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ON HARMONIC NUMBERS AND LUCAS SEQUENCES

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Abstract. Harmonic numbers $H_k = \sum_{0<j\leqslant k} 1/j$ $(k = 0, 1, 2, \ldots)$ arise naturally in many fields of mathematics. In this paper we initiate the study of congruences involving both harmonic numbers and Lucas sequences. One of our three theorems is as follows: Let $u_0 = 0,\ u_1 = 1$, and $u_{n+1} = u_n - 4u_{n-1}$ for $n = 1, 2, 3, \ldots$. Then, for any prime $p > 5$ we have

$$\sum_{k=0}^{p-1} \frac{H_k}{2^k} u_{k+\delta} \equiv 0 \pmod{p},$$

where $\delta = 0$ if $p \equiv 1, 2, 4, 8 \pmod{15}$, and $\delta = 1$ otherwise.

1. Introduction

Harmonic numbers are those rational numbers given by

$$H_n = \sum_{0<k\leqslant n} \frac{1}{k} \quad (n \in \mathbb{N} = \{0, 1, 2, \ldots\}).$$

They play important roles in mathematics; see, e.g., [BPQ] and [BB].

In 1862 J. Wolstenholme [W] (see also [HT]) discovered that for any prime $p > 3$ we have

$$H_{p-1} = \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}.$$

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In a previous paper [Su] the author developed the arithmetic theory of harmonic numbers by proving the following fundamental congruences for primes $p>3$:

$$\sum_{k=1}^{p-1}H_k^2\equiv 2p-2 \pmod{p^2},\quad \sum_{k=1}^{p-1}H_k^3\equiv 6 \pmod{p},$$

and

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^2}\equiv 0 \pmod{p}\qquad \text{provided } p>5.$$

In this paper we initiate the investigation of congruences involving both harmonic numbers and Lucas sequences.

For $A,B\in\mathbb{Z}$ the Lucas sequences $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ and $v_n=v_n(A,B)$ $(n\in\mathbb{N})$ are defined as follows:

$$u_0=0,\ u_1=1,\ \text{and } u_{n+1}=Au_n-Bu_{n-1}\ (n=1,2,3,\ldots);$$

$$v_0=2,\ v_1=A,\ \text{and } v_{n+1}=Av_n-Bv_{n-1}\ (n=1,2,3,\ldots).$$

The sequence $\{v_n\}_{n\geqslant 0}$ is called the companion sequence of $\{u_n\}_{n\geqslant 0}$. The characteristic equation $x^2-Ax+B=0$ of the sequences $\{u_n\}_{n\geqslant 0}$ and $\{v_n\}_{n\geqslant 0}$ has two roots

$$\alpha=\frac{A+\sqrt{\Delta}}{2}\quad\text{and}\quad\beta=\frac{A-\sqrt{\Delta}}{2},$$

where $\Delta=A^2-4B$. It is well known that for any $n\in\mathbb{N}$ we have

$$Au_n+v_n=2u_{n+1},\ (\alpha-\beta)u_n=\alpha^n-\beta^n,\ \text{ and }\ v_n=\alpha^n+\beta^n.$$

(See, e.g., [R, pp. 41–44].) Note that those $F_n=u_n(1,-1)$ and $L_n=v_n(1,-1)$ are well-known Fibonacci numbers and Lucas numbers respectively.

Here is our first theorem.

Theorem 1.1. *Let $p>3$ be a prime and let $A,B\in\mathbb{Z}$ with $p\nmid A$. Then*

$$\sum_{k=1}^{p-1}\frac{v_k(A,B)H_k}{kA^k}\equiv 0 \pmod{p} \tag{1.1}$$

and

$$\sum_{k=1}^{p-1}\frac{u_k(A,B)H_k}{kA^k}\equiv\frac{2}{p}\sum_{k=1}^{p-1}\frac{u_k(A,B)}{kA^k} \pmod{p}. \tag{1.2}$$

Since $v_k(2,1)=2$ for all $k\in\mathbb{N}$, Theorem 1.1 yields the following consequence.

Corollary 1.1 ([Su]). *For any prime $p > 3$ we have*

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod p.$$

Remark. In 1987 S. W. Coffman [C] proved that $\sum_{k=1}^{\infty} H_k/(k2^k) = \pi^2/12$.

Applying Theorem 1.1 with $A = 1$ and $B = -1$ we get the following corollary.

Corollary 1.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{H_k L_k}{k} \equiv 0 \pmod p \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{F_k H_k}{k} \equiv \frac{2}{p} \sum_{k=1}^{p-1} \frac{F_k}{k} \pmod p. \tag{1.3}$$

Let ω denote the cubic root $(-1+\sqrt{-3})/2$. For $n \in \mathbb{N}$ we have

$$u_n(-1,1) = u_n(\omega+\bar{\omega}, \omega\bar{\omega}) = \frac{\omega^n - \bar{\omega}^n}{\sqrt{-3}} = \left(\frac{n}{3}\right)$$

and

$$u_n(1,1) = (-1)^{n-1} u_n(-1,1) = (-1)^{n-1} \left(\frac{n}{3}\right),$$

where $(-)$ denotes the Jacobi symbol. By induction, for any $k \in \mathbb{N}$ we have

$$u_{4k}(2,2) = 0,\ u_{4k+1}(2,2) = (-4)^k \text{ and } u_{4k+2}(2,2) = u_{4k+3}(2,2) = 2(-4)^k.$$

Now we state our second theorem.

