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ON DIVISIBILITY OF BINOMIAL COEFFICIENTS

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In memory of Prof. Alf van der Poorten

Abstract. Motivated by Catalan numbers and higher-order Catalan numbers, we study in this paper factors of products of at most two binomial coefficients.

1. Introduction

There are many papers on divisibility concerning sums of binomial coefficients, see, for example, [C], [CP], [GJZ], [S1], [S2] and [ST].

Via a sophisticated theory of hypergeometric series, J. W. Bober [B] determined all those $a_1,\ldots,a_r,b_1,\ldots,b_{r+1}\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ with $a_1+\cdots+a_r=b_1+\cdots+b_{r+1}$ such that

$$\frac{(a_1n)!\cdots(a_rn)!}{(b_1n)!\cdots(b_{r+1}n)!}$$

is an integer for any $n\in\mathbb{Z}^+$. In particular, if $k,l\in\mathbb{Z}^+$ then

$$\begin{aligned}&\frac{\binom{ln}{n}\binom{kln}{ln}}{\binom{kn}{n}}=\frac{(kln)!((k-1)n)!}{(kn)!((l-1)n)!((k-1)ln)!}\in\mathbb{Z}\ \text{ for all } n\in\mathbb{Z}^+,\\ &\iff k=l,\ \text{or } \{k,l\}\cap\{1,2\}\neq\emptyset,\ \text{or } \{k,l\}=\{3,5\}.\end{aligned}$$

In this paper we study factors of products of at most two binomial coefficients. Our methods are of elementary and combinatorial nature so that many readers could understand the proofs easily.

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For $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, the nth (usual) Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1},$$

The Catalan numbers arise naturally in many enumeration problems in discrete mathematics (see, for example, [St, pp. 219–229]). For $h, n \in \mathbb{N}$ the nth (generalized) Catalan number of order h is defined by

$$C_n^{(h)} = \frac{1}{hn+1}\binom{(h+1)n}{n} = \binom{(h+1)n}{n} - h\binom{(h+1)n}{n-1}.$$

We extend the basic fact $(hn+1) \mid \binom{(h+1)n}{n}$ in the following theorem.

Theorem 1.1. *Let $k, l, n \in \mathbb{Z}^+$. Then*

$$\frac{ln+1}{(k, ln+1)} \,\Big|\, \binom{kn+ln}{kn}, \tag{1.1}$$

where $(k, ln+1)$ denotes the greatest common divisor of k and $ln+1$. In particular, $(ln+1) \mid \binom{kn+ln}{kn}$ if l is divisible by all the prime factors of k.

Our following conjecture seems difficult.

Conjecture 1.1. *Let k and l be positive integers. If $(ln+1) \mid \binom{kn+ln}{kn}$ for all sufficiently large positive integers n, then each prime factor of k divides l. In other words, if k has a prime factor not dividing l then there are infinitely many positive integers n such that $(ln+1) \nmid \binom{kn+ln}{kn}$.*

In view of Conjecture 1.1 we introduce a new function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{N}$ as follows. For positive integers k and l, if $(ln+1) \mid \binom{kn+ln}{kn}$ for all $n \in \mathbb{Z}^+$ (which happens if all prime factors of k divide l) then we set $f(k, l) = 0$, otherwise we define $f(k, l)$ to be the smallest positive integer n such that $(ln+1) \nmid \binom{kn+ln}{kn}$. The following values of f come from our computation via `Mathematica`.

$$f(7, 36) = 279,\ f(10, 192) = 362,\ f(11, 100) = 1187,\ f(22, 200) = 6462;$$
$$f(74, 62) = 885,\ f(213, 3) = 3384,\ f(223, 93) = 13368,\ f(307, 189) = 31915.$$

Now we turn to our results on factors of products of two binomial coefficients. They are related to Catalan numbers or higher-order Catalan numbers. Note that $nC_n^{(h)} = \binom{(h+1)n}{n-1}$ for any $h, n \in \mathbb{Z}^+$.

Theorem 1.2. *Let $k,n\in\mathbb{Z}^+$.*

(i) *We have*

$$\binom{kn}{n}\ \Big|\ (2k-1)C_n\binom{2kn}{2n},\tag{1.2}$$

and $(2k-1)C_n\binom{2kn}{2n}/\binom{kn}{n}$ is odd if and only if $n+1$ is a power of two.

(ii) *Let $(k+1)'$ be the odd part of $k+1$. Then*

$$\binom{2n}{n}\ \Big|\ (k+1)'C_n^{(k-1)}\binom{2kn}{kn},\tag{1.3}$$

and $(k+1)'C_n^{(k-1)}\binom{2kn}{kn}/\binom{2n}{n}$ is odd if and only if $(k-1)n+1$ is a power of two.

By Theorem 1.2(ii), if $n\in\mathbb{Z}^+$ and $k=2^l-1$ for some $l\in\mathbb{N}$ then

$$\binom{2n}{n}\ \Big|\ \binom{2kn}{kn}C_n^{(k-1)},\quad \text{i.e.,}\ n\binom{2n}{n}\ \Big|\ \binom{kn}{n-1}\binom{2kn}{kn}.$$

Via `Mathematica` we find that this can be further strengthened.

