ON DELANNOY NUMBERS
AND SCHRÖDER NUMBERS

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ABSTRACT. The nth Delannoy number and the nth Schröder number given by

$$D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \quad \text{and} \quad S_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}$$

respectively arise naturally from enumerative combinatorics. Let p be an odd prime. We mainly show that

$$\sum_{k=1}^{p-1} \frac{D_k}{k^2} \equiv 2\left(\frac{-1}{p}\right)E_{p-3} \pmod{p}$$

and

$$\sum_{k=1}^{p-1} \frac{S_k}{m^k} \equiv \frac{m^2-6m+1}{2m}\left(1-\left(\frac{m^2-6m+1}{p}\right)\right) \pmod{p},$$

where $(-)$ is the Legendre symbol, $E_0, E_1, E_2, \ldots$ are Euler numbers, and m is any integer not divisible by p. We also conjecture that

$$\sum_{k=1}^{p-1} \frac{D_k^2}{k^2} \equiv -2q_p(2)^2 \pmod{p}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

1. INTRODUCTION

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the (central) Delannoy number D_n denotes the number of lattice paths from the point $(0,0)$ to (n,n) with steps $(1,0), (0,1)$ and $(1,1)$, while the Schröder number S_n represents the number of such paths that never rise above the line $y = x$. It is known that

$$D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}$$

and

$$S_n = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1} = \sum_{k=0}^{n}\binom{n+k}{2k}C_k,$$

where C_k stands for the Catalan number $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$. For information on D_n and S_n, the reader may consult [CHV], [S], and p. 178 and p. 185 of [St].

Despite their combinatorial backgrounds, surprisingly Delannoy numbers and Schröder numbers have some nice number-theoretic properties.

As usual, for an odd prime p we let $(\frac{\cdot}{p})$ denote the Legendre symbol. Recall that Euler numbers $E_0, E_1, E_2, \ldots$ are integers defined by $E_0 = 1$ and the recursion:

$$\sum_{\substack{k=0 \\ 2|k}}^{n}\binom{n}{k}E_{n-k} = 0 \quad \text{for } n = 1, 2, 3, \ldots .$$

Our first theorem is concerned with Delannoy numbers and their generalization.

Theorem 1.1. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{D_k}{k^2} \equiv 2\left(\frac{-1}{p}\right)E_{p-3} \pmod{p} \tag{1.1}$$

and

$$\sum_{k=1}^{p-1}\frac{D_k}{k} \equiv -q_p(2) \pmod{p}, \tag{1.2}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1} - 1)/p$. If we set

$$D_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k \quad (n \in \mathbb{N}),$$

then for any p-adic integer x we have

$$\sum_{k=1}^{p-1}\frac{D_k(x)}{k} \equiv \frac{(-1+\sqrt{-x})^p + (-1-\sqrt{-x})^p + 2}{p} \pmod{p}. \tag{1.3}$$

Corollary 1.1. *Let p be an odd prime. We have*

$$\sum_{k=1}^{p-1} \frac{D_k(3)}{k} \equiv -2q_p(2) \pmod{p} \quad \text{provided } p \neq 3, \tag{1.4}$$

$$\sum_{k=1}^{p-1} \frac{D_k(-4)}{k} \equiv \frac{3 - 3^p}{p} \pmod{p}, \tag{1.5}$$

$$\sum_{k=1}^{p-1} \frac{D_k(-9)}{k} \equiv -6q_p(2) \pmod{p}, \tag{1.6}$$

and also

$$\sum_{k=1}^{p-1} \frac{D_k(-2)}{k} \equiv -\frac{4}{p} P_{p-\left(\frac{2}{p}\right)} \pmod{p}, \tag{1.7}$$

where the Pell sequence $\{P_n\}_{n\geqslant 0}$ is given by

$$P_0 = 0, \ P_1 = 1, \ \text{and } P_{n+1} = 2P_n + P_{n-1} \ (n = 1, 2, 3, \dots).$$

If $p \neq 5$, then

$$\sum_{k=1}^{p-1} \frac{D_k(-5)}{k} \equiv -2q_p(2) - \frac{5}{p} F_{p-\left(\frac{p}{5}\right)} \pmod{p}, \tag{1.8}$$

where the Fibonacci sequence $\{F_n\}_{n\geqslant 0}$ is defined by

$$F_0 = 0, \ F_1 = 1, \ \text{and } F_{n+1} = F_n + F_{n-1} \ (n = 1, 2, 3, \dots).$$

Now we propose two conjectures which seem challenging in the author's opinion.