Theorem 1.2. *Let $p > 3$ be a prime.*

(i) *Let $A, B \in \mathbb{Z}$ with $p \nmid B$ and $(\frac{A^2-4B}{p}) = 1$. Then, for any $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{p-1} (1+B^{-k}) u_k(A,B) H_k^n \equiv 0 \pmod p \tag{1.4}$$

and

$$\sum_{k=0}^{p-1} (1-B^{-k}) v_k(A,B) H_k^n \equiv 0 \pmod p. \tag{1.5}$$

(ii) *We have*

$$\sum_{k=0}^{p-1} (-1)^k \left(\frac{k}{3}\right) H_k \equiv 0 \pmod p \tag{1.6}$$

and

$$\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)H_k \equiv \frac{(\frac{p}{3})-1}{4}q_p(3)\ (\mathrm{mod}\ p), \tag{1.7}$$

where $q_p(3)$ *refers to the Fermat quotient* $(3^{p-1}-1)/p$. *Also,*

$$\sum_{k=0}^{p-1}(-1)^k\left(\frac{k}{3}\right)kH_k \equiv \frac{1-(\frac{p}{3})}{2}\ (\mathrm{mod}\ p) \tag{1.8}$$

and

$$\sum_{k=0}^{p-1}(1+2^{-k})u_k(2,2)H_k \equiv 0\ (\mathrm{mod}\ p). \tag{1.9}$$

Since $F_{2k}=u_k(3,1)$, $F_k=u_k(1,-1)$ and $L_k=v_k(1,-1)$ for all $k\in\mathbb{N}$, Theorem 1.2(i) implies the following result.

Corollary 1.3. *Let* p *be a prime with* $p\equiv\pm1\ (\mathrm{mod}\ 5)$. *Then*

$$\sum_{k=0}^{p-1}F_{2k}H_k^n \equiv \sum_{\substack{k=0\\ 2\mid k}}^{p-1}F_kH_k^n \equiv \sum_{\substack{k=0\\ 2\nmid k}}^{p-1}L_kH_k^n \equiv 0\ (\mathrm{mod}\ p) \tag{1.10}$$

for every $n=0,1,2,\ldots$.

Our third theorem seems curious and unexpected.

Theorem 1.3. *Let* $p>5$ *be a prime. If* $(\frac{p}{15})=1$, *i.e.,* $p\equiv1,2,4,8\ (\mathrm{mod}\ 15)$, *then*

$$\sum_{k=0}^{p-1}\frac{u_k(1,4)}{2^k}H_k \equiv 0\ (\mathrm{mod}\ p). \tag{1.11}$$

If $(\frac{p}{15})=-1$, *i.e.,* $p\equiv7,11,13,14\ (\mathrm{mod}\ 15)$, *then*

$$\sum_{k=0}^{p-1}\frac{u_{k+1}(1,4)}{2^k}H_k \equiv 0\ (\mathrm{mod}\ p). \tag{1.12}$$

In the next section we are going to prove Theorems 1.1 and 1.2. Section 3 is devoted to the proof of Theorem 1.3.

To conclude this section we pose three related conjectures.

Recall that harmonic numbers of the second order are given by

$$H_n^{(2)}=\sum_{0<k\leqslant n}\frac{1}{k^2}\quad (n=0,1,2,\ldots).$$

Conjecture 1.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-2)^k\binom{2k}{k}H_k^{(2)}\equiv\frac{2}{3}q_p(2)^2 \pmod p$$

where $q_p(2)=(2^{p-1}-1)/p$. If $p>5$, then we have

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}H_k^{(2)}\equiv\frac{5}{2}\left(\frac{p}{5}\right)\frac{F^2_{p-(\frac{p}{5})}}{p^2} \pmod p.$$

Conjecture 1.2. *Let p be an odd prime. Then*

$$\begin{aligned}
\sum_{k=0}^{p-1}\frac{u_k(2,-1)}{(-8)^k}\binom{2k}{k}^2 &\equiv 0 \pmod p \quad \text{if } p\equiv 5 \pmod 8,\\
\sum_{k=0}^{p-1}\frac{u_k(2,-1)}{32^k}\binom{2k}{k}^2 &\equiv 0 \pmod p \quad \text{if } p\equiv 7 \pmod 8,\\
\sum_{k=0}^{p-1}\frac{v_k(2,-1)}{(-8)^k}\binom{2k}{k}^2 &\equiv 0 \pmod p \quad \text{if } p\equiv 5,7 \pmod 8,\\
\sum_{k=0}^{p-1}\frac{v_k(2,-1)}{32^k}\binom{2k}{k}^2 &\equiv 0 \pmod p \quad \text{if } p\equiv 5 \pmod 8.
\end{aligned}$$

