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ON SUMS OF APÉRY POLYNOMIALS AND RELATED CONGRUENCES

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Abstract. The Apéry polynomials are given by

$$A_n(x)=\sum_{k=0}^n\binom nk^2\binom{n+k}k^2x^k\ \ (n=0,1,2,\ldots).$$

(Those $A_n=A_n(1)$ are Apéry numbers.) Let p be an odd prime. We show that

$$\sum_{k=0}^{p-1}(-1)^kA_k(x)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}k^3}{16^k}x^k\pmod{p^2},$$

and that

$$\sum_{k=0}^{p-1}A_k(x)\equiv\left(\frac xp\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(256x)^k}\pmod p$$

for any p-adic integer $x\not\equiv0\pmod p$. This enables us to determine explicitly $\sum_{k=0}^{p-1}(\pm1)^kA_k \bmod p$, and $\sum_{k=0}^{p-1}(-1)^kA_k \bmod p^2$ in the case $p\equiv2\pmod3$. Another consequence states that

$$\sum_{k=0}^{p-1}(-1)^kA_k(-2)\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p=x^2+4y^2\ (x,y\in\mathbb Z),\\0\pmod{p^2}&\text{if }p\equiv3\pmod4.\end{cases}$$

We also prove that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}(2k+1)A_k\equiv p+\frac76p^4B_{p-3}\pmod{p^5}$$

where $B_0,B_1,B_2,\ldots$ are Bernoulli numbers.

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1. Introduction

The well-known Apéry numbers given by

$$A_n=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^n\binom{n+k}{2k}^2\binom{2k}{k}^2\ (n\in\mathbb{N}=\{0,1,2,\ldots\}),$$

play a central role in Apéry's proof of the irrationality of $\zeta(3)=\sum_{n=1}^\infty 1/n^3$ (see Apéry [Ap] and van der Poorten [Po]). They also have close connections to modular forms (cf. Ono [O, pp.198–203]). The Dedekind eta function in the theory of modular forms is defined by

$$\eta(\tau)=q^{1/24}\prod_{n=1}^\infty(1-q^n)\quad\text{with } q=e^{2\pi i\tau},$$

where $\tau\in\mathbb{H}=\{z\in\mathbb{C}:\ \mathrm{Im}(z)>0\}$ and hence $|q|<1$. In 1987 Beukers [B] conjectured that

$$A_{(p-1)/2}\equiv a(p)\ (\mathrm{mod}\ p^2)\quad\text{for any prime } p>3,$$

where $a(n)$ $(n=1,2,3,\ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau)=q\prod_{n=1}^\infty(1-q^{2n})^4(1-q^{4n})^4=\sum_{n=1}^\infty a(n)q^n.$$

This was finally confirmed by Ahlgren and Ono [AO] in 2000.

We define Apéry polynomials by

$$A_n(x)=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}^2x^k=\sum_{k=0}^n\binom{n+k}{2k}^2\binom{2k}{k}^2x^k\quad(n\in\mathbb{N}).\quad(1.1)$$

Clearly $A_n(1)=A_n$. Motivated by the Apéry polynomials, we also introduce a new kind of polynomials:

$$W_n(x):=\sum_{k=0}^n\binom{n}{k}^2\binom{n-k}{k}^2x^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}^2\binom{2k}{k}^2x^k\quad(n\in\mathbb{N}).\quad(1.2)$$

Recall that Bernoulli numbers $B_0,B_1,B_2,\ldots$ are rational numbers given by

$$B_0=1\ \text{ and }\ \sum_{k=0}^n\binom{n+1}{k}B_k=0\quad\text{for } n\in\mathbb{Z}^+=\{1,2,3,\ldots\}.$$

It is well known that $B_{2n+1} = 0$ for all $n \in \mathbb{Z}^+$ and

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi)\,.$$

Also, Euler numbers $E_0, E_1, E_2, \ldots$ are integers defined by

$$E_0 = 1 \text{ and } \sum_{\substack{k=0\\ 2\mid k}}^{n} \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{Z}^+.$$

It is well known that $E_{2n+1} = 0$ for all $n \in \mathbb{N}$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \quad \left(|x| < \frac{\pi}{2}\right).$$

Now we state our first theorem.

Theorem 1.1. (i) *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} (-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} (-1)^k W_k(-x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2}. \tag{1.3}$$

Also, for any p-adic integer $x \not\equiv 0 \pmod p$, we have

$$\begin{aligned}\sum_{k=0}^{p-1} A_k(x) &\equiv \sum_{k=0}^{p-1} W_k(x) \pmod{p^2} \\ &\equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p},\end{aligned} \tag{1.4}$$

where $(-)$ denotes the Legendre symbol.

(ii) *For any positive integer n we have*

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) A_k(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \binom{n+k}{2k+1} \binom{2k}{k} x^k. \tag{1.5}$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1} (2k+1) A_k \equiv p + \frac{7}{6} p^4 B_{p-3} \pmod{p^5} \tag{1.6}$$

and

$$\sum_{k=0}^{p-1}(2k+1)A_k(-1) \equiv \left(\frac{-1}{p}\right)p - p^3E_{p-3} \pmod{p^4}. \tag{1.7}$$

(iii) *Given* $\varepsilon \in \{\pm 1\}$ *and* $m \in \mathbb{Z}^+$, *for any prime* p *we have*

$$\sum_{k=0}^{p-1}(2k+1)\varepsilon^k A_k^m \equiv 0 \pmod{p}.$$

Remark 1.1. (i) Let p be an odd prime. The author [Su1, Su2] had conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k \bmod p^2$ with $m = 1, -8, 16, -64, 256, -512, 4096$. Motivated by the author's conjectures on $\sum_{k=0}^{p-1}A_k(x) \bmod p^2$ with $x = 1, -4, 9$ in an initial version of this paper, Guo and Zeng [GZ, Theorem 1.3] recently showed that

$$\sum_{k=0}^{p-1}A_k(x) \equiv \sum_{k=0}^{(p-1)/2}\binom{p+2k}{4k+1}\binom{2k}{k}^2 x^k \pmod{p^2}.$$

(ii) The values of

$$s_n = \frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k \in \mathbb{Z}$$

with $n = 1, \ldots, 8$ are

$$1,\ 8,\ 127,\ 2624,\ 61501,\ 1552760,\ 41186755,\ 1131614720$$

respectively. On June 6, 2011 Richard Penner informed the author an interesting application of (1.5): (1.5) with $x = 1$ implies that s_n is the trace of the inverse of nH_n where H_n refers to the Hilbert matrix $(\frac{1}{i+j-1})_{1\leqslant i,j\leqslant n}$.

Can we find integers $a_0, a_1, a_2, \ldots$ such that $\sum_{k=0}^{p-1}a_k \equiv 4x^2 - 2p \pmod{p^2}$ if $p = x^2 + y^2$ is a prime with x odd and y even? The following corollary provides an affirmative answer!