Conjecture 1.1. *Let $p > 3$ be a prime. We have*

$$\sum_{k=1}^{p-1} \frac{D_k^2}{k^2} \equiv -2q_p(2)^2 \pmod{p}, \tag{1.9}$$

$$\sum_{k=1}^{p-1} \frac{D_k}{k} \equiv -q_p(2) + p\, q_p(2)^2 \pmod{p^2}, \tag{1.10}$$

$$\sum_{k=1}^{p-1} D_k S_k \equiv -2p \sum_{k=1}^{p-1} \frac{(-1)^k + 3}{k} \pmod{p^4},$$

and

$$\sum_{k=1}^{(p-1)/2} D_k S_k \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{4} \ \& \ p = x^2 + y^2 \ (2 \nmid x, \ 2 \mid y), \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Also, $\sum_{n=1}^{p-1} s_n^2/n \equiv -6 \pmod{p}$, *where*

$$s_n := \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k+1} = D_n - S_n.$$

Remark 1.1. Let p be an odd prime. Though there are many congruences for $q_p(2) \bmod p$, (1.9) is curious since its left-hand side is a sum of squares. It is known that $\sum_{k=1}^{p-1} 1/k \equiv -p^2 B_{p-3}/3 \pmod{p^3}$ if $p > 3$, where $B_0, B_1, B_2, \dots$ are Bernoulli numbers. In addition, we can prove that $\sum_{k=0}^{p-1} D_k \equiv (\frac{-1}{p}) - p^2 E_{p-3} \pmod{p^3}$ and $\sum_{k=0}^{p-1} D_k^2 \equiv (\frac{2}{p}) \pmod{p}$.

Conjecture 1.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k D_k(2)^3 \equiv \sum_{k=0}^{p-1}(-1)^k D_k\left(-\frac{1}{4}\right)^3 \equiv \left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}(-1)^k D_k\left(\frac{1}{8}\right)^3$$

$$\equiv \begin{cases} (\frac{-1}{p})(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \ \& \ p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Also,

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}(-1)^k D_k\left(\frac{1}{2}\right)^3$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and } p = x^2 + 6y^2 \ (x, y \in \mathbb{Z}), \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and } p = 2x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p}) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(-1)^k D_k(-4)^3 \equiv \left(\frac{-5}{p}\right)\sum_{k=0}^{p-1}(-1)^k D_k\left(-\frac{1}{16}\right)^3$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and } p = x^2 + 15y^2 \ (x, y \in \mathbb{Z}), \\ 12x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ and } p = 3x^2 + 5y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{-15}{p}) = -1. \end{cases}$$

Remark 1.2. Note that $(-1)^n D_n(x) = D_n(-x-1)$ for any $n \in \mathbb{N}$, since

$$D_n(-x-1) = \sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{k}\sum_{j=0}^{k}\binom{k}{j}x^j$$

$$= \sum_{j=0}^{n}\binom{n}{j}x^j\sum_{k=0}^{n}\binom{-n-1}{k}\binom{n-j}{n-k}$$

$$= \sum_{j=0}^{n}\binom{n}{j}x^j\binom{-j-1}{n} = (-1)^n D_n(x).$$

Concerning Schröder numbers we establish the following result.

Theorem 1.2. *Let p be an odd prime and let m be an integer not divisible by p. Then*

$$\sum_{k=1}^{p-1} \frac{S_k}{m^k} \equiv \frac{m^2 - 6m + 1}{2m}\left(1 - \left(\frac{m^2 - 6m + 1}{p}\right)\right) \pmod{p}. \qquad (1.11)$$

Example 1.1. Theorem 1.2 in the case $m = 6$ gives that

$$\sum_{k=1}^{p-1} \frac{S_k}{6^k} \equiv 0 \pmod{p} \qquad \text{for any prime } p > 3. \qquad (1.12)$$

For technical reasons, we will prove Theorem 1.2 in the next section and show Theorem 1.1 and Corollary 1.1 in Section 3.