Also,

$$\sum_{k=0}^{p-1}\frac{u_k(4,1)}{4^k}\binom{2k}{k}^2 \equiv 0 \pmod p \quad \text{if } p\equiv 2 \pmod 3,$$

$$\sum_{k=0}^{p-1}\frac{u_k(4,1)}{64^k}\binom{2k}{k}^2 \equiv 0 \pmod p \quad \text{if } p\equiv 11 \pmod{12},$$

$$\sum_{k=0}^{p-1}\frac{v_k(4,1)}{4^k}\binom{2k}{k}^2 \equiv \sum_{k=0}^{p-1}\frac{v_k(4,1)}{64^k}\binom{2k}{k}^2 \equiv 0 \pmod p \text{ if } p\equiv 5 \pmod{12}.$$

Conjecture 1.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}((-1)^k-(-3)^{-k})\equiv\left(\frac{p}{3}\right)(3^{p-1}-1) \pmod{p^3}.$$

If $p \equiv \pm 1 \pmod{12}$, *then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}(-1)^k u_k(4,1) \equiv (-1)^{(p-1)/2}u_{p-1}(4,1) \pmod{p^3}.$$

If $p \equiv \pm 1 \pmod 8$, *then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}\frac{u_k(4,2)}{(-2)^k} \equiv (-1)^{(p-1)/2}u_{p-1}(4,2) \pmod{p^3}.$$

2. Proofs of Theorems 1.1 and 1.2

Let p be an odd prime. For $k = 0, 1, \ldots, p-1$ we obviously have

$$(-1)^k\binom{p-1}{k} = \prod_{0<j\leqslant k}\left(1-\frac{p}{j}\right) \equiv 1 - pH_k \pmod{p^2}. \tag{2.1}$$

This basic fact is useful in the study of congruences involving harmonic numbers.

Lemma 2.1. *Let* $n \geqslant j > 0$ *be integers. Then*

$$\sum_{k=j}^{n}\binom{k-1}{j-1} = \binom{n}{j}.$$

Proof. This is a known identity due to Shih-chieh Chu (cf. (5.26) of [GKP, p. 169]). By comparing the coefficients of x^{j-1} on both sides of the identity

$$\sum_{k=1}^{n}(1+x)^{k-1} = \frac{(1+x)^n-1}{(1+x)-1},$$

we get a simple proof of the desired identity. $\square$

Proof of Theorem 1.1. Let α and β be the two roots of the equation

$x^2 - Ax + B = 0$. In view of Lemma 2.1,

$$\begin{aligned}&\sum_{j=1}^{p-1}\frac{v_j(A,B)}{jA^j}(-1)^j\binom{p-1}{j}\\ =&\sum_{j=1}^{p-1}\frac{v_j(A,B)}{jA^j}(-1)^j\sum_{k=j}^{p-1}\binom{k-1}{j-1}\\ =&\sum_{k=1}^{p-1}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{(-1)^jv_j(A,B)}{jA^j}\\ =&\sum_{k=1}^{p-1}\frac{1}{k}\sum_{j=1}^{k}\binom{k}{j}\left(\left(-\frac{\alpha}{A}\right)^j+\left(-\frac{\beta}{A}\right)^j\right)\\ =&\sum_{k=1}^{p-1}\frac{(1-\alpha/A)^k+(1-\beta/A)^k-2}{k}=\sum_{k=1}^{p-1}\frac{\beta^k+\alpha^k}{kA^k}-2\sum_{k=1}^{p-1}\frac{1}{k}\\ \equiv&\sum_{k=1}^{p-1}\frac{v_k(A,B)}{kA^k}\pmod{p^2}.\end{aligned}$$

Since

$$(-1)^k\binom{p-1}{k}-1\equiv -pH_k\pmod{p^2}\quad\text{for } k=1,\ldots,p-1,$$

(1.1) follows from the above.

Similarly,

$$\begin{aligned}&(\alpha-\beta)\sum_{j=1}^{p-1}\frac{u_j(A,B)}{jA^j}(-1)^j\binom{p-1}{j}\\ =&\sum_{j=1}^{p-1}\frac{(\alpha-\beta)u_j(A,B)}{jA^j}(-1)^j\sum_{k=j}^{p-1}\binom{k-1}{j-1}\\ =&\sum_{k=1}^{p-1}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{(-1)^j(\alpha-\beta)u_j(A,B)}{jA^j}\\ =&\sum_{k=1}^{p-1}\frac{1}{k}\sum_{j=1}^{k}\binom{k}{j}\left(\left(-\frac{\alpha}{A}\right)^j-\left(-\frac{\beta}{A}\right)^j\right)\\ =&\sum_{k=1}^{p-1}\frac{(1-\alpha/A)^k-(1-\beta/A)^k}{k}=\sum_{k=1}^{p-1}\frac{\beta^k-\alpha^k}{kA^k}\\ =&(\beta-\alpha)\sum_{k=1}^{p-1}\frac{u_k(A,B)}{kA^k}.\end{aligned}$$

Thus, if $\Delta = A^2 - 4B \neq 0$ then

$$\sum_{k=1}^{p-1} \frac{u_k(A,B)}{kA^k}(1 - pH_k) \equiv -\sum_{k=1}^{p-1} \frac{u_k(A,B)}{kA^k} \pmod{p^2}$$

and hence (1.2) follows.