Corollary 1.1. *Let* p *be any odd prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^kA_k(-2) \equiv \sum_{k=0}^{p-1}(-1)^kA_k\left(\frac{1}{4}\right)\\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 4\ \&\ p = x^2 + y^2\ (2 \nmid x),\\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4.\end{cases}\end{aligned} \tag{1.8}$$

Proof. It is known (cf. Ishikawa [I]) that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1\pmod 4\ \&\ p=x^2+y^2\ (2\nmid x),\\0\pmod{p^2}&\text{if }p\equiv3\pmod 4.\end{cases}$$

The author conjectured that we can replace 64^k by $(-8)^k$ in the congruence, and this was recently confirmed by Z. H. Sun [S3]. So, applying (1.3) with $x=-2,1/4$ we obtain (1.8). □

Corollary 1.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}A_k\equiv c(p)\pmod p \tag{1.9}$$

where

$$c(p):=\begin{cases}4x^2-2p&\text{if }p\equiv1,3\pmod 8\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\0&\text{if }(\frac{-2}{p})=-1,\ i.e.,\ p\equiv5,7\pmod 8.\end{cases}$$

Also,

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^kA_k\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}(-1)^kA_k\left(\frac{1}{16}\right)\\&\equiv\begin{cases}4x^2-2p\pmod p&\text{if }p\equiv1\pmod 3\text{ and }p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }p\equiv2\pmod 3.\end{cases}\end{aligned}\tag{1.10}$$

Proof. By [M05] and [Su4], we have

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{256^k}\equiv c(p)\pmod{p^2}$$

as conjectured in [RV]. (Here we only need the mod p version which was proved in [M05].) So (1.9) follows from (1.4). The author [Su2] conjectured that

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k}\\&\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1\pmod 3\text{ and }p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }p\equiv2\pmod 3.\end{cases}\end{aligned}$$

This was confirmed by Z. H. Sun [S3] in the case $p\equiv2\pmod 3$, and the mod p version in the case $p\equiv1\pmod 3$ follows from (4)-(5) in Ahlgren [A, Theorem 5]. So we get (1.10) by applying (1.3) with $x=1,1/16$. □

Remark 1.2. The author conjectured that (1.9) also holds modulo p^2, and that (1.10) is also valid modulo p^2 in the case $p\equiv1\pmod 3$.

Corollary 1.3. *For any odd prime p and integer x, we have*

$$\sum_{k=0}^{p-1}(2k+1)A_k(x) \equiv p\left(\frac{x}{p}\right) \pmod{p^2}. \tag{1.11}$$

Proof. This follows from (1.5) in the case $n = p$, for, $p \mid \binom{p+k}{2k+1}$ for every $k = 0, \ldots, (p-3)/2$, and $p \mid \binom{2k}{k}$ for all $k = (p+1)/2, \ldots, p-1$. $\square$

We deduce Theorem 1.1(i) from our following result which has its own interest.

Theorem 1.2. *Let p be an odd prime and let x be any p-adic integer.*

(i) *If $x \equiv 2k \pmod{p}$ with $k \in \{0, \ldots, (p-1)/2\}$, then we have*

$$\sum_{r=0}^{p-1}(-1)^r\binom{x}{r}^2 \equiv (-1)^k\binom{x}{k} \pmod{p^2}. \tag{1.12}$$

(ii) *If $x \equiv k \pmod{p}$ with $k \in \{0, \ldots, p-1\}$, then*

$$\sum_{r=0}^{p-1}\binom{x}{r}^2 \equiv \binom{2x}{k} \pmod{p^2}. \tag{1.13}$$

Remark 1.3. In contrast with (1.12) and (1.13), we recall the following identities (cf. [G, (3.32) and (3.66)]):

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^2 = (-1)^n\binom{2n}{n} \quad \text{and} \quad \sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}.$$

Corollary 1.4. *Let p be an odd prime.*

(i) (Conjectured in [RV] and proved in [M03]) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

(ii) (Conjectured by the author [Su1] and confirmed in [S2]) *If $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4}\left(2x - \frac{p}{2x}\right) \pmod{p^2}. \tag{1.14}$$

Proof. Since $\binom{-1/2}{r} = \binom{2r}{r}/(-4)^r$ for all $r = 0, 1, \ldots$, applying (1.13) with $x = -1/2$ and $k = (p-1)/2$ we immediately get the congruence in part (i).

When $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$, by (1.12) with $x = -1/2$ and $k = (p-1)/4$ we have

$$\begin{aligned}\sum_{r=0}^{p-1} \frac{\binom{2r}{r}^2}{(-16)^r} \equiv& (-1)^{(p-1)/4}\binom{-1/2}{(p-1)/4} = \frac{\binom{(p-1)/2}{(p-1)/4}}{4^{(p-1)/4}} = \frac{\binom{(p-1)/2}{(p-1)/4}}{2^{(p-1)/2}} \\ \equiv& \frac{2^{p-1}+1}{2 \times 2^{(p-1)/2}}\left(2x - \frac{p}{2x}\right) \pmod{p^2} \ \text{(by [CDE] or [BEW, (9.0.2)])} \\ \equiv& (-1)^{(p-1)/4}\left(2x - \frac{p}{2x}\right) \pmod{p^2}\end{aligned}$$

since $((-1)^{(p-1)/4}2^{(p-1)/2} - 1)^2 \equiv 0 \pmod{p^2}$. This proves (1.14). $\square$

Corollary 1.5. *Let* $a_n := \sum_{k=0}^n \binom{n}{k}^2 C_k$ *for* $n = 0, 1, 2, \ldots$, *where* C_k *denotes the Catalan number* $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$. *Then, for any odd prime* p *we have*

$$a_1 + \cdots + a_{p-1} \equiv 0 \pmod{p^2}. \tag{1.15}$$

Remark 1.4. We find no prime $p \leqslant 5,000$ with $\sum_{k=1}^{p-1} a_k \equiv 0 \pmod{p^3}$ and no composite number $n \leqslant 70,000$ satisfying $\sum_{k=1}^{n-1} a_k \equiv 0 \pmod{n^2}$. We conjecture that (1.15) holds for no composite $p > 1$.

The author [Su1, Remark 1.2] conjectured that for any prime $p > 5$ with $p \equiv 1 \pmod 4$ we have

$$\sum_{k=0}^{p^a-1} \frac{k^3\binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^{2a}} \quad \text{for } a = 1, 2, 3, \ldots.$$

This was recently confirmed by Z. H. Sun [S3] in the case $a = 1$. Note that

$$\frac{k^3\binom{2k}{k}^3}{64^k} = (-1)^k k^3 \binom{-1/2}{k}^3 = \frac{(-1)^{k-1}}{8}\binom{-3/2}{k-1}^3 \quad \text{for all } k = 1, 2, 3, \ldots.$$

So, for any prime $p > 5$ with $p \equiv 1 \pmod 4$ we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{-3/2}{r}^3 \equiv 0 \pmod{p^2}.$$

Since $-3/2 \equiv -2(p+3)/4 \pmod p$, the result just corresponds to the case $x = -3/2$ of our following general theorem.

Theorem 1.3. *Let $p > 3$ be a prime and let x be a p-adic integer with $x \equiv -2k \pmod p$ for some $k \in \{1, \ldots, \lfloor (p-1)/3 \rfloor\}$. Then we have*

$$\sum_{r=0}^{p-1}(-1)^r \binom{x}{r}^3 \equiv 0 \pmod{p^2}. \tag{1.16}$$

Similar to Apéry numbers, the central Delannoy numbers (see [CHV]) are defined by

$$D_n = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k} = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} \ (n \in \mathbb{N}).$$

Such numbers arise naturally in many enumeration problems in combinatorics (cf. Sloane [S]); for example, D_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$.

Now we give our result on central Delannoy numbers.