2. PROOF OF THEOREM 1.2

Lemma 2.1. *Let p be an odd prime and let m be any integer not divisible by p. Then*

$$\sum_{k=1}^{p-1} \frac{C_k}{m^k} \equiv \frac{m-4}{2}\left(1 - \left(\frac{m(m-4)}{p}\right)\right) \pmod{p}. \qquad (2.1)$$

Proof. This follows from [Su10, Theorem 1.1] in which the author even determined $\sum_{k=1}^{p-1} C_k/m^k \bmod p^2$. However, we will give here a simple proof of (2.1).

For each $k = 1, \ldots, p-1$, we clearly have

$$\binom{(p-1)/2}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod{p}.$$

Note also that

$$C_{p-1} = \frac{1}{2p-1}\prod_{k=1}^{p-1}\frac{p+k}{k} \equiv -1 \pmod{p}.$$

Therefore

$$\sum_{k=1}^{p-1} \frac{C_k}{m^k} \equiv \sum_{0<k<p-1} \binom{(p-1)/2}{k} \frac{1}{k+1} \left(-\frac{4}{m}\right)^k + \frac{C_{p-1}}{m^{p-1}}$$

$$\equiv -\frac{m}{4} \times \frac{2}{p+1} \sum_{k=1}^{(p-1)/2} \binom{(p+1)/2}{k+1} \left(-\frac{4}{m}\right)^{k+1} - 1$$

$$\equiv -\frac{m}{2} \left(\left(1 - \frac{4}{m}\right)^{(p+1)/2} - 1 - \frac{p+1}{2}\left(-\frac{4}{m}\right) \right) - 1$$

$$\equiv -\frac{m}{2} \left(\frac{m-4}{m} \times \frac{(m(m-4))^{(p-1)/2}}{m^{p-1}} - 1 + \frac{2}{m} \right) - 1$$

$$\equiv -\frac{m-4}{2} \left(\frac{m(m-4)}{p} \right) + \frac{m}{2} - 2 \pmod{p}$$

and hence (2.1) follows. $\square$

Lemma 2.2. *For any odd prime p we have*

$$\sum_{k=1}^{p-1} S_k \equiv 2\left(\frac{-1}{p}\right) - 2^p \pmod{p^2}. \qquad (2.2)$$

Proof. Recall the known identity (cf. (5.26) of [GKP, p. 169])

$$\sum_{n=0}^{m} \binom{n}{k} = \binom{m+1}{k+1} \quad (k, m \in \mathbb{N}).$$

Then

$$\sum_{n=0}^{p-1} S_n = \sum_{n=0}^{p-1} \sum_{k=0}^{n} \binom{n+k}{2k} C_k = \sum_{k=0}^{p-1} C_k \sum_{n=k}^{p-1} \binom{n+k}{2k}$$

$$= \sum_{k=0}^{p-1} C_k \binom{p+k}{2k+1} = \sum_{k=0}^{p-1} \frac{p}{k!(k+1)!(2k+1)} \prod_{0<j\leqslant k} (p^2 - j^2)$$

$$\equiv \sum_{k=0}^{p-1} \frac{p(-1)^k (k!)^2}{k!(k+1)!(2k+1)} = p \sum_{k=0}^{p-1} (-1)^k \left(\frac{2}{2k+1} - \frac{1}{k+1} \right) \pmod{p^2}.$$

Observe that

$$2p \sum_{k=0}^{p-1} \frac{(-1)^k}{2k+1} = p \sum_{k=0}^{p-1} \left(\frac{(-1)^k}{2k+1} + \frac{(-1)^{p-1-k}}{2(p-1-k)+1} \right)$$

$$= p \sum_{k=0}^{p-1} (-1)^k \left(\frac{1}{2k+1} + \frac{1}{2p - (2k+1)} \right)$$

$$\equiv p(-1)^{(p-1)/2} \left(\frac{1}{p} + \frac{1}{2p-p} \right) = 2\left(\frac{-1}{p}\right) \pmod{p^2}.$$

Also,

$$-p\sum_{k=0}^{p-1}\frac{(-1)^k}{k+1} = p\sum_{k=1}^{p}\frac{(-1)^k}{k}$$

$$\equiv -\sum_{k=1}^{p-1}\frac{p}{k}\binom{p-1}{k-1} - 1 = -\sum_{k=0}^{p-1}\binom{p}{k} = 1 - 2^p \pmod{p^2}.$$

Combining the above, we obtain

$$\sum_{n=0}^{p-1} S_n \equiv 2\left(\frac{-1}{p}\right) + 1 - 2^p \pmod{p^2}$$

and hence (2.2) holds.　□

Proof of Theorem 1.2. In the case $m \equiv 1 \pmod{p}$, (1.11) reduces to the congruence

$$\sum_{k=1}^{p-1} S_k \equiv -2\left(1 - \left(\frac{-1}{p}\right)\right) \pmod{p}$$

which follows from (2.2) in view of Fermat's little theorem.

