conjectured that the binomial sum

$$F(n,r)=\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2r} \tag{1.1}$$

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is divisible by $2^{2n-\min\{\alpha(n),\alpha(r)\}}$, where $\alpha(n)$ denotes the number of 1s in the binary expansion of n. Note that if $n, r \in \mathbb{Z}^+$ then $F(n,r) = 2\sum_{k=1}^{n}\binom{2n}{n-k}k^{2r}$. Actually the conjecture was motivated by Guo and Zeng's following observations:

$$\begin{aligned}
\sum_{k=1}^{n}\binom{2n}{n-k}k^2 =&2^{2n-2}n,\\
\sum_{k=1}^{n}\binom{2n}{n-k}k^4 =&2^{2n-3}n(3n-1),\\
\sum_{k=1}^{n}\binom{2n}{n-k}k^6 =&2^{2n-4}n(15n^2-15n+4),\\
\sum_{k=1}^{n}\binom{2n}{n-k}k^8 =&2^{2n-5}n(105n^3-210n^2+147n-34).
\end{aligned}$$

In this paper we shall confirm the sophisticated conjecture of Guo and Zeng.

For an integer n and a prime p, the p-adic order of n at p is given by

$$\nu_p(n) = \sup\{v \in \mathbb{N} : p^v \mid n\}.$$

Now we state our main result.

Theorem 1.1. *For any $n, r \in \mathbb{N}$ we have*

$$\nu_2(F(n,r)) \geqslant 2n - \min\{\alpha(n), \alpha(r)\}, \tag{1.2}$$

where $F(n,r)$ is given by (1.1).

Note that (1.2) can be split into two inequalities:

$$\nu_2(F(n,r)) \geqslant 2n - \alpha(n) \tag{1.3}$$

and

$$\nu_2(F(n,r)) \geqslant 2n - \alpha(r). \tag{1.4}$$

In Sections 2 and 3 we will show (1.3) and (1.4) respectively.

2. Proof of (1.3)

Let p be any prime. A useful theorem of Legendre (see, e.g., [2, pp. 22–24]) asserts that for any $n \in \mathbb{N}$ we have

$$\nu_p(n!) = \sum_{i=1}^{\infty}\left\lfloor \frac{n}{p^i}\right\rfloor = \frac{n-\alpha_p(n)}{p-1},$$

where $\alpha_p(n)$ is the sum of the digits of n in the expansion of n in base p. In particular, $\nu_2(n!) = n - \alpha(n)$ for all $n = 0, 1, 2, \ldots$.

Lemma 2.1. (i) *For any* $n \in \mathbb{Z}^+$ *we have*

$$\nu_2(n) - 1 = \alpha(n-1) - \alpha(n). \tag{2.1}$$

(ii) *Let* $s > t \geqslant 0$ *be integers. Then*

$$\nu_2\left(\binom{s}{t}\right) \geqslant \alpha(t) - \alpha(s) + 1. \tag{2.2}$$

Proof. (i) In view of Legendre's theorem, for any positive integer n we have

$$\nu_2(n) = \nu_2(n!) - \nu_2((n-1)!) = n - \alpha(n) - (n-1-\alpha(n-1)) = \alpha(n-1) - \alpha(n) + 1.$$

This proves (2.1).

(ii) With the help of Legendre's theorem,

$$\begin{aligned}\nu_2\left(\binom{s}{t}\right) =&\nu_2(s!) - \nu_2(t!) - \nu_2((s-t)!)\\ =&s - \alpha(s) - (t - \alpha(t)) - (s - t - \alpha(s-t))\\ =&\alpha(t) - \alpha(s) + \alpha(s-t)\\ \geqslant&\alpha(t) - \alpha(s) + 1 \text{ (since } s - t \geqslant 1).\end{aligned}$$

So (2.2) holds. □

Lemma 2.2. *For* $n, r \in \mathbb{Z}^+$ *we have*

$$F(n,r) = n^2 F(n, r-1) - 2n(2n-1)F(n-1, r-1). \tag{2.3}$$

Proof. Since

$$(n^2 - k^2)\binom{2n}{n-k} = 2n(2n-1)\binom{2n-2}{n-1-k},$$

we have

$$\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2r} = n^2\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2r-2} - 2n(2n-1)\sum_{k=-n+1}^{n-1}\binom{2n-2}{n-1-k}k^{2r-2},$$

which gives (2.3). □

Proof of (1.3). We use induction on $n + r$. Clearly (1.3) holds trivially when $n = 0$ or $r = 0$.

Now let $n, r \in \mathbb{Z}^+$ and assume (1.3) for any smaller value of $n + r$. By (2.1), (2.3) and the induction hypothesis, we have

$$\begin{aligned}\nu_2(F(n,r)) \geqslant&\min\{\nu_2(n^2F(n,r-1)), \nu_2(2n(2n-1)F(n-1,r-1))\}\\ =&\min\{2\nu_2(n) + \nu_2(F(n,r-1)), 1 + \nu_2(n) + \nu_2(F(n-1,r-1))\}\\ \geqslant&\min\{2\nu_2(n) + 2n - \alpha(n), 1 + \nu_2(n) + 2(n-1) - \alpha(n-1)\}\\ =&2n - \alpha(n).\end{aligned}$$

This concludes the induction step. □

3. Proof of (1.4)

Lemma 3.1. *For $n, r \in \mathbb{Z}^+$ we have*

$$\begin{aligned}F(n,r) =&4F(n-1,r)-\sum_{i=0}^{r-1}\binom{2r}{2i}F(n,i)-2(2n-1)\sum_{i=0}^{r-1}\binom{2r}{2i+1}F(n-1,i)\\&+n\sum_{i=0}^{r-1}\binom{2r}{2i+1}F(n,i)+2\sum_{i=0}^{r-1}\binom{2r}{2i}F(n-1,i).\end{aligned}\tag{3.1}$$

Proof. Let $n \in \mathbb{N}$ and $r \in \mathbb{Z}^+$. We want to prove (3.1) with n in it replaced by $n+1$.

