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ON SUMS INVOLVING PRODUCTS OF THREE BINOMIAL COEFFICIENTS

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Abstract. In this paper we mainly employ the Zeilberger algorithm to study congruences for sums of terms involving products of three binomial coefficients. Let $p>3$ be a prime. We prove that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}\equiv 0 \pmod{p^2}$$

for all $d\in\{0,\ldots,p-1\}$ with $d\equiv(p+1)/2 \pmod 2$. If $p\equiv 1 \pmod 4$ and $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$, then we show

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k}\equiv 2p-2x^2 \pmod{p^2} \text{ and } \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k}\equiv -2p \pmod{p^2}$$

by means of determining x mod p^2 via

$$(-1)^{(p-1)/4}\,x\equiv\sum_{k=0}^{(p-1)/2}\frac{k+1}{8^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2 \pmod{p^2}.$$

We also solve the remaining open cases of Rodriguez-Villegas' conjectural congruences on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

modulo p^2.

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1. Introduction

Let p be an odd prime. It is known that (see, e.g., S. Ahlgren [A], L. van Hamme [vH] and T. Ishikawa [I])

$$\sum_{k=0}^{(p-1)/2}(-1)^k\binom{-1/2}{k}^3$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p=x^2+y^2\ (2\nmid x\ \&\ 2\mid y),\\0\pmod{p^2}&\text{if }p\equiv3\pmod 4.\end{cases}$$

Clearly,

$$\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\quad\text{for all }k\in\mathbb{N}=\{0,1,2,\ldots\},$$

and

$$\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\equiv0\pmod p\quad\text{for any }k=\frac{p+1}{2},\ldots,p-1.$$

After his determination of $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ mod p^2 (where $m\in\mathbb{Z}$ and $m\not\equiv 0\pmod p$) in [Su1], the author [Su2, Su3] posed some conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ mod p^2 with $m\in\{1,-8,16,-64,256,-512,4096\}$; for example, the author [Su2] conjectured that

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac p7)=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }(\frac p7)=-1,\ \text{i.e., }p\equiv3,5,6\pmod 7,\end{cases}\tag{1.1}$$

where $(-)$ denotes the Legendre symbol. (It is known that if $(\frac p7)=1$ then $p=x^2+7y^2$ for some $x,y\in\mathbb{Z}$; see, e.g., [C, p. 31].) Quite recently Z.-H. Sun [S2] made a certain progress on those conjectures; in particular, he proved (1.1) in the case $(\frac p7)=-1$ and confirmed the author's conjecture on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/(-8)^k$ mod p^2.

Let $p=2n+1$ be an odd prime. It is easy to see that for any $k=0,\ldots,n$ we have

$$\binom{n+k}{2k}=\frac{\prod_{j=1}^k(-(2j-1)^2)}{4^k(2k)!}\prod_{j=1}^k\left(1-\frac{p^2}{(2j-1)^2}\right)\equiv\frac{\binom{2k}{k}}{(-16)^k}\pmod{p^2}.\tag{1.2}$$

Based on this observation Z.-H. Sun [S2] studied the polynomial

$$f_n(x)=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}^2x^k$$

and found the key identity

$$f_n(x(x+1)) = D_n(x)^2 \tag{1.3}$$

in his approach to (1.1), where

$$D_n(x) := \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}x^k = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k.$$

Note that the numbers $D_n = D_n(1)$ $(n \in \mathbb{N})$ are the so-called central Delannoy numbers and $P_n(x) := D_n((x-1)/2)$ is the Legendre polynomial of degree n.

Recall that the Catalan numbers are the integers defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \quad (n \in \mathbb{N})$$

while the Schröder numbers are given by

$$S_n := \sum_{k=0}^{n}\binom{n+k}{2k}C_k = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}.$$

We define the Schröder polynomial of degree n by

$$S_n(x) := \sum_{k=0}^{n}\binom{n+k}{2k}C_k x^k. \tag{1.4}$$

For basic information about D_n and S_n, the reader may consult [CHV], [Sl], [St, pp. 178 and 185], and [Su4].

In combinatorics, Zeilberger's algorithm developed in [Z] (see also Chapter 6 of [PWZ, pp. 101-119]) is an algorithm which finds a polynomial recurrence for a terminating hypergeometric sum. For example, if we use `Mathematica 7` and input `Zb[Binomial[n,k]^3,{k,0,n},n,2]`, then we obtain the following second-order recurrence for $S(n) = \sum_{k=0}^{n}\binom{n}{k}^3$:

$$-8(n+1)^2S(n) - (7n^2+21n+16)S(n+1) + (n+2)^2S(n+2) = 0.$$

Via the Schröder polynomials and the Zeilberger algorithm, we obtain the following result.

Theorem 1.1. *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k} \equiv 0 \pmod{p^2} \tag{1.5}$$

for all $d \in \{0,1,\ldots,p-1\}$ *with* $d \equiv (p+1)/2 \pmod 2$.