Below we assume that $m \not\equiv 1 \pmod{p}$. Then

$$\sum_{n=1}^{p-1}\frac{1}{m^n} \equiv \sum_{n=1}^{p-1} m^{p-1-n} = \frac{m^{p-1}-1}{m-1} \equiv 0 \pmod{p}$$

and hence

$$\sum_{n=1}^{p-1}\frac{S_n}{m^n} \equiv \sum_{n=1}^{p-1}\frac{S_n-1}{m^n} = \sum_{n=1}^{p-1}\frac{\sum_{k=1}^{n}\binom{n+k}{2k}C_k}{m^n} = \sum_{k=1}^{p-1}\frac{C_k}{m^k}\sum_{n=k}^{p-1}\frac{\binom{n+k}{2k}}{m^{n-k}} \pmod{p}.$$

Given $k \in \{1, \ldots, p-1\}$, we have

$$\sum_{n=k}^{p-1}\frac{\binom{n+k}{2k}}{m^{n-k}} = \sum_{r=0}^{p-1-k}\frac{\binom{2k+r}{r}}{m^r} = \sum_{r=0}^{p-1-k}\frac{\binom{-2k-1}{r}}{(-m)^r} \equiv \sum_{r=0}^{p-1-k}\frac{\binom{p-1-2k}{r}}{(-m)^r} \pmod{p}.$$

If $(p-1)/2 < k < p-1$, then

$$C_k = \frac{(2k)!}{k!(k+1)!} \equiv 0 \pmod{p}.$$

Therefore

$$\sum_{n=1}^{p-1} \frac{S_n}{m^n} \equiv \sum_{k=1}^{(p-1)/2} \frac{C_k}{m^k} \left(1 - \frac{1}{m}\right)^{p-1-2k} + \frac{C_{p-1}}{m^{p-1}}$$

$$\equiv \sum_{k=1}^{p-1} \frac{C_k}{m^k} \left(\frac{m}{m-1}\right)^{2k} \equiv \sum_{k=1}^{p-1} \frac{C_k}{m_0^k} \pmod{p},$$

where m_0 is an integer with $m_0 \equiv (m-1)^2/m \pmod{p}$. By Lemma 2.1,

$$\sum_{k=1}^{p-1} \frac{C_k}{m_0^k} \equiv \frac{m_0 - 4}{2} \left(1 - \left(\frac{m_0(m_0 - 4)}{p}\right)\right)$$

$$= \frac{mm_0 - 4m}{2m} \left(1 - \left(\frac{mm_0(mm_0 - 4m)}{p}\right)\right)$$

$$\equiv \frac{(m-1)^2 - 4m}{2m} \left(1 - \left(\frac{(m-1)^2 - 4m}{p}\right)\right) \pmod{p}.$$

So (1.11) follows. We are done. □

3. Proofs of Theorem 1.1 and Corollary 1.1

We need some combinatorial identities.

Lemma 3.1. *For any $n \in \mathbb{N}$, we have*

$$\sum_{r=0}^{2n} \frac{(-1)^r \binom{2n}{r}}{2n+1-2r} = \frac{(-16)^n}{(2n+1)\binom{2n}{n}} \tag{3.1}$$

and

$$\sum_{r=0}^{2n} \frac{(-1)^r \binom{2n}{r}}{(2n+1-2r)^2} = \frac{(-16)^n}{(2n+1)^2\binom{2n}{n}}, \tag{3.2}$$

that is,

$$\sum_{k=-n}^{n} \frac{(-1)^k}{(2k+1)^s} \binom{2n}{n-k} = \frac{16^n}{(2n+1)^s\binom{2n}{n}} \quad \text{for } s = 1, 2. \tag{3.3}$$