Now suppose that $\Delta = 0$. By induction, $u_k = k(A/2)^{k-1}$ for $k = 0, 1, 2, \ldots$. Thus

$$\begin{aligned}
&\sum_{j=1}^{p-1} \frac{u_j(A,B)}{jA^j}(-1)^j \binom{p-1}{j} \\
=&\sum_{j=1}^{p-1} \frac{(-1)^j}{2^{j-1}A} \binom{p-1}{j} = \frac{2}{A} \sum_{j=1}^{p-1} \binom{p-1}{j} \left(-\frac{1}{2}\right)^j \\
=&\frac{2}{A}\left(\left(1 - \frac{1}{2}\right)^{p-1} - 1\right) = \frac{2}{A} \cdot \frac{1 - 2^{p-1}}{2^{p-1}} \\
=&-\frac{2}{A} \sum_{j=0}^{p-2} \frac{2^j}{2^{p-1}} = -\frac{2}{A} \sum_{k=1}^{p-1} \frac{1}{2^k} = -\sum_{k=1}^{p-1} \frac{u_k(A,B)}{kA^k}.
\end{aligned}$$

This yields (1.2) with the help of (2.1).

So far we have completed the proof of Theorem 1.1. $\square$

Lemma 2.2. *Let $A, B \in \mathbb{Z}$ and $\Delta = A^2 - 4B$. Suppose that p is an odd prime with $p \nmid B\Delta$. Then we have the congruence*

$$\left(\frac{A \pm \sqrt{\Delta}}{2}\right)^{p - (\frac{\Delta}{p})} \equiv B^{(1 - (\frac{\Delta}{p}))/2} \pmod{p} \tag{2.2}$$

in the ring of algebraic integers.

Proof. Both $\alpha = (A + \sqrt{\Delta})/2$ and $\beta = (A - \sqrt{\Delta})/2$ are roots of the equation $x^2 - Ax + B = 0$. Observe that

$$2\alpha^p \equiv 2^p \alpha^p = (A + \sqrt{\Delta})^p \equiv A^p + (\sqrt{\Delta})^p \equiv A + \left(\frac{\Delta}{p}\right)\sqrt{\Delta} \pmod{p}.$$

Similarly,

$$2\beta^p \equiv A - \left(\frac{\Delta}{p}\right)\sqrt{\Delta} \pmod{p}.$$

Thus, if $(\frac{\Delta}{p}) = 1$, then

$$\alpha^{p-1}B = \alpha^p\beta \equiv \alpha\beta = B \pmod{p} \text{ and } \beta^{p-1}B = \alpha\beta^p \equiv \alpha\beta = B \pmod{p},$$

hence $\alpha^{p-1} \equiv 1 \equiv \beta^{p-1} \pmod p$. When $(\frac{\Delta}{p}) = -1$, by the above we have

$$\alpha^{p+1} = \alpha\alpha^p \equiv \alpha\beta = B \pmod p \quad \text{and} \quad \beta^{p+1} = \beta^p\beta \equiv \alpha\beta = B \pmod p.$$

This concludes the proof. □

Proof of Theorem 1.2. (i) The equation $x^2 - Ax + B = 0$ has two roots

$$\alpha = \frac{A + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{\Delta}}{2},$$

where $\Delta = A^2 - 4B$. Also,

$$H_{p-1-k} = H_{p-1} - \sum_{0<j\leqslant k} \frac{1}{p-j} \equiv H_k \pmod p \quad \text{for } k = 0, 1, \ldots, p-1.$$

As $(\frac{\Delta}{p}) = 1$, with the help of Lemma 2.2, for any $n \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{k=0}^{p-1} v_k(A,B)H_k^n =& \sum_{k=0}^{p-1} v_{p-1-k}(A,B)H_{p-1-k}^n \\ \equiv& \sum_{k=0}^{p-1} \left(\alpha^{p-1-k} + \beta^{p-1-k}\right) H_k^n \\ \equiv& \sum_{k=0}^{p-1} \left(\frac{\beta^k}{B^k} + \frac{\alpha^k}{B^k}\right) H_k^n = \sum_{k=0}^{p-1} B^{-k} v_k(A,B) H_k^n \pmod p.\end{aligned}$$

This proves (1.5). Similarly,

$$\begin{aligned}(\alpha - \beta)\sum_{k=0}^{p-1} u_k(A,B)H_k^n =& \sum_{k=0}^{p-1} (\alpha^{p-1-k} - \beta^{p-1-k})H_{p-1-k}^n \\ \equiv \sum_{k=0}^{p-1} \left(\frac{\beta^k}{B^k} - \frac{\alpha^k}{B^k}\right) H_k^n =& (\beta - \alpha)\sum_{k=0}^{p-1} B^{-k} u_k(A,B) H_k^n \pmod p.\end{aligned}$$

As $(\alpha - \beta)^2 = \Delta \not\equiv 0 \pmod p$, (1.4) follows.