(ii) *If* $p \equiv 3 \pmod 4$, *then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{2k}{k+1}}{64^k} \equiv (2p+2-2^{p-1})\binom{(p-1)/2}{(p+1)/4}^2 \pmod{p^2} \tag{1.6}$$

Now we state our second theorem the first part of which plays a key role in our proof of the second part.

Theorem 1.2. *Let* $p \equiv 1 \pmod 4$ *be a prime and write* $p = x^2 + y^2$ *with* $x \equiv 1 \pmod 4$ *and* $y \equiv 0 \pmod 2$.

(i) *We can determine* x *mod* p^2 *in the following way:*

$$(-1)^{(p-1)/4}\,x \equiv \sum_{k=0}^{(p-1)/2} \frac{k+1}{8^k}\binom{2k}{k}^2 \equiv \sum_{k=0}^{(p-1)/2} \frac{2k+1}{(-16)^k}\binom{2k}{k}^2 \pmod{p^2}. \tag{1.7}$$

Also,

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}C_k}{8^k} \equiv -2\sum_{k=0}^{p-1} \frac{k\binom{2k}{k}^2}{8^k} \equiv (-1)^{(p-1)/4}\left(2x - \frac{p}{x}\right) \pmod{p^2}, \tag{1.8}$$

$$\begin{aligned} S_{(p-1)/2} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}C_k}{(-16)^k} \equiv -8 \sum_{k=0}^{(p-1)/2} \frac{k\binom{2k}{k}^2}{(-16)^k} \\ \equiv (-1)^{(p-1)/4}\,2\left(2x - \frac{p}{x}\right) \pmod{p^2}, \end{aligned} \tag{1.9}$$

$$\sum_{k=0}^{(p-1)/2} \frac{k^2\binom{2k}{k}^2}{8^k} \equiv (-1)^{(p-1)/4}\left(x - \frac{3p}{4x}\right) \pmod{p^2}, \tag{1.10}$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{k^2\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p+3)/4}\frac{p}{16x} \pmod{p^2}. \tag{1.11}$$

(ii) *We have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k} \equiv 2p - 2x^2 \pmod{p^2} \tag{1.12}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k} \equiv -2p \pmod{p^2}. \tag{1.13}$$

Remark 1.1. Let p be an odd prime. We conjecture that

$$\sum_{k=0}^{p-1}\frac{k+1}{8^k}\binom{2k}{k}^2+\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2 \equiv\begin{cases}2(\frac{2}{p})x \pmod{p^3} & \text{if } p=x^2+y^2\ (4\mid x-1\ \&\ 2\mid y),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Motivated by his study of Gaussian hypergeometric series and Calabi-Yau manifolds, in 2003 Rodriguez-Villegas [RV] raised some conjectures on congruences. In particular, he conjectured that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv b(p)\pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv c(p)\pmod{p^2},\tag{1.14}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\left(\frac{p}{3}\right)a(p)\pmod{p^2},\tag{1.15}$$

where

$$\sum_{n=1}^{\infty}a(n)q^n=q\prod_{n=1}^{\infty}(1-q^{4n})^6=\eta(4z)^6,$$

$$\sum_{n=1}^{\infty}b(n)q^n=q\prod_{n=1}^{\infty}(1-q^{6n})^3(1-q^{2n})^3=\eta^3(6z)\eta^3(2z),$$

$$\sum_{n=1}^{\infty}c(n)q^n=q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2=\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z),$$

and the Dedekind η-function is given by

$$\eta(z)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\quad (\mathrm{Im}(z)>0 \text{ and } q=e^{2\pi iz}).$$

In 1892 F. Klein and R. Fricke [KF] proved that (see also [SB])

$$a(p)=\begin{cases}4x^2-2p & \text{if } p\equiv 1 \pmod 4 \text{ and } p=x^2+y^2\ (2\nmid x),\\ 0 & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

By [SB] we also have

$$b(p)=\begin{cases}4x^2-2p & \text{if } p\equiv 1 \pmod 3 \text{ and } p=x^2+3y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

and

$$c(p) = \begin{cases} 4x^2 - 2p & \text{if } (\frac{-2}{p}) = 1 \text{ and } p = x^2 + 2y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 & \text{if } (\frac{-2}{p}) = -1, \text{ i.e., } p \equiv 5, 7 \pmod 8. \end{cases}$$

Via an advanced approach involving the p-adic Gamma function and Gauss and Jacobi sums (see K. Ono [O, Chapter 11] for an introduction to this method), E. Mortenson [M] managed to provide a partial solution of (1.14) and (1.15), with the following congruences still open:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv b(p) = 0 \pmod{p^2} \quad \text{if } p \equiv 5 \pmod 6, \tag{1.16}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \equiv c(p) \pmod{p^2} \quad \text{if } p \equiv 3 \pmod 4, \tag{1.17}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}} \equiv -a(p) \pmod{p^2} \quad \text{if } p \equiv 5 \pmod 6. \tag{1.18}$$

Concerning (1.16)-(1.18), Mortenson's approach [M] only allowed him to show that for each of them the squares of both sides of the congruence are congruent modulo p^2.