Theorem 1.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} D_k \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3}, \tag{1.17}$$

We also have

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k D_k \equiv p - \frac{7}{12}p^4 B_{p-3} \pmod{p^5} \tag{1.18}$$

and

$$\sum_{k=0}^{p-1}(2k+1)D_k \equiv p + 2p^2 q_p(2) - p^3 q_p(2)^2 \pmod{p^4}, \tag{1.19}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Remark 1.5. In [Su3] the author determined $\sum_{k=1}^{p-1} D_k/k$ and $\sum_{k=1}^{p-1} D_k/k^2$ modulo an odd prime p.

In the next section we will show Theorems 1.1-1.2 and Corollary 1.5. Section 3 is devoted to our proofs of Theorems 1.3 and 1.4. In Section 4 we are going to raise some related conjectures for further research.

2. Proofs of Theorems 1.1-1.2 and Corollary 1.5

We first prove Theorem 1.2.

Proof of Theorem 1.2. (i) We now consider the first part of Theorem 1.2. Set

$$f_k(y) := \sum_{r=0}^{p-1}(-1)^r\binom{2k+py}{r}^2 \qquad \text{for } k\in\mathbb{N}. \tag{2.1}$$

We want to prove that

$$f_k(y) \equiv (-1)^k\binom{2k+py}{k} \pmod{p^2} \tag{2.2}$$

for any p-adic integer y and $k\in\{0,1,\ldots,(p-1)/2\}$.

Applying the Zeilberger algorithm (cf. [PWZ]) via `Mathematica 7`, we find that

$$\begin{aligned}&(py+2k+2)f_{k+1}(y)+4(py+2k+1)f_k(y)\\ =&\frac{(p(y-1)+2k+3)^2F_k(y)}{(py+2k+1)(py+2k+2)^2}\binom{py+2k+2}{p-1}^2,\end{aligned} \tag{2.3}$$

where

$$F_k(y)=14+34k+20k^2-10p-12kp+2p^2+17py+20kpy-6p^2y+5p^2y^2.$$

Now fix a p-adic integer y. Observe that

$$\begin{aligned}f_{(p-1)/2}(y)=&\sum_{r=0}^{p-1}(-1)^r\binom{p-1+py}{r}^2=\sum_{r=0}^{p-1}(-1)^r\prod_{0<s\leqslant r}\left(1-\frac{p(y+1)}{s}\right)^2\\ \equiv&\sum_{r=0}^{p-1}(-1)^r\left(1-\sum_{0<s\leqslant r}\frac{2p(y+1)}{s}\right)=1-\sum_{r=1}^{p-1}(-1)^r\sum_{s=1}^{r}\frac{2p(y+1)}{s}\\ =&1-2p(y+1)\sum_{s=1}^{p-1}\frac{1}{s}\sum_{r=s}^{p-1}(-1)^r=1-p(y+1)\sum_{j=1}^{(p-1)/2}\frac{1}{j}\\ \equiv&(-1)^{(p-1)/2}\binom{p-1+py}{(p-1)/2} \pmod{p^2}.\end{aligned}$$

For each $k\in\{0,\ldots,(p-3)/2\}$, clearly $py+2k+1, py+2k+2\not\equiv 0 \pmod p$, and also

$$(p(y-1)+2k+3)^2\binom{py+2k+2}{p-1}^2\equiv 0 \pmod{p^2}$$

since $\binom{py+2k+2}{p-1} = \frac{p}{py+2k+3}\binom{py+2k+3}{p} \equiv 0 \pmod{p}$ if $0 \leqslant k < (p-3)/2$. Thus, by (2.3) we have

$$f_k(y) \equiv -\frac{py+2k+2}{4(py+2k+1)} f_{k+1}(y) \pmod{p^2} \quad \text{for } k = 0, \ldots, \frac{p-3}{2}.$$

If $0 \leqslant k < (p-1)/2$ and

$$f_{k+1}(y) \equiv (-1)^{k+1}\binom{2(k+1)+py}{k+1} \pmod{p^2},$$

then

$$\begin{aligned} f_k(y) \equiv& -\frac{py+2k+2}{4(py+2k+1)}(-1)^{k+1}\binom{2(k+1)+py}{k+1} \\ =&\frac{(-10^k(py+2k+2)^2}{4(k+1)(py+k+1)}\binom{2k+py}{k} \equiv (-1)^k\binom{2k+py}{k} \pmod{p^2}. \end{aligned}$$

Therefore (2.2) holds for all $k = 0, 1, \ldots, (p-1)/2$. This proves Theorem 1.2(i).

(ii) The second part of Theorem 1.2 can be proved in a similar way. Here we mention that if we define

$$g_k(y) := \sum_{r=0}^{p-1}\binom{k+py}{r}^2 \quad \text{for } k \in \mathbb{N} \tag{2.4}$$

then by the Zeilberger algorithm (cf. [PWZ]) we have the recursion

$$\begin{aligned} &(py+k+1)g_{k+1}(y) - 2(2py+2k+1)g_k(y) \\ =& -\frac{(p(y-1)+k+2)^2(3py-2p+3k+3)}{(py+k+1)^2}\binom{py+k+1}{p-1}^2. \end{aligned}$$

It follows that if $k \in \{0, \ldots, p-2\}$ and y is a p-adic integer then

$$\begin{aligned} &g_{k+1}(y) \equiv \binom{2(k+1)+2py}{k+1} \pmod{p^2} \\ \Longrightarrow\ & g_k(y) \equiv \binom{2k+2py}{k} \pmod{p^2}. \end{aligned} \tag{2.5}$$

In view of this, we have the second part of Theorem 1.2 by induction.

The proof of Theorem 1.2 is now complete. □

Proof of Corollary 1.5. Observe that

$$\sum_{n=0}^{p-1} a_n = \sum_{k=0}^{p-1} C_k \sum_{n=k}^{p-1}\binom{n}{k}^2 = \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1-k}\binom{k+j}{k}^2.$$

If $0 \leqslant k \leqslant p-1$ and $p-k \leqslant j \leqslant p-1$, then

$$\binom{k+j}{k} = \frac{(k+j)!}{k!j!} \equiv 0 \pmod p.$$

Therefore

$$\sum_{n=0}^{p-1} a_n \equiv \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1} \binom{k+j}{k}^2 = \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1} \binom{x_k}{j}^2,$$

where $x_k = -k-1 \equiv p-1-k \pmod p$. Applying Theorem 1.2(ii) we get

$$\sum_{n=0}^{p-1} a_n \equiv \sum_{k=0}^{p-1} C_k \binom{2x_k}{p-1-k} = \sum_{k=0}^{p-1} (-1)^k \binom{p+k}{2k+1} C_k \pmod{p^2}.$$

So it suffices to show that for any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} C_k = 1. \tag{2.6}$$

We prove (2.6) by induction. Clearly, (2.6) holds for $n=1$. Let n be any positive integer. By the Chu-Vandermonde identity

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}$$

(see, e.g., [GKP, p. 169]), we have

$$\sum_{k=0}^{n-1} \binom{n+1}{k+1}\binom{n+k}{k}(-1)^k = \sum_{k=0}^{n-1} \binom{n+1}{n-k}\binom{-n-1}{k} = -\binom{-n-1}{n}.$$

Thus

$$\begin{aligned}
&\sum_{k=0}^{n} (-1)^k \binom{n+1+k}{2k+1} C_k - \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} C_k \\
=&(-1)^n C_n + \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k} C_k \\
=&(-1)^n C_n + \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k+1}\binom{n+k}{k}(-1)^k \\
=&(-1)^n C_n - \frac{1}{n+1}\binom{-n-1}{n} = 0.
\end{aligned}$$

This concludes the induction step. We are done. □

Now we can apply Theorem 1.2 to deduce the first part of Theorem 1.1.

