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SOME q-CONGRUENCES RELATED TO 3-ADIC VALUATIONS

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Abstract. In 1992, Strauss, Shallit and Zagier proved that for any positive integer a,

$$\sum_{k=0}^{3^a-1}\binom{2k}{k}\equiv 0 \pmod{3^{2a}}$$

and furthermore

$$\frac{1}{3^{2a}}\sum_{k=0}^{3^a-1}\binom{2k}{k}\equiv 1 \pmod 3.$$

Recently a q-analogue of the first congruence was conjectured by Guo and Zeng. In this paper we prove the conjecture of Guo and Zeng, and also give a q-analogue of the second congruence.

1. Introduction

Partially motivated by the work of Pan and Sun [PS], Sun and Tauraso [ST2] proved that for any prime p and $a\in\mathbb{Z}^+=\{1,2,3,\dots\}$,

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right) \pmod{p^2},$$

where $(-)$ is the Legendre symbol. (See also [ST1] and [ZPS] for related results.) When checking whether there are composite numbers n such that

$$\sum_{k=0}^{n-1}\binom{2k}{k}\equiv\left(\frac{n}{3}\right) \pmod{n^2},$$

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Sun and Tauraso found that

$$\nu_3\bigg(\sum_{k=0}^{3^a-1}\binom{2k}{k}\bigg) \geqslant 2a \quad \text{for } a=1,2,3,\dots, \tag{1.1}$$

where $\nu_3(m)$ denotes the 3-adic valuation of an integer m (i.e., $\nu_3(m) = \sup\{a \in \mathbb{N} :\ 3^a \mid m\}$ with $\mathbb{N} = \{0,1,2,\dots\}$). However, a refinement of this was proved earlier by Strauss, Shallit and Zagier [SSZ] in 1992.

Theorem 1.1 (Strauss, Shallit and Zagier [SSZ]). *For any $a \in \mathbb{Z}^+$ we have*

$$\sum_{k=0}^{3^a-1}\binom{2k}{k} \equiv 3^{2a} \pmod{3^{2a+1}}. \tag{1.2}$$

Furthermore,

$$\frac{\sum_{k=0}^{n-1}\binom{2k}{k}}{n^2\binom{2n}{n}} \equiv -1 \pmod 3 \quad \textit{for all } n \in \mathbb{Z}^+.$$

Recall that the usual q-analogue of a natural number n is

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{0\leqslant k<n} q^k$$

which tends to n as $q \to 1$. For $d \in \mathbb{Z}^+$ the d-th cyclotomic polynomial in the variable q is given by

$$\Phi_d(q) = \prod_{\substack{r=1\\(r,d)=1}}^{d}\left(q - e^{2\pi i r/d}\right).$$

The polynomial $\Phi_d(q)$ has integer coefficients. Given a positive integer $n > 1$, it is clear that

$$[n]_q = \frac{q^n-1}{q-1} = \prod_{k=1}^{n-1}\left(q - e^{2\pi i k/n}\right) = \prod_{\substack{d\mid n\\ d>1}}\Phi_d(x).$$

It is well known that if $d_1, d_2 \in \mathbb{Z}^+$ are distinct then $\Phi_{d_1}(q)$ and $\Phi_{d_2}(q)$ are relatively prime in the polynomial ring $\mathbb{Z}[q]$. If p is a prime and a is a positive integer, then

$$\Phi_{p^a}(q) = \frac{q^{p^a}-1}{q^{p^{a-1}}-1} = [p]_{q^{p^{a-1}}} \quad\text{and}\quad [p^a]_q = \prod_{j=1}^{a}\Phi_{p^j}(q).$$

For $n,k\in\mathbb{N}$ the usual q-analogue of the binomial coefficient $\binom{n}{k}$ is the following q-binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} ([n]_q\cdots[n-k+1]_q)/([1]_q\cdots[k]_q) & \text{if } 0<k\leqslant n, \\ 1 & \text{if } k=0, \\ 0 & \text{if } k>n. \end{cases}$$

Recently Guo and Zeng [GZ] conjectured the following q-analogue of (1.1).