Our following theorem confirms (1.16)-(1.18) and hence completes the proof of (1.14) and (1.15). So far, all conjectures of Rodriguez-Villegas [RV] involving at most three products of binomial coefficients have been proved!

Theorem 1.3. *Let $p > 3$ be a prime.*

(i) *Given $d \in \{0, \ldots, p-1\}$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+d}\binom{2k}{k}\binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2} \quad \text{if } d \equiv \frac{1 + (\frac{p}{3})}{2} \pmod 2, \tag{1.19}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+d}\binom{2k}{k}\binom{4k}{2k}}{256^k} \equiv 0 \pmod{p^2} \quad \text{if } d \equiv \frac{1 + (\frac{-2}{p})}{2} \pmod 2, \tag{1.20}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k+d}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}} \equiv 0 \pmod{p^2} \quad \text{if } d \equiv \frac{1 + (\frac{-1}{p})}{2} \pmod 2. \tag{1.21}$$

(ii) *If $p \equiv 3 \pmod 8$ and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \equiv 4x^2 - 2p \pmod{p^2}. \tag{1.22}$$

(iii) *If* $p \equiv 5 \pmod{12}$ *and* $p = x^2 + y^2$ *with* $2 \nmid x$ *and* $2 \mid y$, *then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}} \equiv 2p - 4x^2 \pmod{p^2}. \tag{1.23}$$

In the case $d = 1$, Theorem 1.3(i) yields the following new result. (Note that $\binom{2k}{k}\binom{3k}{k+1} = 2\binom{2k}{k+1}\binom{3k}{k}$ for any $k \in \mathbb{N}$.)

Corollary 1.1. *Let* $p > 3$ *be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k+1}}{108^k} \equiv 0 \pmod{p^2} \quad \textit{if } p \equiv 1 \pmod 3, \tag{1.24}$$

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}\binom{2k}{k+1}}{256^k} \equiv 0 \pmod{p^2} \quad \textit{if } p \equiv 1, 3 \pmod 8, \tag{1.25}$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k+1}}{12^{3k}} \equiv 0 \pmod{p^2} \quad \textit{if } p \equiv 1 \pmod 4. \tag{1.26}$$

We will prove Theorems 1.1-1.3 in Sections 2-4 respectively.

2. Proof of Theorem 1.1

Lemma 2.1. *For any positive integer* n *we have*

$$\sum_{k=1}^{n} \binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1} x^{k-1}(x+1)^{k+1} = n(n+1)S_n(x)^2. \tag{2.1}$$

Proof. Observe that

$$S_n(x)^2 = \sum_{k=0}^{n} \binom{n+k}{2k} C_k x^k \sum_{l=0}^{n} \binom{n+l}{2l} C_l x^l = \sum_{m=0}^{2n} a_m(n) x^m,$$

where

$$a_m(n) := \sum_{k=0}^{m} \binom{n+k}{2k} C_k \binom{n+m-k}{2m-2k} C_{m-k}.$$

Also, the coefficient of x^m on the left-hand side of (2.1) coincides with

$$\begin{aligned} b_m(n) :=& \sum_{k=1}^{m+1} \binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\binom{k+1}{m+1-k} \\ =& \sum_{k=0}^{m} \binom{n+k+1}{2k+2}\binom{2k+2}{k+1}\binom{2k+2}{k}\binom{k+2}{m-k}. \end{aligned}$$

Thus, for the validity of (2.1) it suffices to show that $b_m(n)=n(n+1)a_m(n)$ for all $m=0,1,\ldots$. Obviously, $a_0(n)=1$ and $b_0(n)=n(n+1)$. Also, $a_1(n)=n(n+1)$ and $b_1(n)=n^2(n+1)^2$. By the Zeilberger algorithm via `Mathematica 7` we find that both $u_m=a_m(n)$ and $u_m=b_m(n)$ satisfy the following recursion:

$$\begin{aligned}&(m+2)(m+3)(m+4)u_{m+2}\\=&2(2mn^2+5n^2+2mn+5n-m^3-6m^2-11m-6)u_{m+1}\\&-(m+1)(m-2n)(m+2n+2)u_m.\end{aligned}$$

So $b_m(n)=n(n+1)a_m(n)$ for all $m\in\mathbb{N}$. This proves (2.1). $\square$

Proof of Theorem 1.1. We first determine $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{2k}{k+1}/64^k \bmod p^2$ via Lemma 2.1, which actually led the author to the study of (1.5).