Proof of Theorem 1.1(i). Let $\varepsilon\in\{\pm1\}$. Then

$$\begin{aligned}\sum_{m=0}^{p-1}\varepsilon^mA_m(x)&=\sum_{m=0}^{p-1}\varepsilon^m\sum_{k=0}^{m}\binom{m+k}{2k}^2\binom{2k}{k}^2x^k\\&=\sum_{k=0}^{p-1}\binom{2k}{k}^2x^k\sum_{m=k}^{p-1}\varepsilon^m\binom{m+k}{2k}^2\\&=\sum_{k=0}^{p-1}\binom{2k}{k}^2x^k\sum_{r=0}^{p-1-k}\varepsilon^{k+r}\binom{2k+r}{r}^2\\&=\sum_{k=0}^{p-1}\binom{2k}{k}^2\varepsilon^kx^k\sum_{r=0}^{p-1-k}\varepsilon^r\binom{p-1-2k-p}{r}^2\end{aligned}$$

Set $n=(p-1)/2$. Clearly $\binom{2k}{k}\equiv0\pmod p$ for $k=n+1,\ldots,p-1$, and

$$\binom{p-1-2k-p}{r}\equiv\binom{p-1-2k}{r}=0\pmod p$$

if $0\leqslant k\leqslant n$ and $p-1-2k<r\leqslant p-1$. Therefore

$$\sum_{m=0}^{p-1}\varepsilon^mA_m(x)\equiv\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^k\sum_{r=0}^{p-1}\varepsilon^r\binom{2(n-k)-p}{r}^2\pmod{p^2}.$$

Similarly,

$$\begin{aligned}\sum_{m=0}^{p-1}\varepsilon^mW_m(\varepsilon x)&=\sum_{m=0}^{p-1}\varepsilon^m\sum_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}^2\binom{2k}{k}^2(\varepsilon x)^k\\&=\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^k\sum_{m=2k}^{p-1}\varepsilon^m\binom{m}{2k}^2\\&=\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^k\sum_{r=0}^{p-1-2k}\varepsilon^{2k+r}\binom{2k+r}{r}^2\\&\equiv\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^k\sum_{r=0}^{p-1}\varepsilon^r\binom{2(n-k)-p}{r}^2\pmod{p^2}.\end{aligned}$$

So we have

$$\sum_{m=0}^{p-1}\varepsilon^mA_m(x)\equiv\sum_{m=0}^{p-1}\varepsilon^mW_m(\varepsilon x)\equiv\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^kS_k(\varepsilon)\pmod{p^2},\tag{2.7}$$

where

$$S_k(\varepsilon) := \sum_{r=0}^{p-1} \varepsilon^r \binom{2(n-k)-p}{r}^2.$$

Applying Theorem 1.2(i) we get

$$\begin{aligned} S_k(-1) \equiv&(-1)^{n-k}\binom{2(n-k)-p}{n-k} = (-1)^{n-k}\binom{-2k-1}{n-k} \\ =&\binom{n+k}{n-k} = \binom{n+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}. \end{aligned}$$

(The last congruence can be easily deduced, see. e.g., [S2, Lemma 2.2].) Combining this with (2.7) in the case $\varepsilon = -1$ we immediately obtain (1.3).

In view of Theorem 1.2(ii),

$$S_k(1) \equiv \binom{4(n-k)-2p}{2(n-k)} \pmod{p^2}.$$

Recall that $\binom{n+k}{n-k}(-16)^k \equiv \binom{2k}{k} \pmod{p^2}$. So, in view of (2.7) with $\varepsilon = 1$, we have

$$\begin{aligned} \sum_{m=0}^{p-1} A_m(x) \equiv \sum_{m=0}^{p-1} W_m(x) \equiv& \sum_{k=0}^{n} \binom{n+k}{n-k}^2 (-16)^{2k} x^k \binom{4(n-k)-2p}{2(n-k)} \\ =&\sum_{j=0}^{n} \binom{n+(n-j)}{j}^2 256^{n-j} x^{n-j} \binom{4j-2p}{2j} \\ =&16^{p-1} \sum_{k=0}^{n} \frac{\binom{4k-2p}{2k}\binom{2k-p}{k}^2}{256^k} x^{n-k} \pmod{p^2} \end{aligned}$$

If x is a p-adic integer with $x \not\equiv 0 \pmod{p}$, then

$$\begin{aligned} &16^{p-1} \sum_{k=0}^{n} \frac{\binom{4k-2p}{2k}\binom{2k-p}{k}^2}{256^k} x^{n-k} \\ \equiv& \left(\frac{x}{p}\right) \sum_{k=0}^{n} \frac{\binom{4k}{2k}\binom{2k}{k}^2}{(256x)^k} \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}, \end{aligned}$$

and therefore (1.4) holds. $\square$

Lemma 2.1. *Let $k \in \mathbb{N}$. Then, for any $n \in \mathbb{Z}^+$ we have the identity*

$$\sum_{m=0}^{n-1} (2m+1)\binom{m+k}{2k}^2 = \frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2. \tag{2.8}$$

Proof. Obviously (2.8) holds when $n = 1$.

Now assume that $n > 1$ and (2.8) holds. Then

$$\begin{aligned}&\sum_{m=0}^{n}(2m+1)\binom{m+k}{2k}^2\\=&\frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2+(2n+1)\binom{n+k}{2k}^2\\=&\frac{(n+k+1)^2}{2k+1}\binom{n+k}{2k}^2=\frac{(n+1-k)^2}{2k+1}\binom{(n+1)+k}{2k}^2.\end{aligned}$$

Combining the above, we have proved the desired result by induction. $\square$

Lemma 2.2. *Let $p > 3$ be a prime. Then*

$$\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{2k+1}\equiv -pE_{p-3}\pmod{p^2}. \tag{2.9}$$

Proof. Observe that

$$\begin{aligned}\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{2k+1}=&\frac{1}{2}\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\left(\frac{(-1)^k}{2k+1}+\frac{(-1)^{p-1-k}}{(2(p-1-k)+1)}\right)\\=&-p\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{(2k+1)(2k+1-2p)}\\\equiv&-\frac{p}{4}\sum_{k=0}^{p-1}(-1)^k\left(k+\frac{1}{2}\right)^{p-3}\pmod{p^2}.\end{aligned}$$

So we have reduced (2.9) to the following congruence

$$\sum_{k=0}^{p-1}(-1)^k\left(k+\frac{1}{2}\right)^{p-3}\equiv 4E_{p-3}\pmod{p}. \tag{2.10}$$

Recall that the Euler polynomial of degree n is defined by

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}.$$

It is well known that

$$E_n(x)+E_n(x+1)=2x^n.$$

Thus

$$\begin{aligned}&2\sum_{k=0}^{p-1}(-1)^k\left(k+\frac12\right)^{p-3}\\
=&\sum_{k=0}^{p-1}\left((-1)^kE_{p-3}\left(k+\frac12\right)-(-1)^{k+1}E_{p-3}\left(k+1+\frac12\right)\right)\\
=&E_{p-3}\left(\frac12\right)-(-1)^pE_{p-3}\left(p+\frac12\right)\\
\equiv&2E_{p-3}\left(\frac12\right)=2\frac{E_{p-3}}{2^{p-3}}\equiv 8E_{p-3}\ (\mathrm{mod}\ p)\end{aligned}$$

and hence (2.10) follows. We are done. □

For each $m=1,2,3,\ldots$ those rational numbers

$$H_n^{(m)}:=\sum_{0<k\leqslant n}\frac{1}{k^m}\quad(n=0,1,2,\ldots)$$

are called harmonic numbers of order m. We simply write H_n for $H_n^{(1)}$. A well-known theorem of Wolstenholme asserts that $H_{p-1}\equiv 0\pmod{p^2}$ and $H_{p-1}^{(2)}\equiv 0\pmod p$ for any prime $p>3$.