Conjecture 1.2 (Guo and Zeng [GZ, Conjecture 3.5])**.** *Let a be a positive integer. Then*

$$\sum_{k=0}^{3^a m-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv 0 \pmod{[3^a]_q^2} \quad \textit{for any } m\in\mathbb{Z}^+. \tag{1.3}$$

Concerning this conjecture, Guo and Zeng [GZ] were able to show congruence (1.3) with the modulus $[3^a]_q^2$ replaced by $[3^a]_q$.

In this paper we prove Conjecture 1.2 as well as a q-analogue of congruence (1.2).

Theorem 1.3. *Let $a\in\mathbb{Z}^+$. Then* (1.3) *holds. Furthermore, we have the following q-analogue of* (1.2)*:*

$$\frac{1}{[3^a]_q^2}\sum_{k=0}^{3^a-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv 2R(a,q) \pmod{\Phi_{3^a}(q)} \tag{1.4}$$

where

$$R(a,q) := \sum_{\substack{k=1 \\ 3\mid k-1}}^{3^a-1} q^{\frac{(k+2)(k-1)}{6}} \frac{(-1)^k}{[k]_q^2}\left(1+\left(\frac{k-1}{3}-\frac{3^{a-1}+1}{2}\right)(1-q^k)\right). \tag{1.5}$$

We remark that if $a\in\mathbb{Z}^+$, then $\lim_{q\to1}R(a,q)\equiv -1 \pmod 3$. This follows from

$$\sum_{\substack{k=1 \\ 3\mid k-1}}^{3^a-1}\frac{(-1)^k}{k^2} = \sum_{j=0}^{3^{a-1}-1}\frac{(-1)^{3j+1}}{(3j+1)^2} \equiv -\sum_{j=0}^{3^{a-1}-1}(-1)^j = -1 \pmod 3.$$

Also, for $k\in\mathbb{Z}^+$ with $k\equiv 1 \pmod 3$, $[k]_q$ is relatively prime to $[3^a]_q$ since k is relatively prime to 3^a. Therefore congruence (1.4) implies both congruences (1.2) and (1.3) in the case $m=1$.

We will prove an auxiliary result in the next section and then show Theorem 1.3 in Section 3.

2. An auxiliary theorem

Theorem 2.1. *Let $a, m \in \mathbb{Z}^+$ and let $\psi : \mathbb{Z} \to \mathbb{Z}$ be a function such that for any $k \in \mathbb{Z}$ and $j = 1, \ldots, a$ we have*

$$\psi(k) \equiv \psi(-k) \pmod{3^a} \quad \text{and} \quad \psi(k + 3^j) \equiv \psi(k) \pmod{3^j}.$$

Then

$$\sum_{k=1}^{3^a m-1} q^{\psi(k)} \left(\frac{k}{3}\right) \begin{bmatrix} 2 \cdot 3^a m \\ k \end{bmatrix}_q \equiv 0 \pmod{[3^a]_q^2}. \tag{2.1}$$

In particular,

$$\sum_{k=1}^{3^a m-1} \left(\frac{k}{3}\right) \begin{bmatrix} 2 \cdot 3^a m \\ k \end{bmatrix}_q \equiv 0 \pmod{[3^a]_q^2}. \tag{2.2}$$

Proof. Clearly $[x]_q \equiv [y]_q \pmod{\Phi_d(q)}$ provided that $x \equiv y \pmod d$. By the q-Lucas congruence (cf. [Sa]),

$$\begin{bmatrix} x_1 d + y_1 \\ x_2 d + y_2 \end{bmatrix}_q \equiv \binom{x_1}{x_2} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_q \pmod{\Phi_d(q)}$$

for $x_1, x_2, y_1, y_2 \in \mathbb{N}$ with $0 \le y_1, y_2 \le d - 1$. Recall that

$$[3^a]_q = \prod_{j=1}^{a} \Phi_{3^j}(q).$$

Since these $\Phi_{3^j}(q)$ are relatively prime and $[2 \cdot 3^a m]_q \equiv 0 \pmod{[3^a]_q}$, we only need to show that

$$\sum_{k=1}^{3^a m-1} \left(\frac{k}{3}\right) \frac{q^{\psi(k)}}{[k]_q} \begin{bmatrix} 2 \cdot 3^a m - 1 \\ k - 1 \end{bmatrix}_q \equiv 0 \pmod{\Phi_{3^j}(q)}$$

for every $j = 1, \ldots, a$.