Recall the following combinatorial identity (cf. [Su2, (4.3)]):

$$\sum_{k=0}^{n}\binom{n+k}{2k}\frac{C_k}{(-2)^k}=\begin{cases}(-1)^{(n-1)/2}C_{(n-1)/2}/2^n & \text{if } 2\nmid n,\\ 0 & \text{if } 2\mid n.\end{cases}\tag{2.2}$$

Set $n=(p-1)/2$. Applying (2.1) with $x=-1/2$ we get

$$\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\frac{1}{(-2)^{k-1}2^{k+1}}=n(n+1)S_n\left(-\frac12\right)^2.$$

Thus, with the helps from (1.2) and (2.2), we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{64^k}\equiv&\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\frac{1}{(-4)^k}\\=&-n(n+1)S_n\left(-\frac12\right)^2\\\equiv&\begin{cases}0 \pmod{p^2} & \text{if } p\equiv1 \pmod 4\\ C_{(n-1)/2}^2/2^{2n+2} \pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned}$$

Therefore (1.5) with $d=1$ holds if $p\equiv1 \pmod 4$. In the case $p\equiv 3 \pmod 4$, clearly

$$\begin{aligned}\frac{C_{(n-1)/2}^2}{2^{2n+2}}=&\frac{\left(\binom{(p-1)/2}{(p+1)/4}\frac{2}{p-1}\right)^2}{4\times2^{p-1}}\\\equiv&\frac{1}{(1-2p)(1+p\,q_p(2))}\binom{(p-1)/2}{(p+1)/4}^2\\\equiv&(1+2p-p\,q_p(2))\binom{(p-1)/2}{(p+1)/4}^2 \pmod{p^2}\end{aligned}$$

where $q_p(2) = (2^{p-1} - 1)/p$, and hence (1.6) holds.

For $d = 0, 1, 2, \dots$ set

$$u_d = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{2k}{k+d}}{64^k} = \sum_{d \leqslant k < p} \frac{\binom{2k}{k}^2 \binom{2k}{k+d}}{64^k}.$$

By the Zeilberger algorithm we find the recursion

$$(2d+1)^2 u_d - (2d+3)^2 u_{d+2} = \frac{(2p-1)^2(d+1)}{64^{p-1}p} \binom{2p}{p+d+1} \binom{2p-2}{p-1}^2.$$

Note that

$$\binom{2p-2}{p-1} = pC_{p-1} \equiv 0 \pmod{p}.$$

If $0 \leqslant d < p-2$, then

$$\binom{2p}{p+d+1} = \frac{2p}{p+d+1} \binom{2p-1}{p+d} \equiv 0 \pmod{p}$$

and hence

$$(2d+1)^2 u_d \equiv (2d+3)^2 u_{d+2} \pmod{p^2}.$$

For $d \in \{0, \dots, p-3\}$ with $d \equiv (p+1)/2 \pmod 2$, clearly $p \neq 2d+1 < 2p$ and hence

$$u_{d+2} \equiv 0 \pmod{p^2} \implies u_d \equiv 0 \pmod{p^2}.$$

If $d \in \{p-1, p-2\}$ and $d \equiv (p+1)/2 \pmod 2$, then $d \geqslant (p+1)/2$ and hence $u_d \equiv 0 \pmod{p^2}$. So (1.5) holds for all $d \in \{0, \dots, p-1\}$ with $d \equiv (p+1)/2 \pmod 2$.

So far we have completed the proof of Theorem 1.1. $\square$

3. Proof of Theorem 1.2

Lemma 3.1. *For any $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n} \binom{2k}{k}^3 \binom{k}{n-k} (-16)^{n-k} = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2. \tag{3.1}$$

Proof. For $n = 0, 1$, both sides of (3.1) take the values 1 and 8 respectively. Let u_n denote the left-hand side of (3.1) or the right-hand side of (3.1). Applying the Zeilberger algorithm via `Mathematica 7`, we obtain the recursion

$$(n+2)^3 u_{n+2} = 8(2n+3)(2n^2+6n+5)u_{n+1} - 256(n+1)^3 u_n \ (n \in \mathbb{N}).$$

So, by induction (3.1) holds for all $n = 0, 1, 2, \dots$. $\square$

Lemma 3.2. *Let p be an odd prime. Then*

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\ \equiv&\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\ \equiv&p\left(\frac{-1}{p}\right)\pmod{p^3}.\end{aligned}$$

Proof. In view of Lemma 3.1, we have

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\ =&\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^n\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}\\ =&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{8^k}\sum_{j=0}^{p-1-k}(k+j+1)\binom{k}{j}\frac{(-16)^j}{8^j}\\ \equiv&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{8^k}\left((k+1)\sum_{j=0}^k\binom{k}{j}(-2)^j-2k\sum_{j=1}^k\binom{k-1}{j-1}(-2)^{j-1}\right)\\ =&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{8^k}\left((k+1)(-1)^k-2k(-1)^{k-1}\right)\\ \equiv&\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\pmod{p^3}.\end{aligned}$$