Lemma 2.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1}\equiv-\frac74B_{p-3}\pmod p. \tag{2.11}$$

Proof. Clearly,

$$\sum_{k=1}^{p-1}\frac{1}{k^3}=\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^3}+\frac{1}{(p-k)^3}\right)\equiv 0\pmod p.$$

By [ST, (5.4)], $\sum_{k=1}^{p-1}H_k/k^2\equiv B_{p-3}\pmod p$. Therefore

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k}=&\sum_{k=1}^{p-1}\frac1k\sum_{j=1}^{k}\frac{1}{j^2}=\sum_{j=1}^{p-1}\frac{H_{p-1}-H_{j-1}}{j^2}\\
\equiv&-\sum_{k=1}^{p-1}\frac{H_k}{k^2}+\sum_{k=1}^{p-1}\frac{1}{k^3}\equiv-B_{p-3}\ (\mathrm{mod}\ p).\end{aligned}$$

On the other hand,

$$\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k}=\sum_{k=1}^{(p-1)/2}\left(\frac{H_k^{(2)}}{k}+\frac{H_{p-k}^{(2)}}{p-k}\right)$$

$$\equiv\sum_{k=1}^{(p-1)/2}\left(\frac{H_k^{(2)}}{k}+\frac{H_{p-1}^{(2)}-H_{k-1}^{(2)}}{-k}\right)\equiv 2\sum_{k=1}^{(p-1)/2}\frac{H_k^{(2)}}{k}-H_{(p-1)/2}^{(3)}\pmod p.$$

It is known (see, e.g., [S1, Corollary 5.2]) that

$$H_{(p-1)/2}^{(3)}=\sum_{k=1}^{(p-1)/2}\frac{1}{k^3}\equiv -2B_{p-3}\pmod p.$$

So we have

$$\sum_{k=1}^{(p-1)/2}\frac{H_k^{(2)}}{k}\equiv\frac12\bigg(\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k}+H_{(p-1)/2}^{(3)}\bigg)\equiv\frac{-B_{p-3}-2B_{p-3}}{2}=-\frac32 B_{p-3}\pmod p.$$

Clearly

$$H_{(p-1)/2}^{(2)}\equiv\frac12\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=\frac12 H_{p-1}^{(2)}\equiv 0\pmod p.$$

Observe that

$$\begin{aligned}\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1}\equiv&-\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{p-1-2k}=-\sum_{k=1}^{(p-1)/2}\frac{H_{(p-1)/2-k}^{(2)}}{2k}\\ \equiv&-\frac12\sum_{k=1}^{(p-1)/2}\frac1k\bigg(H_{(p-1)/2}^{(2)}-\sum_{j=0}^{k-1}\frac{1}{((p-1)/2-j)^2}\bigg)\\ \equiv&2\sum_{k=1}^{(p-1)/2}\frac1k\sum_{j=0}^{k-1}\frac{1}{(2j+1)^2}\equiv 2\sum_{k=1}^{(p-1)/2}\frac1k\bigg(H_{2k}^{(2)}-\sum_{j=1}^{k}\frac{1}{(2j)^2}\bigg)\\ =&4\sum_{k=1}^{(p-1)/2}\frac{H_{2k}^{(2)}}{2k}-\frac12\sum_{k=1}^{(p-1)/2}\frac{H_k^{(2)}}{k}\pmod p\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{H_{2k}^{(2)}}{2k}=&\sum_{\substack{k=1\\2\mid k}}^{p-1}\frac1k\sum_{j=1}^{k}\frac{1}{j^2}=\sum_{\substack{k=1\\2\mid k}}^{p-1}\frac{1}{k^3}+\sum_{\substack{1\leqslant j<k\leqslant p-1\\2\mid k}}\frac{1}{j^2k}\\ \equiv&\frac18 H_{(p-1)/2}^{(3)}-\frac38 B_{p-3}\quad\text{(by Pan [P, (2.4)])}\\ \equiv&\frac18(-2B_{p-3})-\frac38 B_{p-3}=-\frac58 B_{p-3}\pmod p.\end{aligned}$$

So we finally get

$$\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1}\equiv 4\left(-\frac{5}{8}B_{p-3}\right)-\frac{1}{2}\left(-\frac{3}{2}B_{p-3}\right)=-\frac{7}{4}B_{p-3}\pmod p.$$

This concludes the proof of (2.11). □

Proof of Theorem 1.1(ii). (i) Let n be any positive integer. Then

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)A_m(x)&=\sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{m+k}{2k}^2\binom{2k}{k}^2x^k\\&=\sum_{k=0}^{n-1}\binom{2k}{k}^2x^k\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}^2\\&=\sum_{k=0}^{n-1}\binom{2k}{k}^2x^k\frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2\quad\text{(by (2.8))}\\&=\sum_{k=0}^{n-1}\frac{(n-k)^2}{2k+1}\binom{n}{k}^2\binom{n+k}{k}^2x^k.\end{aligned}$$

Since

$$(n-k)\binom{n}{k}=n\binom{n-1}{k}\quad\text{for all }\ k=0,\ldots,n-1,$$

we have

$$\begin{aligned}\frac{1}{n}\sum_{m=0}^{n-1}(2m+1)A_m(x)&=\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{n-k}{2k+1}\binom{n}{k}\binom{n+k}{k}^2x^k\\&=\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{n-k}{2k+1}\binom{n+k}{2k}\binom{2k}{k}\binom{n+k}{k}x^k\\&=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}\binom{n+k}{2k+1}\binom{2k}{k}x^k.\end{aligned}$$

This proves (1.5).

Now we fix a prime $p>3$. By the above,

$$\sum_{m=0}^{p-1}(2m+1)A_m(x)=\sum_{k=0}^{p-1}\frac{p^2}{2k+1}\binom{p-1}{k}^2\binom{p+k}{k}^2x^k.\tag{2.12}$$

For $k\in\{0,\ldots,p-1\}$, clearly

$$\begin{aligned}\binom{p-1}{k}^2\binom{p+k}{k}^2&=\prod_{0<j\leqslant k}\left(\frac{p-j}{j}\cdot\frac{p+j}{j}\right)^2=\prod_{0<j\leqslant k}\left(1-\frac{p^2}{j^2}\right)^2\\&\equiv\prod_{0<j\leqslant k}\left(1-\frac{2p^2}{j^2}\right)\equiv 1-2p^2H_k^{(2)}\pmod{p^4}.\end{aligned}$$

Thus (2.12) implies that

$$\sum_{m=0}^{p-1}(2m+1)A_m(x)=\sum_{k=0}^{p-1}\frac{p^2}{2k+1}\left(1-2p^2H_k^{(2)}\right)x^k \pmod{p^5}. \quad (2.13)$$

Since $H_{(p-1)/2}^{(2)}\equiv 0 \pmod p$, taking $x=-1$ in (2.13) and applying (2.9) we obtain

$$\sum_{m=0}^{p-1}(2m+1)A_m(-1)\equiv\sum_{k=0}^{p-1}\frac{p^2(-1)^k}{2k+1}\equiv\frac{p^2(-1)^{(p-1)/2}}{2(p-1)/2+1}-p^3E_{p-3} \pmod{p^4}$$

and hence (1.7) holds.