For any $1 \le j \le a$ and $1 \le k \le 3^a m - 1$ with $3 \nmid k$, write $k = 3^j s + t$ where $1 \le t \le 3^j - 1$. Then, by the q-Lucas congruence,

$$\begin{bmatrix} 2 \cdot 3^a m - 1 \\ k - 1 \end{bmatrix}_q \equiv \binom{2 \cdot 3^{a-j} m - 1}{s} \begin{bmatrix} 3^j - 1 \\ t - 1 \end{bmatrix}_q \pmod{\Phi_{3^j}(q)}.$$

And we have

$$\begin{aligned} \begin{bmatrix} 3^j - 1 \\ t - 1 \end{bmatrix}_q &= \prod_{j=1}^{t-1} \frac{[3^j - j]_q}{[j]_q} \\ &= \prod_{j=1}^{t-1} \frac{q^{-j}([3^j]_q - [j]_q)}{[j]_q} \equiv (-1)^{t-1} q^{-\binom{t}{2}} \pmod{\Phi_{3^j}(q)}. \end{aligned}$$

Hence

$$
\begin{aligned}
&\sum_{k=1}^{3^a m-1}\left(\frac{k}{3}\right)\frac{q^{\psi(k)}}{[k]_q}{2\cdot 3^a m-1 \brack k-1}_q\\
&=\sum_{s=0}^{3^{a-j}m-1}\sum_{t=1}^{3^j-1}\left(\frac{3^j s+t}{3}\right)\frac{q^{\psi(3^j s+t)}}{[3^j s+t]_q}{2\cdot 3^a m-1 \brack 3^j s+t-1}_q\\
&\equiv\sum_{s=0}^{3^{a-j}m-1}\binom{2\cdot 3^{a-j}m-1}{s}\sum_{t=1}^{3^j-1}\left(\frac{t}{3}\right)\frac{(-1)^{t-1}q^{\psi(t)-\binom{t}{2}}}{[t]_q}\pmod{\Phi_{3^j}(q)}.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
&2\sum_{t=1}^{3^j-1}\left(\frac{t}{3}\right)\frac{(-1)^{t-1}q^{\psi(t)-\binom{t}{2}}}{[t]_q}\\
&=\sum_{t=1}^{3^j-1}\left(\left(\frac{t}{3}\right)\frac{(-1)^{t-1}q^{\psi(t)-\binom{t}{2}}}{[t]_q}+\left(\frac{3^j-t}{3}\right)\frac{(-1)^{3^j-t-1}q^{\psi(3^j-t)-\binom{3^j-t}{2}}}{[3^j-t]_q}\right)\\
&\equiv\sum_{\substack{t=1\\3\nmid t}}^{3^j-1}\left(\frac{t}{3}\right)\left(\frac{(-1)^{t-1}q^{\psi(t)-\binom{t}{2}}}{[t]_q}+\frac{(-1)^{t-1}q^{\psi(t)-\binom{-t}{2}}}{-q^{-t}[t]_q}\right)=0\pmod{\Phi_{3^j}(q)}.
\end{aligned}
$$

So (2.1) holds.

Note that (2.2) is just (2.1) with ψ replaced by the zero function from $\mathbb{Z}\to\mathbb{Z}$. So (2.2) is also valid. This concludes the proof. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let $a\in\mathbb{Z}^+$ and let ψ be a function as in Theorem 2.1. Then*

$$
\begin{aligned}
&\frac{1}{2[3^a]_q^2}\sum_{k=1}^{3^a-1}q^{\psi(k)}\left(\frac{k}{3}\right){2\cdot 3^a \brack k}_q\\
&\equiv\sum_{\substack{k=1\\3\mid k-1}}^{3^a-1}q^{\psi(k)-\binom{k}{2}}\frac{(-1)^{k-1}}{[k]_q^2}(1+\Psi_a(k)(1-q^k))\pmod{\Phi_{3^a}(q)},
\end{aligned}
\tag{3.1}
$$