In [Su3] the author conjectured that

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3}\pmod{p^4}$$

provided $p>3$, where $E_0,E_1,E_2,\ldots$ are Euler numbers given by

$$E_0=1\quad\text{and}\quad\sum_{\substack{k=0\\2\mid k}}^n\binom{n}{k}E_{n-k}=0\quad(n=1,2,3,\ldots).$$

The last congruence is still open but [GZ] confirmed that

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)\pmod{p^3}.$$

So we have

$$\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\equiv p\left(\frac{-1}{p}\right)\pmod{p^3}.$$

Similarly,

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\ =&\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}\\ \equiv&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-16)^k}\left((2k+1)\sum_{j=0}^{k}\binom{k}{j}+2k\sum_{j=1}^{k}\binom{k-1}{j-1}\right)\\ \equiv&\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)\pmod{p^3}.\end{aligned}$$

This concludes the proof. $\square$

Lemma 3.3. *Let p be an odd prime. Then*

$$\begin{aligned}2\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{8^k}+\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{8^k}&\equiv 2p^2\left(\frac{2}{p}\right)\pmod{p^3},\\ 8\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{(-16)^k}+\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{(-16)^k}&\equiv 2p^2\left(\frac{-1}{p}\right)\pmod{p^3},\\ \sum_{k=0}^{(p-1)/2}(2k^2+4k+1)\frac{\binom{2k}{k}^2}{8^k}&\equiv p^2\left(\frac{2}{p}\right)\pmod{p^3},\\ \sum_{k=0}^{(p-1)/2}(8k^2+4k+1)\frac{\binom{2k}{k}^2}{(-16)^k}&\equiv p^2\left(\frac{-1}{p}\right)\pmod{p^3}.\end{aligned}$$

Proof. By induction, for every $n=0,1,2,\ldots$ we have

$$\begin{aligned}\sum_{k=0}^n\left(2k+\frac{1}{k+1}\right)\frac{\binom{2k}{k}^2}{8^k}&=\frac{(2n+1)^2}{(n+1)8^n}\binom{2n}{n}^2,\\ \sum_{k=0}^n\left(8k+\frac{1}{k+1}\right)\frac{\binom{2k}{k}^2}{(-16)^k}&=\frac{(2n+1)^2}{(n+1)(-16)^n}\binom{2n}{n}^2,\\ \sum_{k=0}^n(2k^2+4k+1)\frac{\binom{2k}{k}^2}{8^k}&=\frac{(2n+1)^2}{8^n}\binom{2n}{n}^2,\\ \sum_{k=0}^n(8k^2+4k+1)\frac{\binom{2k}{k}^2}{(-16)^k}&=\frac{(2n+1)^2}{(-16)^n}\binom{2n}{n}^2.\end{aligned}$$

Applying these identities with $n=(p-1)/2$ we immediately get the desired congruences. □

Let $p\equiv 1\pmod 4$ be a prime and write $p=x^2+y^2$ with $x\equiv 1\pmod 4$ and $y\equiv 0\pmod 2$. In 1828 Gauss showed the congruence $\binom{(p-1)/2}{(p-1)/4}\equiv 2x\pmod p$. In 1986, S. Chowla, B. Dwork and R. J. Evans [CDE] used Gauss and Jacobi sums to prove that

$$\binom{(p-1)/2}{(p-1)/4}\equiv\frac{2^{p-1}+1}{2}\left(2x-\frac{p}{2x}\right)\pmod{p^2},\tag{3.2}$$

which was first conjectured by F. Beukers. (See also [BEW, Chapter 9] and [HW] for further related results.) In 2009, the author (see [Su2]) conjectured that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right)\pmod{p^2},\tag{3.3}$$

and this was confirmed by Z.-H. Sun [S1] via (3.2) and the Legendre polynomials.

Proof of Theorem 1.2(i). By (1.2),

$$S_{(p-1)/2}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{(-16)^k}\pmod{p^2}.$$

In view of this and Lemma 3.3 and (3.3), it suffices to show (1.7).

As $p \mid \binom{2k}{k}$ for all $k=(p+1)/2,\ldots,p-1$, we have

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\&=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\sum_{n=k}^{p-1}\frac{n+1}{8^{n-k}}\binom{2(n-k)}{n-k}^2\\&=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\sum_{j=0}^{p-1-k}\frac{k+j+1}{8^j}\binom{2j}{j}^2\\&\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{8^k}\sum_{j=0}^{(p-1)/2}\frac{(k+1)+(j+1)-1}{8^j}\binom{2j}{j}^2\\&=2\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{8^k}\sum_{j=0}^{(p-1)/2}\frac{(j+1)\binom{2j}{j}^2}{8^j}-\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\right)^2\pmod{p^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\&\equiv 2\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(-16)^k}\sum_{j=0}^{(p-1)/2}\frac{(2j+1)\binom{2j}{j}^2}{(-16)^j}-\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\right)^2\pmod{p^2}.\end{aligned}$$