(ii) If $p \equiv 1 \pmod 3$ (i.e., $(\frac{-3}{p}) = 1$), then by putting $A = \pm 1$ and $B = 1$ in (1.4) we get (1.6) and (1.7). Now we prove (1.6) and (1.7) in the case $p \equiv 2 \pmod 3$. Observe that

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^k\left(\frac{k}{3}\right)\left(\binom{p-1}{k}(-1)^k-1\right)\\=&\sum_{k=0}^{p-1}\binom{p-1}{k}\left(\frac{k}{3}\right)-\sum_{k=0}^{p-1}(-1)^k\left(\frac{k}{3}\right)\\=&\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\omega^k-\bar\omega^k}{\sqrt{-3}}-\sum_{k=0}^{p-1}(-1)^k\frac{\omega^k-\bar\omega^k}{\sqrt{-3}}\\=&\frac{1}{\sqrt{-3}}\left((1+\omega)^{p-1}-(1+\bar\omega)^{p-1}\right)-\frac{1}{\sqrt{-3}}\left(\frac{1+\omega^p}{1+\omega}-\frac{1+\bar\omega^p}{1+\bar\omega}\right)\\=&\frac{1}{\sqrt{-3}}\left((-\omega^2)^{p-1}-(-\omega)^{p-1}-\frac{-\omega^{2p}}{-\omega^2}+\frac{-\omega^p}{-\omega}\right)=0.\end{aligned}$$

Also,

$$\begin{aligned}&\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)\left(\binom{p-1}{k}(-1)^k-1\right)\\=&\sum_{k=0}^{p-1}\binom{p-1}{k}(-1)^k\frac{\omega^k-\bar\omega^k}{\sqrt{-3}}-\sum_{j=1}^{(p-2)/3}\sum_{d=0}^{2}\left(\frac{3j-d}{3}\right)-\left(\frac{p-1}{3}\right)\\=&\frac{1}{\sqrt{-3}}\left((1-\omega)^{p-1}-(1-\omega^2)^{p-1}\right)-\left(\frac{p-1}{3}\right)\\=&\frac{1}{\sqrt{-3}}(1-\omega)^{p-1}\left(1-(-\omega^2)^{p-1}\right)-1\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)\left(\binom{p-1}{k}(-1)^k-1\right)\\=&\frac{1-\omega^2}{\sqrt{-3}}(1-\omega)^{p-1}-1=\frac{-\omega^2}{\sqrt{-3}}(1-\omega)^p-1\\=&\frac{-\omega^2}{\sqrt{-3}}(\sqrt{-3}\,\omega^2)^p-1=-(-3)^{(p-1)/2}\omega^{2+2p}-1\\=&-\left((-3)^{(p-1)/2}-\left(\frac{-3}{p}\right)\right)\\\equiv&-\frac{(\frac{-3}{p})}{2}\left((-3)^{p-1}-\left(\frac{-3}{p}\right)^2\right)=\frac{3^{p-1}-1}{2}\pmod{p^2}.\end{aligned}$$

Combining these with (2.1) we immediately obtain (1.6) and (1.7).

Next we show (1.8). Observe that

$$\begin{aligned}&\sum_{k=0}^{p-(\frac{p}{3})}(-1)^k\left(\frac{k}{3}\right)k=\sum_{q=1}^{(p-(\frac{p}{3}))/6}\sum_{r=0}^{5}(-1)^{6q-r}\left(\frac{6q-r}{3}\right)(6q-r)\\=&\sum_{q=1}^{(p-(\frac{p}{3}))/6}((6q-1)+(6q-2)-(6q-4)-(6q-5))=p-\left(\frac{p}{3}\right)\end{aligned}$$

and hence

$$\sum_{k=0}^{p-1}(-1)^k\left(\frac{k}{3}\right)k=\frac{1+(\frac{p}{3})}{2}p-\left(\frac{p}{3}\right).$$

Also,

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{p-1}{k}\left(\frac{k}{3}\right)k=(p-1)\sum_{k=1}^{p-1}\binom{p-2}{k-1}\left(\frac{k}{3}\right)\\=&(p-1)\sum_{j=0}^{p-2}\binom{p-2}{j}\frac{\omega^{j+1}-\bar\omega^{j+1}}{\sqrt{-3}}=\frac{p-1}{\sqrt{-3}}\left(\omega(1+\omega)^{p-2}-\bar\omega(1+\bar\omega)^{p-2}\right)\\=&\frac{p-1}{\sqrt{-3}}\left(\omega(-\omega^2)^{p-2}-\omega^2(-\omega)^{p-2}\right)=\frac{p-1}{\sqrt{-3}}\left(\omega^p-\omega^{2p}\right)=(p-1)\left(\frac{p}{3}\right).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^k\left(\frac{k}{3}\right)k\left(\binom{p-1}{k}(-1)^k-1\right)\\=&(p-1)\left(\frac{p}{3}\right)-\left(\frac{1+(\frac{p}{3})}{2}p-\left(\frac{p}{3}\right)\right)=\frac{(\frac{p}{3})-1}{2}p.\end{aligned}$$

This implies (1.8) due to (2.1).