Now we prove (1.6). In view of (2.13) with $x=1$, we have

$$\begin{aligned}\sum_{m=0}^{p-1}(2m+1)A_m \equiv&\frac{p^2}{2(p-1)/2+1}\left(1-2p^2H_{(p-1)/2}^{(2)}\right)\\ &+p^2\sum_{k=0}^{(p-3)/2}\left(\frac{1-2p^2H_k^{(2)}}{2k+1}+\frac{1-2p^2H_{p-1-k}^{(2)}}{2(p-1-k)+1}\right)\\ =&p-2p^3H_{(p-1)/2}^{(2)}+2p^3\sum_{k=0}^{(p-3)/2}\frac{2p+2k+1}{(2k+1)(4p^2-(2k+1)^2)}\\ &-2p^4\sum_{k=0}^{(p-3)/2}\left(\frac{H_k^{(2)}}{2k+1}+\frac{H_{p-1}^{(2)}-\sum_{0<j\leqslant k}(p-j)^{-2}}{2p-(2k+1)}\right)\\ \equiv&p-2p^3H_{(p-1)/2}^{(2)}-4p^4\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^3}\\ &-2p^3\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^2}-4p^4\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1} \pmod{p^5}.\end{aligned}$$

By [S1, Corollaries 5.1 and 5.2],

$$H_{p-1}^{(2)}\equiv\frac{2}{3}pB_{p-3} \pmod{p^2},\quad H_{(p-1)/2}^{(2)}\equiv\frac{7}{3}pB_{p-3} \pmod{p^2},$$

$$\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^2}=H_{p-1}^{(2)}-\frac{H_{(p-1)/2}^{(2)}}{4}\equiv\frac{p}{12}B_{p-3} \pmod{p^2},$$

and

$$\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^3}=H_{p-1}^{(3)}-\frac{H_{(p-1)/2}^{(3)}}{8}\equiv 0-\frac{-2B_{p-3}}{8}=\frac{B_{p-3}}{4} \pmod{p}.$$

Combining these with Lemma 2.3, we finally obtain

$$\begin{aligned}\sum_{k=0}^{p-1}(2k+1)A_k \equiv& p-2p^3\frac{7}{3}pB_{p-3}-4p^4\frac{B_{p-3}}{4}-2p^3\frac{p}{12}B_{p-3}-4p^4\left(-\frac{7}{4}B_{p-3}\right)\\ =& p+\frac{7}{6}p^4B_{p-3} \pmod{p^5}\end{aligned}$$

So far we have proved the second part of Theorem 1.1. □

Part (iii) of Theorem 1.1 is easy.

Proof of Theorem 1.1(iii). As $A_0=1$ and $A_1=3$, the desired congruence with $p=2$ holds trivially.

Below we assume that p is an odd prime. If $k\in\{0,1\ldots,p-1\}$, then

$$\begin{aligned}&A_{p-1-k}=\sum_{j=0}^{p-1}\binom{(p-1-k)+j}{2j}^2\binom{2j}{j}^2\\ \equiv&\sum_{j=0}^{p-1}\binom{j-k-1}{2j}^2\binom{2j}{j}^2=\sum_{j=0}^{k}\binom{j+k}{2j}^2\binom{2j}{j}^2=A_k \pmod p\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=0}^{p-1}(2k+1)\varepsilon^kA_k^m=&\sum_{k=0}^{p-1}(2(p-1-k)+1)\varepsilon^{p-1-k}A_{p-1-k}^m\\ \equiv&-\sum_{k=0}^{p-1}(2k+1)\varepsilon^kA_k^m \pmod p\end{aligned}$$

and hence we have the desired congruence. □

3. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Define

$$w_k(y):=\sum_{r=0}^{p-1}(-1)^r\binom{py-2k}{r}^3\quad\text{for } k\in\mathbb{N}. \tag{3.1}$$

We want to show that $w_k(y)\equiv 0 \pmod{p^2}$ for any p-adic integer y and $k\in\{1,\ldots,\lfloor(p-1)/3\rfloor\}$.

By the Zeilberger algorithm (cf. [PWZ]), for $k=0,1,2,\ldots$ we have

$$\begin{aligned}&(py-2k)^2w_k(y)+3(3py-2(3k+1))(3py-2(3k+2))w_{k+1}(y)\\ &\quad=\frac{P(k,p,y)(p(1-y)+2k-1)^3}{(py-2k)^3(py-2k-1)^3}\binom{py-2k}{p-1}^3\end{aligned} \tag{3.2}$$

where $P(k,p,y)$ is a suitable polynomial in k,p,y with integer coefficients such that $P(0,p,y)\equiv 0 \pmod{p^2}$. (Here we omit the explicit expression of $P(k,p,y)$ since it is complicated.) Note also that

$$w_1(0)=\sum_{r=0}^{p-1}(-1)^r\binom{-2}{r}^3=\sum_{r=0}^{p-1}(r+1)^3=\frac{p^2(p+1)^2}{4}\equiv 0 \pmod{p^2}.$$

Fix a p-adic integer y. If $y\neq 0$, then (3.2) with $k=0$ yields

$$\begin{aligned}&3(3py-2)(3py-4)w_1(y)\\ \equiv&\frac{P(0,p,y)(p(1-y)-1)^3}{(py)^3(py-1)^3}\left(\frac{py}{p-1}\binom{p(y-1)+p-1}{p-2}\right)^3\equiv 0 \pmod{p^2}\end{aligned}$$

and hence $w_1(y)\equiv 0 \pmod{p^2}$. If $1<k+1\leqslant\lfloor(p-1)/3\rfloor$, then by (3.2) we have

$$(py-2k)^2w_k(y)+3(3py-2(3k+1))(3py-2(3k+2))w_{k+1}(y)\equiv 0 \pmod{p^3}$$

since

$$\binom{py-2k}{p-1}=\frac{p}{py-2k+1}\binom{py-2k+1}{p}\equiv 0 \pmod{p}.$$

Thus, when $1<k+1\leqslant\lfloor(p-1)/3\rfloor$ we have

$$w_k(y)\equiv 0 \pmod{p^2}\implies w_{k+1}(y)\equiv 0 \pmod{p^2}.$$

So, by induction, $w_k(y)\equiv 0 \pmod{p^2}$ for all $k=1,\ldots,\lfloor(p-1)/3\rfloor$.