Proof. If we denote by a_n the left-hand side of (3.1), then the well-known Zeilberger algorithm (cf. [PWZ]) yields the recursion

$$a_{n+1} = -\frac{8(n+1)}{2n+3} a_n \quad (n = 0, 1, 2, \dots).$$

So (3.1) can be easily proved by induction. (3.2) is equivalent to [Su11, (2.5)] which was shown by a similar method. Clearly (3.3) is just a combination of (3.1) and (3.2). We are done. $\square$

Proof of Theorem 1.1. Let $s \in \{1, 2\}$ and let x be any p-adic integer. We claim that

$$\delta_{s,2}\,\delta_{p,3} + \sum_{n=1}^{p-1} \frac{D_n(x)}{n^s} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-x)^k}{k^s} \pmod{p}. \qquad (3.4)$$

Clearly,

$$\sum_{n=1}^{p-1} \frac{D_n(x)-1}{n^s} = \sum_{n=1}^{p-1} \frac{\sum_{k=1}^{n} \binom{n+k}{2k}\binom{2k}{k} x^k}{n^s} = \sum_{k=1}^{p-1} \binom{2k}{k} x^k \sum_{n=k}^{p-1} \frac{\binom{n+k}{2k}}{n^s}.$$

Note that $\sum_{n=1}^{p-1} 1/n^s \equiv -\delta_{s,2}\,\delta_{p,3} \pmod{p}$ since

$$\sum_{k=1}^{p-1} \frac{1}{(2k)^s} \equiv \sum_{n=1}^{p-1} \frac{1}{n^s} \pmod{p}.$$

As $p \mid \binom{2k}{k}$ for $k = (p+1)/2, \ldots, p-1$, and

$$\sum_{n=k}^{p-1} \frac{\binom{n+k}{2k}}{n^s} = \sum_{r=0}^{p-1-k} \frac{\binom{2k+r}{r}}{(k+r)^s} \equiv (-2)^s \sum_{r=0}^{p-1-k} \frac{(-1)^r \binom{p-1-2k}{r}}{(p-2k-2r)^s} \pmod{p}$$

for $k = 1, \ldots, (p-1)/2$, by applying Lemma 3.1 we obtain from the above that

$$\delta_{s,2}\,\delta_{p,3} + \sum_{n=1}^{p-1} \frac{D_n(x)}{n^s} \equiv (-2)^s \sum_{k=1}^{(p-1)/2} \binom{2k}{k} x^k \frac{(-16)^{(p-1)/2-k}}{(p-2k)^s \binom{p-1-2k}{(p-1)/2-k}}$$

$$\equiv \sum_{k=1}^{(p-1)/2} \binom{2k}{k} \frac{x^k}{k^s} 4^{(p-1)/2-k} \binom{-1/2}{(p-1)/2-k}^{-1}$$

$$\equiv \sum_{k=1}^{(p-1)/2} \binom{2k}{k} \frac{x^k}{k^s 4^k} \binom{(p-1)/2}{k}^{-1}$$

$$\equiv \sum_{k=1}^{(p-1)/2} \binom{2k}{k} \frac{x^k}{k^s 4^k} \binom{-1/2}{k}^{-1} = \sum_{k=1}^{(p-1)/2} \frac{(-x)^k}{k^s} \pmod{p}.$$

In the case $s = 2$ and $x = 1$, (3.4) yields the congruence

$$\delta_{p,3} + \sum_{n=1}^{p-1} \frac{D_n}{n^2} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} \pmod{p}.$$

By Lehmer [L, (20)],

$$\sum_{\substack{k=1 \\ 2 \mid k}}^{(p-1)/2} \frac{1}{k^2} \equiv \delta_{p,3} + \left(\frac{-1}{p}\right) E_{p-3} \pmod{p}$$

and hence

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} = 2 \sum_{\substack{k=1 \\ 2 \mid k}}^{(p-1)/2} \frac{1}{k^2} - \sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv \delta_{p,3} + 2 \left(\frac{-1}{p}\right) E_{p-3} \pmod{p}$$

since $\sum_{k=1}^{(p-1)/2} (1/k^2 + 1/(p-k)^2) = \sum_{k=1}^{p-1} 1/k^2 \equiv 0 \pmod{p}$ if $p > 3$. So (1.1) follows.