Finally we prove (1.9). If $p\equiv 1 \pmod 4$ (i.e., $(\frac{-4}{p})=1$), then (1.4) in the case $A=B=2$ yields (1.9). Below we assume that $p\equiv 3 \pmod 4$. Note that

$$u_k(2,2)=\frac{(1+i)^k-(1-i)^k}{2i}\quad\text{for all }k\in\mathbb{N}.$$

Thus

$$\begin{aligned}
&2i\sum_{k=0}^{p-1}\binom{p-1}{k}(-1)^k(2^{-k}+1)u_k(2,2)\\
=&\sum_{k=0}^{p-1}\binom{p-1}{k}((-2)^{-k}+(-1)^k)\left((1+i)^k-(1-i)^k\right)\\
=&\left(1-\frac{1+i}{2}\right)^{p-1}-\left(1-\frac{1-i}{2}\right)^{p-1}+(1-(1+i))^{p-1}-(1-(1-i))^{p-1}\\
=&\left(\frac{1-i}{2}\right)^{p-1}-\left(\frac{1+i}{2}\right)^{p-1}+(-i)^{p-1}-i^{p-1}\\
=&\left(\frac{-2i}{4}\right)^{(p-1)/2}-\left(\frac{2i}{4}\right)^{(p-1)/2}=2i\frac{i^{(p+1)/2}}{2^{(p-1)/2}}
\end{aligned}$$

and hence

$$\sum_{k=0}^{p-1}\binom{p-1}{k}(-1)^k(2^{-k}+1)u_k(2,2)=\frac{(-1)^{(p+1)/4}}{2^{(p-1)/2}}. \tag{2.3}$$

Also,

$$\begin{aligned}
&2i\sum_{k=0}^{p-1}(2^{-k}+1)u_k(2,2)\\
=&\sum_{k=0}^{p-1}(2^{-k}+1)\left((1+i)^k-(1-i)^k\right)\\
=&\frac{1-((1+i)/2)^p}{1-(1+i)/2}-\frac{1-((1-i)/2)^p}{1-(1-i)/2}+\frac{1-(1+i)^p}{1-(1+i)}-\frac{1-(1-i)^p}{1-(1-i)}\\
=&(1+i)-2\left(\frac{1+i}{2}\right)^{p+1}-\left((1-i)-2\left(\frac{1-i}{2}\right)^{p+1}\right)\\
&+i-i(1+i)(1+i)^{p-1}+i-i(1-i)(1-i)^{p-1}\\
=&2i-2\left(\frac{2i}{4}\right)^{(p+1)/2}+2\left(\frac{-2i}{4}\right)^{(p+1)/2}\\
&+2i+(1-i)(2i)^{(p-1)/2}-(1+i)(-2i)^{(p-1)/2}\\
=&4i+2(2i)^{(p-1)/2}=2i\left(2+i^{(p-3)/2}2^{(p-1)/2}\right)
\end{aligned}$$

and hence

$$\sum_{k=0}^{p-1}(2^{-k}+1)u_k(2,2)=2-(-1)^{(p+1)/4}2^{(p-1)/2}. \tag{2.4}$$

Combining (2.1), (2.3) and (2.4) we obtain

$$\begin{aligned}
&-p\sum_{k=0}^{p-1}(2^{-k}+1)u_k(2,2)H_k\\
\equiv&\frac{(-1)^{(p+1)/4}}{2^{(p-1)/2}}-2+(-1)^{(p+1)/4}2^{(p-1)/2}\\
\equiv&\frac{(-1)^{(p+1)/4}}{2^{(p-1)/2}}\left(2^{(p-1)/2}-(-1)^{(p+1)/4}\right)^2\equiv 0 \pmod{p^2}
\end{aligned}$$

since

$$\left(\frac{2}{p}\right)=(-1)^{(p^2-1)/8}=(-1)^{(p+1)/4\times(p-1)/2}=(-1)^{(p+1)/4}.$$

Therefore (1.9) holds.

By the above, we have completed the proof of Theorem 1.2. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let $A,B\in\mathbb{Z}$. Let p be an odd prime with $(\frac{B}{p})=1$. Suppose that $b^2\equiv B \pmod p$ where $b\in\mathbb{Z}$. Then*

$$u_{(p-1)/2}(A,B)\equiv\begin{cases}0 \pmod p & \text{if } (\frac{A^2-4B}{p})=1,\\ \frac{1}{b}(\frac{A-2b}{p}) \pmod p & \text{if } (\frac{A^2-4B}{p})=-1;\end{cases}$$

and

$$u_{(p+1)/2}(A,B)\equiv\begin{cases}(\frac{A-2b}{p}) \pmod p & \text{if } (\frac{A^2-4B}{p})=1,\\ 0 \pmod p & \text{if } (\frac{A^2-4B}{p})=-1.\end{cases}$$

Proof. The congruences are known results, see, e.g., [S]. □

Lemma 3.2. *Let $u_n=u_n(1,4)$ for $n\in\mathbb{N}$. Then, for any prime $p>5$ we have*

$$u_p-2^{p-1}\left(\frac{p}{15}\right)\equiv 2^{(\frac{p}{15})-2}u_{p-(\frac{p}{15})} \pmod{p^2}. \tag{3.1}$$