Clearly

$$\begin{aligned}F(n,r)=&\sum_{k=-n-1}^{n-1}\binom{2n}{n-1-k}(k+1)^{2r}\\=&\sum_{k=-n-1}^{n}\left(\binom{2n+1}{n-k}-\binom{2n}{n-k}\right)\sum_{j=0}^{2r}\binom{2r}{j}k^j\\=&\sum_{j=0}^{2r}\binom{2r}{j}\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^j-\sum_{i=0}^{r}\binom{2r}{2i}\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2i};\end{aligned}\tag{3.2}$$

in the last step we use the fact that if j is odd then

$$\begin{aligned}\sum_{k=-n}^{n}\binom{2n}{n-k}k^j=&\frac{1}{2}\left(\sum_{k=-n}^{n}\binom{2n}{n-k}k^j+\sum_{k=-n}^{n}\binom{2n}{n+k}(-k)^j\right)\\=&\sum_{k=-n}^{n}\binom{2n}{n-k}(k^j+(-k)^j)=0.\end{aligned}$$

When j is even, we have

$$\begin{aligned}2\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^j=&\sum_{k=-n-1}^{n+1}\left(\binom{2n+1}{n-k}+\binom{2n+1}{n+k}\right)k^j\\=&\sum_{k=-n-1}^{n+1}\binom{2n+2}{n+1-k}k^j=F\left(n+1,\frac{j}{2}\right).\end{aligned}\tag{3.3}$$

If j is odd, then

$$\begin{aligned}&(n+1)\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^{j-1}+\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^j\\=&\sum_{k=-n-1}^{n}(n+1+k)\binom{2n+1}{n-k}k^{j-1}=(2n+1)\sum_{k=-n}^{n}\binom{2n}{n-k}k^{j-1},\end{aligned}$$

i.e.,

$$\begin{aligned}\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^j =&(2n+1)\sum_{k=-n}^{n}\binom{2n}{n-k}k^{j-1}-(n+1)\sum_{k=-n-1}^{n}\binom{2n+1}{n-k}k^{j-1}\\ =&(2n+1)F\left(n,\frac{j-1}{2}\right)-\frac{n+1}{2}F\left(n+1,\frac{j-1}{2}\right),\end{aligned}\tag{3.4}$$

where we use (3.3) in the last step. Combining (3.2)-(3.4), we get

$$\begin{aligned}F(n,r)=&\frac{1}{2}\sum_{i=0}^{r}\binom{2r}{2i}F(n+1,i)+(2n+1)\sum_{i=0}^{r-1}\binom{2r}{2i+1}F(n,i)\\ &-\frac{n+1}{2}\sum_{i=0}^{r-1}\binom{2r}{2i+1}F(n+1,i)-\sum_{i=0}^{r}\binom{2r}{2i}F(n,i),\end{aligned}$$

which yields the desired result. □

Proof of (1.4). We still use induction on $n+r$. There is nothing to do if $n=0$ or $r=0$. Assume that $n,r\geqslant 1$ and (1.4) holds for any smaller value of $n+r$. In view of Lemma 3.1, $\nu_2(F(n,r))$ is not smaller than the minimum of the following numbers:

$$\begin{aligned}&2+\nu_2(F(n-1,r)),\ \min_{0\leqslant i<r}\nu_2\left(\binom{2r}{2i}F(n,i)\right),\ \min_{0\leqslant i<r}\nu_2\left(n\binom{2r}{2i+1}F(n,i)\right)\\ &1+\min_{0\leqslant i<r}\nu_2\left(\binom{2r}{2i+1}F(n-1,i)\right),\ 1+\min_{0\leqslant i<r}\nu_2\left(\binom{2r}{2i}F(n-1,i)\right).\end{aligned}$$

By the induction hypothesis and Lemma 2.1(ii), we have $\nu_2(F(n-1,r))\geqslant 2n-2-\alpha(r)$, and also

$$\nu_2\left(\binom{2r}{2i}F(n,i)\right)\geqslant 2n-\alpha(i)+\alpha(2i)-\alpha(2r)+1=2n-\alpha(r)+1,$$

$$\nu_2\left(\binom{2r}{2i+1}F(n-1,i)\right)\geqslant 2n-2-\alpha(i)+\alpha(2i+1)-\alpha(2r)+1=2n-\alpha(r),$$

$$\nu_2\left(n\binom{2r}{2i+1}F(n,i)\right)\geqslant 2n-\alpha(i)+\alpha(2i+1)-\alpha(2r)+1=2n-\alpha(r)+2,$$

and

$$\nu_2\left(\binom{2r}{2i}F(n-1,i)\right)\geqslant 2n-2-\alpha(i)+\alpha(2i)-\alpha(2r)+1=2n-\alpha(r)-1.$$

Thus (1.4) follows. □

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