where

$$
\Psi_a(k):=\frac{\psi(3^a-k)-\psi(k)}{3^a}+\frac{3^a-1}{2}-k. \tag{3.2}
$$

Proof. We have

$$\begin{aligned}
&\sum_{k=1}^{3^a-1}\left(\frac{k}{3}\right)\frac{q^{\psi(k)}}{[k]_q}\begin{bmatrix}2\cdot 3^a-1\\ k-1\end{bmatrix}_q\\
=&\sum_{k=1}^{3^a-1}\left(\frac{k}{3}\right)\frac{q^{\psi(k)}}{[k]_q}\prod_{j=1}^{k-1}\frac{q^{-j}([2\cdot 3^a]_q-[j]_q)}{[j]_q}\\
\equiv&\sum_{k=1}^{3^a-1}\left(\frac{k}{3}\right)\frac{(-1)^{k-1}q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(1-2\sum_{j=1}^{k-1}\frac{[3^a]_q}{[j]_q}\right)\ (\mathrm{mod}\ \Phi_{3^a}(q)^2),
\end{aligned}$$

since

$$[2\cdot 3^a]_q=[3^a]_q(1+q^{3^a})=[3^a]_q(2+q^{3^a}-1)\equiv 2[3^a]_q\ (\mathrm{mod}\ [3^a]_q^2).$$

Note that for $s=0,1,2,\dots$ we have

$$\begin{aligned}
q^{3^as}=&1+(q^{3^a}-1)\sum_{j=0}^{s-1}q^{3^aj}=1+(q^{3^a}-1)\left(s+\sum_{j=0}^{s-1}(q^{3^aj}-1)\right)\\
\equiv&1+s(q^{3^a}-1)\ (\mathrm{mod}\ \Phi_{3^a}(q)^2)
\end{aligned}$$

and

$$q^{-3^as}\equiv\frac{1}{1+s(q^{3^a}-1)}=\frac{1-s(q^{3^a}-1)}{1-s^2(q^{3^a}-1)^2}\equiv 1-s(q^{3^a}-1)\ (\mathrm{mod}\ \Phi_{3^a}(q)^2).$$

Also, for each $1\le k\le 3^a-1$, we have

$$\begin{aligned}
&\frac{q^{\psi(3^a-k)-\binom{3^a-k}{2}}}{[3^a-k]_q}\left(1-2\sum_{j=1}^{3^a-k-1}\frac{[3^a]_q}{[j]_q}\right)\\
=&\frac{q^{\psi(3^a-k)-\binom{3^a}{2}+3^ak-\binom{k+1}{2}}([3^a]_q+[k]_q)}{q^{-k}([3^a]_q^2-[k]_q^2)}\left(1-2\sum_{j=k+1}^{3^a-1}\frac{[3^a]_q}{[3^a-j]_q}\right)\\
\equiv&\frac{q^{\psi(k)-\binom{k}{2}}(1+(\frac{\psi(3^a-k)-\psi(k)}{3^a}+k-\frac{3^a-1}{2})(q^{3^a}-1))([3^a]_q+[k]_q)}{-[k]_q^2}\\
&\times\left(1+2\sum_{j=k+1}^{3^a-1}\frac{q^j[3^a]_q}{[j]_q}\right)\\
\equiv&-\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(1+\left(\frac{\psi(3^a-k)-\psi(k)}{3^a}+k-\frac{3^a-1}{2}\right)(q^{3^a}-1)\right)\\
&-2\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\sum_{j=k+1}^{3^a-1}\frac{q^j[3^a]_q}{[j]_q}-\frac{q^{\psi(k)-\binom{k}{2}}[3^a]_q}{[k]_q^2}\ (\mathrm{mod}\ \Phi_{3^a}(q)^2).
\end{aligned}$$

Clearly,

$$\sum_{j=k+1}^{3^a-1}\frac{q^j}{[j]_q}=\sum_{j=k+1}^{3^a-1}\frac{1+q^j-1}{[j]_q}=-(3^a-1-k)(1-q)+\sum_{j=k+1}^{3^a-1}\frac{1}{[j]_q},$$

and

$$\begin{aligned}&\sum_{j=1}^{3^a-1}\frac{1}{[j]_q}=\frac{1}{2}\sum_{j=1}^{3^a-1}\left(\frac{1}{[j]_q}+\frac{1}{[3^a-j]_q}\right)\\ \equiv&\frac{1}{2}\sum_{j=1}^{3^a-1}\left(\frac{1}{[j]_q}-\frac{q^j}{[j]_q}\right)=\frac{3^a-1}{2}(1-q)\pmod{\Phi_{3^a}(q)}.\end{aligned}$$