Proof. The two roots

$$\alpha=\frac{1+\sqrt{-15}}{2}\quad\text{and}\quad\beta=\frac{1-\sqrt{-15}}{2}$$

of the equation $x^2-x+4=0$ are algebraic integers. Clearly

$$-15u_p=(\alpha-\beta)^2u_p=(\alpha-\beta)(\alpha^p-\beta^p)\equiv(\alpha-\beta)^{p+1}=(-15)^{(p+1)/2} \pmod p$$

and hence

$$u_p \equiv (-15)^{(p-1)/2} \equiv \left(\frac{-15}{p}\right) = \left(\frac{p}{15}\right) \pmod{p}.$$

Also,

$$v_p = \alpha^p + \beta^p \equiv (\alpha+\beta)^p = 1 \pmod{p},$$

where v_n refers to $v_n(1,4)$. (In fact, both $u_p(A,B)$ and $v_p(A,B)$ modulo p are known for any $A, B \in \mathbb{Z}$.) By induction, $u_n + v_n = 2u_{n+1}$ for any $n \in \mathbb{N}$.

Case 1. $(\frac{p}{15}) = 1$. In this case,

$$4u_{p-1} = u_p - u_{p+1} = \frac{u_p - v_p}{2} \equiv \frac{1-1}{2} = 0 \pmod{p}$$

and

$$v_{p-1} = 2u_p - u_{p-1} \equiv 2 \equiv 2^p \pmod{p}.$$

Since

$$\begin{aligned}(v_{p-1} - 2^p)(v_{p-1} + 2^p) =&(\alpha^{p-1} + \beta^{p-1})^2 - 4(\alpha\beta)^{p-1}\\ =&(\alpha^{p-1} - \beta^{p-1})^2 = -15u_{p-1}^2 \equiv 0 \pmod{p^2},\end{aligned}$$

we must have $v_{p-1} \equiv 2^p \pmod{p^2}$ and hence

$$2u_p = u_{p-1} + v_{p-1} \equiv u_{p-1} + 2^p \pmod{p^2}.$$

Case 2. $(\frac{p}{15}) = -1$. In this case,

$$2u_{p+1} = u_p + v_p \equiv -1 + 1 = 0 \pmod{p}$$

and

$$v_{p+1} = 2u_{p+2} - u_{p+1} = u_{p+1} - 8u_p \equiv 8 \equiv 2^{p+2} \pmod{p}.$$

As

$$\begin{aligned}(v_{p+1} - 2^{p+2})(v_{p+1} + 2^{p+2}) =&(\alpha^{p+1} + \beta^{p+1})^2 - 4(\alpha\beta)^{p+1}\\ =&(\alpha^{p+1} - \beta^{p+1})^2 = -15u_{p+1}^2 \equiv 0 \pmod{p^2},\end{aligned}$$

we must have $v_{p+1} \equiv 2^{p+2} \pmod{p^2}$ and hence

$$\begin{aligned}8u_p =&2(u_{p+1} - u_{p+2}) = 2u_{p+1} - (u_{p+1} + v_{p+1})\\ =&u_{p+1} - v_{p+1} \equiv u_{p+1} - 2^{p+2} \pmod{p^2}.\end{aligned}$$

Combining the above, we immediately obtain the desired result. □

Proof of Theorem 1.3. Set $\delta = (1 - (\frac{p}{15}))/2$. Then

$$\sum_{k=0}^{p-1} \frac{u_{k+\delta}}{2^k} H_k \equiv \sum_{k=0}^{p-1} \frac{u_{k+\delta}}{2^k} \cdot \frac{1 - (-1)^k \binom{p-1}{k}}{p} \pmod{p}.$$

So it suffices to show

$$\sum_{k=0}^{p-1} u_{k+\delta} 2^{p-1-k} \equiv \sum_{k=0}^{p-1} \binom{p-1}{k} u_{k+\delta} (-2)^{p-1-k} \pmod{p^2}, \tag{3.2}$$

which implies (1.11) and (1.12) in the cases $\delta = 0, 1$ respectively. Recall that

$$u_{k+\delta} = \frac{\alpha^{k+\delta} - \beta^{k+\delta}}{\alpha - \beta},$$

where

$$\alpha = \frac{1 + \sqrt{-15}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{-15}}{2}$$

are the two roots of the equation $x^2 - x + 4 = 0$. Since

$$\sum_{k=0}^{p-1} x^k y^{p-1-k} = \frac{x^p - y^p}{x - y} \quad \text{and} \quad \sum_{k=0}^{p-1} \binom{p-1}{k} x^k y^{p-1-k} = (x+y)^{p-1},$$

(3.2) can be rewritten as follows:

$$\begin{aligned} &\frac{1}{\alpha - \beta}\left(\alpha^\delta \frac{\alpha^p - 2^p}{\alpha - 2} - \beta^\delta \frac{\beta^p - 2^p}{\beta - 2}\right) \\ \equiv &\frac{\alpha^\delta (\alpha - 2)^{p-1} - \beta^\delta (\beta - 2)^{p-1}}{\alpha - \beta} \pmod{p^2}. \end{aligned} \tag{3.3}$$