Combining these with Lemma 3.2 and (3.3), we immediately obtain (1.7). □

Lemma 3.4. *Let $p\equiv 1 \pmod 4$ be a prime. Write $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$. Then*

$$D_{(p-1)/2}\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right)\pmod{p^2}. \tag{3.4}$$

Proof. By (1.2),

$$D_{(p-1)/2}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(-16)^k}\pmod{p^2}.$$

So (3.4) follows from (3.3). □

Remark 3.1. If p is a prime with $p\equiv 3 \pmod 4$, then $n=(p-1)/2$ is odd and hence

$$\begin{aligned}D_n&\equiv\sum_{k=0}^{n}(-1)^k\frac{\binom{2k}{k}^2}{16^k}=\sum_{k=0}^{n}(-1)^k\binom{-1/2}{k}^2\\&\equiv\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}^2=0\pmod p.\end{aligned}$$

The following result was conjectured by the author [Su2] and confirmed by Z.-H. Sun [S2].

Lemma 3.5. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } 4\mid p-1\ \&\ p=x^2+y^2\ (2\nmid x),\\ 0\pmod{p^2}&\text{if } p\equiv 3\pmod 4.\end{cases}\tag{3.5}$$

Remark 3.2. Fix an odd prime $p=2n+1$. By (1.2) and (1.3) we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2 2^k=D_n^2\pmod{p^2}.$$

Hence (3.5) follows from Lemma 3.4 and Remark 3.1.

Lemma 3.6. *For any positive integer n we have*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\frac{2k+1}{(k+1)^2}x^k(x+1)^{k+1}=\frac{S_n(x)}{2}(D_{n-1}(x)+D_{n+1}(x)).\tag{3.6}$$

Proof. Note that

$$S_n(x)(D_{n-1}(x)+D_{n+1}(x))=\sum_{m=0}^{2n+1}c_m(n)x^m$$

where

$$\begin{aligned}c_m(n)=&\sum_{k=0}^{m}\binom{n+k}{2k}C_k\binom{2m-2k}{m-k}\left(\binom{n-1+m-k}{2m-2k}+\binom{n+1+m-k}{2m-2k}\right)\\ =&2\sum_{k=0}^{m}\binom{n+k}{2k}C_k\binom{n+m-k}{2m-2k}\binom{2m-2k}{m-k}\frac{(m+n-k)^2-n(2m-2k-1)}{(m+n-k)(n-m+k+1)}.\end{aligned}$$

By the Zeilberger algorithm we find that $u_m=c_m(n)/2$ satisfies the recursion

$$\begin{aligned}&(m+2)(m+3)^2(m^2+5m+6+4n(n+1))u_{m+2}+2P(m,n)u_{m+1}\\ =&(m+2)((2n+1)^2-m^2)(m^2+7m+12+4n(n+1))u_m\end{aligned}\tag{3.7}$$

where $P(m,n)$ denotes the polynomial

$$\begin{aligned}&m^5+11m^4+45m^3+83m^2+64m+12+20n^4-40n^3-58n^2-38n\\ &-25mn+m^2n+2m^3n-33mn^2+m^2n^2+2m^3n^2-16mn^3-8mn^4.\end{aligned}$$

Clearly the coefficient of x^m on the left-hand side of (3.6) coincides with

$$d_m(n) = \sum_{k=0}^{m} \binom{n+k}{2k}\binom{2k}{k}^2\binom{k+1}{m-k}\frac{2k+1}{(k+1)^2}.$$

By the Zeilberger algorithm $u_m = d_m(n)$ also satisfies the recursion (3.7). Thus we have $d_m(n) = c_m(n)$ by induction on m. So (3.6) holds. □

Proof of Theorem 1.2(ii). Write $p = 2n+1$. By (2.1),

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}2^k = \frac{n(n+1)}{2}S_n^2.$$

Thus, by (1.2) and (1.9) we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k} &\equiv \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}2^k\\ &\equiv \frac{p^2-1}{8}4(4x^2-4p) \pmod{p^2}\end{aligned}$$

and hence (1.12) holds.

Now we consider (1.13). Observe that

$$\binom{2k}{k+1}^2 = \left(1-\frac{2k+1}{(k+1)^2}\right)\binom{2k}{k}^2 \quad \text{for } k = 0, 1, 2, \dots,$$

and

$$\binom{2(p-1)}{p-1}\binom{2(p-1)}{(p-1)+1}^2 = \frac{p}{2p-1}\binom{2p-1}{p-1}\binom{2p-2}{p-2}^2 \equiv -p \pmod{p^2}.$$

Thus we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k} \equiv -p + \sum_{k=0}^{n}\frac{\binom{2k}{k}^3}{(-8)^k} - \sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k} \pmod{p^2}. \tag{3.8}$$