With the help of (3.4) in the case $s = x = 1$, we have

$$\sum_{n=1}^{p-1} \frac{D_n}{n} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} \left(\frac{(-1)^k}{k} + \frac{(-1)^{p-k}}{p-k}\right)$$

$$\equiv -\frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{k} \binom{p-1}{k-1} = -\frac{1}{2p} \sum_{k=1}^{p-1} \binom{p}{k} = -q_p(2) \pmod{p}.$$

This proves (1.2).

Now fix a p-adic integer x. Observe that

$$p \sum_{k=1}^{(p-1)/2} \frac{(-x)^k}{k} \equiv -2 \sum_{k=1}^{(p-1)/2} \frac{p}{2k} \binom{p-1}{2k-1} (-x)^k$$

$$= \sum_{\substack{j=1 \\ 2 \mid j}}^{p} \binom{p}{j} (-1)^{p-j} ((\sqrt{-x})^j + (-\sqrt{-x})^j)$$

$$= (-1 + \sqrt{-x})^p + (-1 - \sqrt{-x})^p + 2 \pmod{p^2}.$$

Combining this with (3.4) in the case $s = 1$ we immediately get (1.3).

The proof of Theorem 1.1 is now complete. □

Remark 3.1. By modifying our proof of (1.2) and using the new identity $\sum_{r=0}^{2n} \binom{2n}{r}/(2n+1-2r) = 2^{2n}/(2n+1)$, we can prove the congruence $\sum_{k=1}^{p-1} (-1)^k s_k/k \equiv 4((\frac{2}{p}) - 1) \pmod{p}$ for any odd prime p. Combining this with $\sum_{k=1}^{p-1} (-1)^k D_k/k \equiv -4P_{p-(\frac{2}{p})}/p \pmod{p}$ (an equivalent form of (1.7)) we obtain that $\sum_{k=1}^{p-1} (-1)^k S_k/k \equiv 4(1 - (\frac{2}{p}) - P_{p-(\frac{2}{p})}/p) \pmod{p}$.

Proof of Corollary 1.1. Note that $\omega = (-1 + \sqrt{-3})/2$ is a primitiv e cubic root of unity. If $p \neq 3$, then

$$(-1 + \sqrt{-3})^p + (-1 - \sqrt{-3})^p = (2\omega)^p + (2\omega^2)^p = -2^p$$

and hence (1.3) with $x = 3$ yields the congruence in (1.4).

Clearly (1.5) follows from (1.3) with $x = -4$.

Since $2^p - 4^p + 2 = (2 - 2^p)(2^p + 1) \equiv 6(1 - 2^{p-1}) \pmod{p^2}$, (1.3) in the case $x = -9$ yields (1.6).

The companion sequence $\{Q_n\}_{n \geqslant 0}$ of the Pell sequence is defined by $Q_0 = Q_1 = 2$ and $Q_{n+1} = 2Q_n + Q_{n-1}$ $(n = 1, 2, 3, \dots)$. It is well known that

$$Q_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n \quad \text{for all } n \in \mathbb{N}.$$

(1.3) with $x = -2$ yields the congruence

$$\sum_{k=1}^{p-1} \frac{D_k(-2)}{k} \equiv \frac{2 - Q_p}{p} \pmod{p}.$$

Since $Q_p - 2 \equiv 4P_{p - (\frac{2}{p})} \pmod{p^2}$ by the proof of [ST, Corollary 1.3], (1.7) follows immediately.

Recall that the Lucas sequence $\{L_n\}_{n \geqslant 0}$ is given by

$$L_0 = 2, \ L_1 = 1, \ \text{and } L_{n+1} = L_n + L_{n-1} \ (n = 1, 2, 3, \dots).$$

It is well known that

$$L_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n \quad \text{for all } n \in \mathbb{N}.$$

Putting $x = -5$ in (1.3) we get

$$\sum_{k=1}^{p-1} \frac{D_k(-5)}{k} \equiv \frac{2 - 2^p L_p}{p} = \frac{2^p(1 - L_p) + 2 - 2^p}{p}$$

$$\equiv -\frac{2}{p}(L_p - 1) - 2q_p(2) \pmod{p}.$$

It is known that $2(L_p - 1) \equiv 5F_{p - (\frac{p}{5})} \pmod{p^2}$ provided $p \neq 5$ (see the proof of [ST, Corollary 1.3]). So (1.8) holds if $p \neq 5$. We are done. $\square$

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