In view of the above, we have completed the proof of Theorem 1.3. □

Lemma 3.1. *Let $n\in\mathbb{N}$. Then we have*

$$\sum_{k=0}^{n}\binom{x+k-1}{k}=\binom{x+n}{n}. \tag{3.3}$$

Proof. By the Chu-Vandermonde identity (see, e.g., [GKP, p. 169]),

$$\sum_{k=0}^{n}\binom{-x}{k}\binom{-1}{n-k}=\binom{-x-1}{n}$$

which is equivalent to (3.3). Of course, it is easy to prove (3.3) by induction. □

Proof of Theorem 1.4. (i) Observe that

$$\begin{aligned}\sum_{n=0}^{p-1}D_n&=\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}=\sum_{k=0}^{p-1}\binom{2k}{k}\sum_{n=k}^{p-1}\binom{n+k}{2k}\\&=\sum_{k=0}^{p-1}\binom{2k}{k}\sum_{j=0}^{p-1-k}\binom{j+2k}{j}\\&=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{2k+1+p-1-k}{p-1-k}\ \text{(by Lemma 3.1)}\\&=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{p+k}{2k+1}=\sum_{k=0}^{p-1}\frac{k+1}{2k+1}\binom{2k+1}{k}\binom{p+k}{2k+1}\end{aligned}$$

and thus

$$\sum_{n=0}^{p-1}D_n=\sum_{k=0}^{p-1}\frac{k+1}{2k+1}\binom{p+k}{k}\binom{p}{k+1}=p+\sum_{k=1}^{p-1}\frac{p}{2k+1}\binom{p-1}{k}\binom{p+k}{k}.$$

For $k=1,\ldots,p-1$ we clearly have

$$\binom{p-1}{k}\binom{p+k}{k}=(-1)^k\prod_{j=1}^{k}\left(1-\frac{p^2}{j^2}\right)\equiv(-1)^k(1-p^2H_k^{(2)})\pmod{p^4};\tag{3.4}$$

in particular,

$$\binom{p-1}{(p-1)/2}\binom{p+(p-1)/2}{(p-1)/2}\equiv(-1)^{(p-1)/2}=\left(\frac{-1}{p}\right)\pmod{p^3}$$

since $H_{(p-1)/2}^{(2)}\equiv 0\pmod{p}$. Therefore

$$\begin{aligned}\sum_{n=0}^{p-1}D_n&\equiv\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\frac{p}{2k+1}(-1)^k+\left(\frac{-1}{p}\right)\\&\equiv\left(\frac{-1}{p}\right)-p^2E_{p-3}\pmod{p^3}\quad\text{(by (2.9)).}\end{aligned}$$

This proves (1.17).

(ii) Now we prove (1.18) and (1.19).

Let n be any positive integer. Then

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)(-1)^mD_m&=\sum_{m=0}^{n-1}(2m+1)(-1)^m\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}\\&=\sum_{k=0}^{n-1}\binom{2k}{k}\sum_{m=0}^{n-1}(2m+1)(-1)^m\binom{m+k}{2k}\end{aligned}$$

By induction, we have the identity

$$\sum_{m=0}^{n-1}(2m+1)(-1)^m\binom{m+k}{2k}=(-1)^n(k-n)\binom{n+k}{2k}. \tag{3.5}$$

Thus

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)(-1)^mD_m=&(-1)^{n-1}\sum_{k=0}^{n-1}\binom{2k}{k}(n-k)\binom{n+k}{2k}\\=&(-1)^{n-1}\sum_{k=0}^{n-1}(n-k)\binom{n}{k}\binom{n+k}{k}\\=&(-1)^{n-1}n\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)D_m&=\sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}\\&=\sum_{k=0}^{n-1}\binom{2k}{k}\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}\\&=n\sum_{k=0}^{n-1}C_k(n-k)\binom{n+k}{2k}=\sum_{k=0}^{n-1}\frac{n^2}{k+1}\binom{n-1}{k}\binom{n+k}{k}.\end{aligned}$$

In view of (3.4) and the above,

$$\begin{aligned}&\frac{1}{p}\sum_{m=0}^{p-1}(2m+1)(-1)^mD_m=\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{p+k}{k}\\\equiv&\sum_{k=0}^{p-1}(-1)^k-p^2\sum_{k=1}^{p-1}\sum_{0<j\leqslant k}\frac{(-1)^k}{j^2}=1-p^2\sum_{j=1}^{p-1}\frac{1}{j^2}\sum_{k=j}^{p-1}(-1)^k\\\equiv&1-p^2\sum_{i=1}^{(p-1)/2}\frac{1}{(2i)^2}=1-\frac{p^2}{4}H^{(2)}_{(p-1)/2}\equiv1-\frac{7}{12}p^3B_{p-3}\ (\mathrm{mod}\ p^4)\end{aligned}$$

and hence (1.18) holds. Similarly,

$$\begin{aligned}&\frac{1}{p}\sum_{m=0}^{p-1}(2m+1)D_m=\sum_{k=0}^{p-1}\frac{p}{k+1}\binom{p-1}{k}\binom{p+k}{k}\\\equiv&\binom{p+(p-1)}{p-1}+p\sum_{k=0}^{p-2}\frac{(-1)^k}{k+1}\left(1-p^2H_k^{(2)}\right)\ (\mathrm{mod}\ p^5)\\\equiv&\binom{2p-1}{p-1}-p\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k}\equiv1-pH_{(p-1)/2}\ (\mathrm{mod}\ p^3).\end{aligned}$$

(We have employed Wolstenholme's congruences $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ and $H_{p-1} \equiv 0 \pmod{p^2}$.) To obtain (1.19) it suffices to apply Lehmer's congruence (cf. [L])

$$H_{(p-1)/2} \equiv -2q_p(2) + p\,q_p^2(2) \pmod{p^2}.$$

The proof of Theorem 1.4 is now complete. □

4. Some related conjectures

Our following conjecture was motivated by Theorem 1.1(i).

Conjecture 4.1. *Let $p > 3$ be a prime.*

(i) *If $p \equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{p-1}(-1)^k A_k \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \pmod{p^3}. \tag{4.1}$$

If $p \equiv 1, 3 \pmod 8$, then

$$\sum_{k=0}^{p-1} A_k \equiv \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{256^k} \pmod{p^3}. \tag{4.2}$$

(ii) *If x belongs to the set*

$$\{1, -4, 9, -48, 81, -324, 2401, 9801, -25920, -777924, 96059601\} \bigcup \left\{\frac{81}{256}, -\frac{9}{16}, \frac{81}{32}, -\frac{3969}{256}\right\}$$

and $x \not\equiv 0 \pmod p$, then we must have

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p^2}.$$

Remark 4.1. For those

$$x = -4, 9, -48, 81, -324, 2401, 9801, -25920, -777924, 96059601, \frac{81}{256},$$

the author (cf. [Su2]) had conjectures on $\sum_{k=0}^{p-1}\binom{4k}{k,k,k,k}/(256x)^k \bmod p^2$. Motivated by this, Z. H. Sun [S2] guessed $\sum_{k=0}^{p-1}\binom{4k}{k,k,k,k}/(256x)^k \bmod p^2$ for $x = -9/16,\ 81/32,\ -3969/256$ in a similar way.

We have checked Conjecture 4.1 as well as all the other conjectures in this paper via `Mathematica 7`. Below we provide numerical evidences for (4.1) and (4.2).