By (1.2) and (3.6) with $x = 1$,

$$\begin{aligned}\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k} &\equiv \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\frac{(2k+1)2^k}{(k+1)^2}\\ &= \frac{S_n}{4}(D_{n-1}+D_{n+1}) \pmod{p^2}.\end{aligned}$$

It is known (cf. [Sl] and [St, p. 191]) that

$$(n+1)D_{n+1}=3(2n+1)D_n-nD_{n-1}\quad\text{and}\quad D_{n+1}-3D_n=2nS_n.$$

Thus

$$\begin{aligned}n(D_{n-1}+D_{n+1})=&3(2n+1)D_n-D_{n+1}\\=&3(2n+1)D_n-(3D_n+2nS_n)=2n(3D_n-S_n)\end{aligned}$$

and hence

$$\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\equiv\frac{S_n}{2}(3D_n-S_n)\pmod{p^2}.$$

With the help of (1.9) and (3.4), we have

$$\frac{S_n}{2}(3D_n-S_n)\equiv\left(2x-\frac{p}{x}\right)\left(3\left(2x-\frac{p}{2x}\right)-\left(4x-\frac{2p}{x}\right)\right)\pmod{p^2}$$

and hence

$$\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\equiv 4x^2-p\pmod{p^2}.$$

Combining this with (3.5) and (3.8), we immediately obtain (1.13). □

4. Proof of Theorem 1.3

Lemma 4.1. *Let p be an odd prime. Then, for any p-adic integer $x\not\equiv 0,-1\pmod p$ we have*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\left(\frac{-x}{64}\right)^k\equiv\left(\frac{x+1}{p}\right)\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\left(\frac{x}{64(x+1)^2}\right)^k\pmod p.\tag{4.1}$$

Proof. Taking $n=(p-1)/2$ in the following identity of MacMahon (see, e.g., [G, (6.7)])

$$\sum_{k=0}^{n}\binom{n}{k}^3x^k=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{n-k}{k}x^k(1+x)^{n-2k}$$

and noting (1.2) and the basic facts

$$\binom{n}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod p$$

and

$$\binom{n-k}{k} \equiv \binom{-1/2-k}{k} = \frac{\binom{4k}{2k}}{(-4)^k} \pmod p,$$

we immediately get (4.1). $\square$

Proof of Theorem 1.3. (i) For $d = 0, 1, 2, \ldots$, we define

$$f(d) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+d}\binom{2k}{k}\binom{3k}{k}}{108^k}, \quad g(d) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+d}\binom{2k}{k}\binom{4k}{2k}}{256^k},$$

and

$$h(d) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+d}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}.$$

By the Zeilberger algorithm, we find the recursive relations:

$$\begin{aligned}&(3d+1)(3d+2)f(d) - (3d+4)(3d+5)f(d+2)\\ =&\frac{(3p-1)(3p-2)(d+1)}{108^{p-1}p}\binom{2p}{p+d+1}\binom{2p-2}{p-1}\binom{3p-3}{p-1},\end{aligned} \tag{4.2}$$

$$\begin{aligned}&(4d+1)(4d+3)g(d) - (4d+5)(4d+7)g(d+2)\\ =&\frac{(4p-1)(4p-3)(d+1)}{256^{p-1}p}\binom{2p}{p+d+1}\binom{2p-2}{p-1}\binom{4p-4}{2p-2},\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&(6d+1)(6d+5)h(d) - (6d+7)(6d+11)h(d+2)\\ =&\frac{(6p-1)(6p-5)(d+1)}{1728^{p-1}p}\binom{2p}{p+d+1}\binom{3p-3}{p-1}\binom{6p-6}{3p-3}.\end{aligned} \tag{4.4}$$

Recall that $\binom{2p-2}{p-1} = pC_{p-1} \equiv 0 \pmod p$. Also,

$$\begin{aligned}(3p-2)\binom{3p-3}{p-1} =& p\binom{3p-2}{p} \equiv 0 \pmod p,\\ (4p-3)\binom{4p-4}{2p-2} =& p\binom{4p-2}{2p} \equiv 0 \pmod p,\\ (6p-5)\binom{6p-6}{3p-3} =& \frac{3p(3p-1)(3p-2)}{(6p-3)(6p-4)}\binom{6p-3}{3p} \equiv 0 \pmod p.\end{aligned}$$