Note that $(\alpha - 2)(\beta - 2) = 4 + \alpha\beta - 2(\alpha + \beta) = 4 + 4 - 2 = 6$ and

$$\begin{aligned} &\alpha^\delta(\alpha^p - 2^p)(\beta - 2) - \beta^\delta(\alpha - 2)(\beta^p - 2^p) \\ =&(2^p - \alpha^p)(\alpha^{\delta+1} + \alpha^\delta) + (\beta^{\delta+1} + \beta^\delta)(\beta^p - 2^p) \\ =&2^p(\alpha^{\delta+1} - \beta^{\delta+1} + \alpha^\delta - \beta^\delta) - (\alpha^{p+\delta+1} - \beta^{p+\delta+1}) - (\alpha^{p+\delta} - \beta^{\beta+\delta}) \\ =&(\alpha - \beta)\left(2^p(u_{\delta+1} + u_\delta) - (u_{p+\delta+1} + u_{p+\delta})\right) \\ =&(\alpha - \beta)\left(2^{p+\delta} - 2u_{p+\delta} + 4u_{p+\delta-1}\right). \end{aligned}$$

So the left-hand side of (3.3) coincides with

$$\frac{2^{p+\delta} - 2u_{p+\delta} + 4u_{p+\delta-1}}{6} = \frac{2^{p+\delta-1} - u_{p+\delta} + 2u_{p+\delta-1}}{3}.$$

Since $(\alpha-2)^2=-3\alpha$ and $(\beta-2)^2=-3\beta$, we have

$$\begin{aligned}&\frac{\alpha^\delta(\alpha-2)^{p-1}-\beta^\delta(\beta-2)^{p-1}}{\alpha-\beta}\\ =&\frac{\alpha^\delta(-3\alpha)^{(p-1)/2}-\beta^\delta(-3\beta)^{(p-1)/2}}{\alpha-\beta}\\ =&(-3)^{(p-1)/2}\frac{\alpha^{(p-1)/2+\delta}-\beta^{(p-1)/2+\delta}}{\alpha-\beta}=(-3)^{(p-1)/2}u_{(p-(\frac{p}{15}))/2}.\end{aligned}$$

Applying Lemma 3.1 with $A=1$ and $B=4$, we get that

$$u_{(p-(\frac{p}{15}))/2}\equiv 0 \pmod p\quad\text{and}\quad u_{(p+(\frac{p}{15}))/2}\equiv\left(\frac{-3}{p}\right)2^{((\frac{p}{15})-1)/2} \pmod p. \tag{3.4}$$

For any $n\in\mathbb{N}$ we have

$$u_{2n}=\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}=\frac{\alpha^n-\beta^n}{\alpha-\beta}(\alpha^n+\beta^n)=u_nv_n.$$

If $(\frac{p}{15})=1$, then by (3.4) we have $u_{(p-1)/2}\equiv 0 \pmod p$ and

$$v_{(p-1)/2}=2u_{(p+1)/2}-u_{(p-1)/2}\equiv 2\left(\frac{-3}{p}\right) \pmod p,$$

hence

$$\begin{aligned}u_{p-1}=&u_{(p-1)/2}v_{(p-1)/2}\\ \equiv&2\left(\frac{-3}{p}\right)u_{(p-1)/2}\equiv 2(-3)^{(p-1)/2}u_{(p-1)/2} \pmod{p^2}.\end{aligned}$$

If $(\frac{p}{15})=-1$, then by (3.4) we have $u_{(p+1)/2}\equiv 0 \pmod p$ and

$$\begin{aligned}v_{(p+1)/2}=&2u_{(p+3)/2}-u_{(p+1)/2}=u_{(p+1)/2}-8u_{(p-1)/2}\\ \equiv&-8\left(\frac{-3}{p}\right)2^{-1}=-4\left(\frac{-3}{p}\right) \pmod p,\end{aligned}$$

hence

$$\begin{aligned}u_{p+1}=&u_{(p+1)/2}v_{(p+1)/2}\\ \equiv&-4\left(\frac{-3}{p}\right)u_{(p+1)/2}\equiv -4(-3)^{(p-1)/2}u_{(p+1)/2} \pmod{p^2}.\end{aligned}$$

Thus the right-hand side of (3.3) is congruent to $u_{p-(\frac{p}{15})}/(2(-2)^\delta)$ mod p^2.

By the above, (3.3) is equivalent to the following congruence

$$\frac{2^{p+\delta-1} - u_{p+\delta} + 2u_{p+\delta-1}}{3} \equiv \frac{u_{p-(\frac{p}{15})}}{2(-2)^{\delta}} \pmod{p^2}. \tag{3.5}$$

If $(\frac{p}{15}) = 1$, then $\delta = 0$, and hence (3.5) reduces to the congruence

$$2(2^{p-1} - u_p + 2u_{p-1}) \equiv 3u_{p-1} \pmod{p^2}$$

which is equivalent to (3.1) since $(\frac{p}{15}) = 1$. When $(\frac{p}{15}) = -1$, we have $\delta = 1$ and hence (3.5) can be rewritten as

$$-4(2^p - u_{p+1} + 2u_p) \equiv 3u_{p+1} \pmod{p^2}$$

which follows from (3.1) since $(\frac{p}{15}) = -1$. This concludes the proof. $\square$

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