Example 4.1. The values of $A_0, A_1, \ldots, A_{10}$ are given by

$$\begin{gathered}1,\ 5,\ 73,\ 1445,\ 33001,\ 819005,\ 21460825,\\ 584307365,\ 16367912425,\ 468690849005,\ 13657436403073\end{gathered}$$

respectively. Via computation we find that

$$\sum_{k=0}^{6}(-1)^k A_k = 20673445,\quad \sum_{k=0}^{6}\frac{\binom{2k}{k}^3}{16^k} = \frac{18825543}{262144},$$

and also

$$20673445 \times 262144 - 18825543 = 7^3 \times 15800002159.$$

This verifies (4.1) for $p = 7$. By computation we have

$$\sum_{k=0}^{10} A_k = 14143101786223,\quad \sum_{k=0}^{10}\frac{\binom{4k}{k,k,k,k}}{256^k} = \frac{22821835381970859184405}{1888946593147858085478 4},$$

and also

$$\begin{aligned}&14143101786223 \times 18889465931478580854784 - 22821835381970859184405\\ &\quad = 11^3 \times 200717985992690007194123778899817.\end{aligned}$$

This verifies (4.2) for $p = 11$.

Inspired by parts (ii) and (iii) of Theorem 1.1, we raise the following conjecture.

Conjecture 4.2. *For any $\varepsilon \in \{\pm 1\}$, $m, n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{n-1}(2k+1)\varepsilon^k A_k(x)^m \equiv 0 \pmod{n}. \tag{4.3}$$

If $p > 5$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p - \frac{7}{2}p^2 H_{p-1} \pmod{p^6}. \tag{4.4}$$

Remark 4.2. After reading an initial version of this paper, Guo and Zeng [GZ] proved the author's following conjectural results:

(a) For any $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$ we have

$$\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k(x) \equiv 0 \pmod{n}.$$

If p is an odd prime and x is an integer, then

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k(x) \equiv p\left(\frac{1-4x}{p}\right) \pmod{p^2}.$$

(b) For any prime $p > 3$ we have

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k \equiv p\left(\frac{p}{3}\right) \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k(-2) \equiv p - \frac{4}{3}p^2 q_p(2) \pmod{p^3}.$$

Example 4.2. It is easy to check that

$$\sum_{k=0}^{9}(2k+1)A_k^2 = 4178310699572329761604780 \equiv 0 \pmod{10}$$

and

$$\sum_{k=0}^{9}(2k+1)(-1)^k A_k^2 = -4169201796654383947725970 \equiv 0 \pmod{10}.$$

So (4.3) holds when $m = 2,\ n = 10,\ x = 1$ and $\varepsilon \in \{\pm 1\}$. Via computation we find that

$$\sum_{k=0}^{10}(2k+1)A_k = 295998598024613, \qquad H_{10} = \frac{7381}{2520},$$

and

$$295998598024613 - 11 + \frac{7}{2} \times 11^2 \times \frac{7381}{2520} = 11^6 \times \frac{120300114181}{720}.$$

This verifies (4.4) for $p = 11$.

Recall that for a prime p and a rational number x, the *p-adic valuation* of x is given by

$$\nu_p(x) = \sup\{a \in \mathbb{Z}: \text{ the denominator of } p^{-a}x \text{ is not divisible by } p\}.$$

Just like the Apéry polynomial $A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 x^k$ we define

$$D_n(x) = \sum_{k=0}^n \binom{n}{k}\binom{n+k}{k} x^k.$$

Actually $D_n((x-1)/2)$ coincides with the Legendre polynomial $P_n(x)$ of degree n.

Our following conjecture involves p-adic valuations.

Conjecture 4.3. (i) *For any $n \in \mathbb{Z}^+$ the numbers*

$$s(n) = \frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k\left(\frac{1}{4}\right)$$

and

$$t(n) = \frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)(-1)^k D_k\left(-\frac{1}{4}\right)^3$$

are rational numbers with denominators $2^{2\nu_2(n!)}$ and $2^{3(n-1+\nu_2(n!))-\nu_2(n)}$ respectively. Moreover, the numerators of $s(1), s(3), s(5), \ldots$ are congruent to 1 modulo 12 and the numerators of $s(2), s(4), s(6), \ldots$ are congruent to 7 modulo 12. If p is an odd prime and $a \in \mathbb{Z}^+$, then

$$s(p^a) \equiv t(p^a) \equiv 1 \pmod p.$$

For $p = 3$ and $a \in \mathbb{Z}^+$ we have

$$s(3^a) \equiv 4 \pmod{3^2} \quad \text{and} \quad t(3^a) \equiv -8 \pmod{3^5}.$$

(ii) *Let p be a prime. For any positive integer n and p-adic integer x, we have*

$$\nu_p\left(\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k(x)\right) \geqslant \min\{\nu_p(n),\, \nu_p(4x-1)\} \tag{4.5}$$

and

$$\nu_p\left(\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k D_k(x)^3\right) \geqslant \min\{\nu_p(n),\, \nu_p(4x+1)\}. \tag{4.6}$$

Example 4.3. We check Conjecture 4.3 with $n = p = 5$. For $n = 5$ we have

$$2^{2\nu_2(n!)} = 2^{2\nu_2(120)} = 2^6 = 64$$

and

$$2^{3(n-1+\nu_2(n!))-\nu_2(n)} = 2^{3(5-1+3)-0} = 2^{21} = 2097152.$$

Via computation we find that

$$s(5) = \frac{19849}{64} \quad \text{and} \quad t(5) = \frac{82547}{2097152}.$$

Note that $19849 \equiv 1 \pmod{12}$ and $s(5) \equiv t(5) \equiv 1 \pmod 5$. Also,

$$\nu_5\left(\frac{1}{5}\sum_{k=0}^{4}(2k+1)(-1)^k A_k(-1)\right) = \nu_5(-13095) = 1 = \nu_5(\pm 5)$$

and

$$\nu_5\left(\frac{1}{5}\sum_{k=0}^{4}(2k+1)(-1)^k D_k(1)^3\right) = \nu_5(59189205) = 1 = \nu_5(5).$$

Motivated by Theorem 1.3, we pose the following conjecture.

Conjecture 4.4. *Let p be an odd prime and let $n \geqslant 2$ be an integer. Suppose that x is a p-adic integer with $x \equiv -2k \pmod p$ for some $k \in \{1, \ldots, \lfloor (p+1)/(2n+1) \rfloor\}$. Then we have*

$$\sum_{r=0}^{p-1}(-1)^r\binom{x}{r}^{2n+1} \equiv 0 \pmod{p^2}. \tag{4.7}$$

Example 4.4. Clearly $1/3 \equiv -2 \pmod 7$ and $\lfloor (7+1)/5 \rfloor = 1$. Via computation we find that

$$\sum_{r=0}^{6}(-1)^r\binom{1/3}{r}^5 = \frac{12107415300799972328}{12157665459056928801} = 7^2 \times \frac{247090108179591272}{12157665459056928801}.$$

So (4.7) holds when $p = 7$, $n = 2$ and $x = 1/3$.

Conjecture 4.5. *Let $p \equiv 3 \pmod 4$ be a prime and let $m \in \mathbb{Z}$ with $p \nmid m(4m+1)$. Then*

$$\sum_{k=0}^{p-1}\frac{W_k(-m^2)}{(4m+1)^k} \equiv \sum_{k=0}^{p-1}\frac{A_k(-m^2/(4m+1))}{(4m+1)^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}D_{2k}\left(\frac{1}{4m}\right) \pmod{p^2}.$$

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