If $0 \leqslant d < p-1$, then

$$\binom{2p}{p+d+1} = \binom{2p}{p-1-d} \equiv 0 \pmod p.$$

So, by (4.2)-(4.4), for any $d\in\{0,\ldots,p-1\}$ we have

$$(3d+1)(3d+2)f(d)\equiv(3d+4)(3d+5)f(d+2)\pmod{p^2},\tag{4.5}$$

$$(4d+1)(4d+3)g(d)\equiv(4d+5)(4d+7)g(d+2)\pmod{p^2},\tag{4.6}$$

$$(6d+1)(6d+5)h(d)\equiv(6d+7)(6d+11)h(d+2)\pmod{p^2}.\tag{4.7}$$

Fix $0\leqslant d\leqslant p-1$. If $d\equiv(1+(\frac{p}{3}))/2\pmod 2$, then it is easy to verify that $\{3d+1,3d+2\}\cap\{p,2p\}=\emptyset$, hence $(3d+1)(3d+2)\not\equiv 0\pmod p$ and thus by (4.5) we have

$$f(d+2)\equiv 0\ (\mathrm{mod}\ p^2)\implies f(d)\equiv 0\ (\mathrm{mod}\ p^2).$$

If $d\equiv(1+(\frac{-2}{p}))/2\pmod 2$, then $\{4d+1,4d+3\}\cap\{p,3p\}=\emptyset$, hence $(4d+1)(4d+3)\not\equiv 0\pmod p$ and thus by (4.6) we have

$$g(d+2)\equiv 0\ (\mathrm{mod}\ p^2)\implies g(d)\equiv 0\ (\mathrm{mod}\ p^2).$$

If $d\equiv(1+(\frac{-1}{p}))/2\pmod 2$, then $\{6d+1,6d+3\}\cap\{p,3p,5p\}=\emptyset$, hence $(6d+1)(6d+3)\not\equiv 0\pmod p$ and thus (4.7) yields

$$h(d+2)\equiv 0\ (\mathrm{mod}\ p^2)\implies h(d)\equiv 0\ (\mathrm{mod}\ p^2).$$

Since

$$f(p)=f(p+1)=g(p)=g(p+1)=h(p)=h(p+1)=0,$$

by the last paragraph, for every $d=p+1,p,\ldots,0$ we have the desired (1.19)-(1.21).

(ii) Assume that $p\equiv 3\pmod 8$ and $p=x^2+2y^2$ with $x,y\in\mathbb{Z}$. Since $4x^2\not\equiv 0\pmod p$ and Mortenson [M] already proved that the squares of both sides of (1.22) are congruent modulo p^2, (1.22) is reduced to its mod p form. Applying (4.1) with $x=1$ we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\pmod p.$$

By [A, Theorem 5(3)], we have

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}(-1)^k\equiv 4x^2-2p\pmod p,$$

where $n = (p-1)/2$. For $k = 0, \ldots, n$ clearly

$$\begin{aligned}\binom{n}{k}^2\binom{n+k}{k}(-1)^k =&\binom{(p-1)/2}{k}^2\binom{-(p+1)/2}{k}\\ \equiv&\binom{-1/2}{k}^3 = \frac{\binom{2k}{k}^3}{(-64)^k} \pmod p,\end{aligned}$$

therefore

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{-1}{p}\right)(4x^2-2p) \pmod p$$

and hence (1.22) follows.

(iii) Finally we suppose $p \equiv 5 \pmod{12}$ and write $p = x^2 + y^2$ with x odd and y even. Once again it suffices to show the mod p form of (1.23) in view of Mortenson's work [M]. As Z.-H. Sun observed,

$$\binom{(p-5)/6+k}{2k}\binom{2k}{k} \equiv \binom{k-5/6}{2k}\binom{2k}{k} = \frac{\binom{3k}{k}\binom{6k}{3k}}{(-432)^k} \pmod p$$

for all $k = 0, 1, 2, \ldots$. If $p/6 < k < p/3$ then $p \mid \binom{6k}{3k}$; if $p/3 < k < p/2$ then $p \mid \binom{3k}{k}$; if $p/2 < k < p$ then $p \mid \binom{2k}{k}$. Thus

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}} \equiv& \sum_{k=0}^{(p-5)/6}\binom{(p-5)/6+k}{2k}\binom{2k}{k}^2\left(-\frac{1}{4}\right)^k\\ =&D_{2n}\left(-\frac{1}{2}\right)^2 \pmod p \quad \text{(by (1.3))},\end{aligned}$$

where $n = (p-5)/12$. Note that

$$D_{2n}\left(-\frac{1}{2}\right) = \frac{1}{(-4)^n}\binom{2n}{n}$$

by [G, (3.133) and (3.135)], and

$$\binom{(p-1)/2}{(p-1)/4} \equiv 12(-432)^n\binom{2n}{n} \pmod p$$

by P. Morton [Mo]. Therefore

$$D_{2n}\left(-\frac{1}{2}\right)^2 = \frac{1}{16^n}\binom{2n}{n}^2 \equiv \frac{\binom{(p-1)/2}{(p-1)/4}^2}{12^{6n+2}} \equiv \left(\frac{12}{p}\right)\binom{(p-1)/2}{(p-1)/4}^2 \pmod p.$$

Thus, by applying Gauss' congruence $\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p$ (cf. [BEW, (9.0.1)] or [HW]) we immediately get the mod p form of (1.23) from the above.

The proof of Theorem 1.3 is now complete. □

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