Thus we get

$$\begin{aligned}&\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(1-2\sum_{j=1}^{k-1}\frac{[3^a]_q}{[j]_q}\right)+\frac{q^{\psi(3^a-k)-\binom{3^a-k}{2}}}{[3^a-k]_q}\left(1-2\sum_{j=1}^{3^a-k-1}\frac{[3^a]_q}{[j]_q}\right)\\ \equiv&-\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(\left(\frac{\psi(3^a-k)-\psi(k)}{3^a}+\frac{3}{2}(3^a-1)-k\right)(q^{3^a}-1)\right)\\ &-\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(2\sum_{j=1}^{k-1}\frac{[3^a]_q}{[j]_q}+2\sum_{j=k+1}^{3^a-1}\frac{[3^a]_q}{[j]_q}\right)-\frac{q^{\psi(k)-\binom{k}{2}}[3^a]_q}{[k]_q^2}\\ \equiv&\frac{q^{\psi(k)-\binom{k}{2}}[3^a]_q}{[k]_q^2}+\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\Psi_a(k)(1-q^{3^a})\pmod{\Phi_{3^a}(q)^2}.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{k=1}^{3^a-1}\left(\frac{k}{3}\right)\frac{q^{\psi(k)}}{[k]_q}\begin{bmatrix}2\cdot 3^a-1\\ k-1\end{bmatrix}_q\\ \equiv&\sum_{\substack{k=1\\ 3\mid k-1}}^{3^a-1}(-1)^{k-1}\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(\frac{[3^a]_q}{[k]_q}+\Psi_a(k)(1-q^{3^a})\right)\pmod{\Phi_{3^a}(q)^2}.\end{aligned}$$

Noting that $[3^a]_q$ divides both sides of the above congruence by Theorem 2.1 and $[2\cdot 3^a]_q\equiv 2[3^a]_q \pmod{[3^a]_q^2}$, we are done. $\square$

Proof of Theorem 1.3. Let $m\in\mathbb{Z}^+$. By [T, (4.3)] in the case $d=0$, we have

$$\sum_{k=0}^{3^am-1}q^k\begin{bmatrix}2k\\ k\end{bmatrix}_q=-\sum_{k=1}^{3^am-1}q^{\psi_m(k)}\left(\frac{k}{3}\right)\begin{bmatrix}2\cdot 3^am\\ k\end{bmatrix}_q,\qquad(3.3)$$

where

$$\psi_m(k) = \frac{2(3^a m-k)^2 - (3^a m-k)\left(\frac{3^a m-k}{3}\right) - \left(\frac{k}{3}\right)^2}{3} \in \mathbb{Z}.$$

Whenever $k \equiv l \pmod{3^j}$ with $k, l \in \mathbb{Z}$ and $1 \leqslant j \leqslant a$, we have

$$2(k+l) - \left(\frac{k}{3}\right) \equiv 4k - \left(\frac{k}{3}\right) \equiv 0 \pmod 3$$

and hence

$$\begin{aligned}&2k^2 - k\left(\frac{k}{3}\right) - \left(2l^2 - l\left(\frac{l}{3}\right)\right)\\ =&(k-l)\left(2(k+l) - \left(\frac{k}{3}\right)\right) \equiv 0 \pmod{3^{j+1}}.\end{aligned}$$

So the function $\psi = \psi_m$ has the property described in Theorem 2.1. Combining (2.1) with (3.3) we get (1.3).

Now it remains to prove (1.4). By (3.3) and Lemma 3.1, we finally obtain

$$\begin{aligned}&\sum_{k=0}^{3^a-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q = -\sum_{k=1}^{3^a-1} q^{\psi_1(k)} \left(\frac{k}{3}\right) \begin{bmatrix} 2\cdot 3^a \\ k \end{bmatrix}_q\\ \equiv&2[3^a]_q^2 \sum_{\substack{k=1\\ 3\mid k-1}}^{3^a-1} q^{\frac{(k+2)(k-1)}{6}} \frac{(-1)^k}{[k]_q^2}\left(1 + \left(\frac{k-1}{3} - \frac{3^{a-1}+1}{2}\right)(1-q^k)\right)\\ &\pmod{\Phi_{3^a}(q)[3^a]_q^2}.\end{aligned}$$

This concludes our proof. $\square$

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