Theorem 1.3. *For any $k,n\in\mathbb{Z}^+$ we have*

$$2^{k-1}\binom{2n}{n}\ \Big|\ \binom{2(2^k-1)n}{(2^k-1)n}C_n^{(2^k-2)}.\tag{1.4}$$

A key step in our proof of (1.4) is to show the first assertion in our following conjecture with m a prime.

Conjecture 1.2. *Let $m>1$ be an integer and let k and n be positive integers. Then the sum of all digits in the expansion of $(m^k-1)n$ in base m is at least $k(m-1)$. Also, the expansion of $\frac{m^k-1}{m-1}n$ in base m has at least k nonzero digits.*

Our following result involves certain particular properties of 3 and 5.

Theorem 1.4. *For any $n\in\mathbb{Z}^+$ we have*

$$(6n+1)\binom{5n}{n}\ \Big|\ \binom{3n-1}{n-1}C_{3n}^{(4)}\tag{1.5}$$

and

$$\binom{3n}{n}\ \Big|\ \binom{5n-1}{n-1}C_{5n}^{(2)}.\tag{1.6}$$

Define two new sequences $\{s_n\}_{n\geqslant1}$ and $\{t_n\}_{n\geqslant1}$ of integers by

$$s_n=\frac{\binom{3n-1}{n-1}C_{3n}^{(4)}}{(6n+1)\binom{5n}{n}}=\frac{\binom{3n-1}{n-1}\binom{15n}{3n}}{(6n+1)(12n+1)\binom{5n}{n}}=\frac{\binom{3n}{n}\binom{15n}{3n-1}}{9n(6n+1)\binom{5n}{n}}\tag{1.7}$$

and

$$t_n=\frac{\binom{5n-1}{n-1}C_{5n}^{(2)}}{\binom{3n}{n}}=\frac{\binom{5n-1}{n-1}\binom{15n}{5n}}{(10n+1)\binom{3n}{n}}=\frac{\binom{5n}{n}\binom{15n}{5n-1}}{25n\binom{3n}{n}}. \tag{1.8}$$

It would be interesting to find recursion formulae or combinatorial interpretations for s_n and t_n.

Based on our computation via `Mathematica`, we formulate the following conjecture on the sequence $\{t_n\}_{n\geqslant 1}$.

Conjecture 1.3. *For any $n\in\mathbb{Z}^+$ we have $(10n+3)\mid 21t_n$.*

For a prime p, the p-adic evaluation of an integer m is given by

$$\nu_p(m)=\sup\{a\in\mathbb{N}:\ p^a\mid m\}.$$

For a rational number $x=m/n$ with $m\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$, we set $\nu_p(x)=\nu_p(m)-\nu_p(n)$ for any prime p.

In this paper the following lemma is fundamental for our approach.

Lemma 1.1. (i) *A rational number x is an integer if and only if $\nu_p(x)\geqslant 0$ for all primes p.*

(ii) (Legendre's theorem) For any prime p and $n\in\mathbb{N}$, we have

$$\nu_p(n!)=\sum_{i=1}^{\infty}\left\lfloor\frac{n}{p^i}\right\rfloor=\frac{n-\rho_p(n)}{p-1},$$

where $\rho_p(n)$ is the sum of the digits in the expansion of n in base p.

(iii) Let n be a positive integer. Then $\nu_2(n!)\leqslant n-1$. Also, $\nu_2(n!)=n-1$ if and only if n is a power of two.

Part (i) is obvious. Part (ii) is well known and it can be found in [R, pp. 22–24]. And the third part follows immediately from Part (ii), see also [SD, Lemma 4.1].

Example 1.1. Let $m\in\mathbb{N}$ and $n\in\mathbb{Z}^+$, and set

$$Q(m,n):=\frac{\binom{2n}{n}\binom{2m+2n}{2n}}{2\binom{m+n}{n}}.$$

Then

$$Q(m,n)=\frac{2^{n-1}}{n!}\prod_{j=1}^{n}(2m+2j-1)=(-1)^n2^{2n-1}\binom{-m-1/2}{n}.$$

Applying Lemma 1.1 we see that $Q(m,n)\in\mathbb{Z}$ and that $2\nmid Q(m,n)$ if and only if n is a power of two. When $n>1$ we have

$$\frac{\binom{2n}{n}\binom{2m+2n}{2n-1}}{8\binom{m+n}{n}}=Q(m+1,n-1)\in\mathbb{Z}.$$

Also, $\binom{2n}{n}\binom{2m+2n}{2n-1}/\left(8\binom{m+n}{n}\right)$ is odd if and only if $n-1$ is a power of two.

By the latter part of Example 1.1, $\binom{kn}{n}\mid\binom{2n}{n}\binom{2kn}{2n-1}$ for any $k,n\in\mathbb{Z}^+$. In view of this and Theorems 1.2-1.4, we raise the following conjecture.

Conjecture 1.4. *Let k and l be integers greater than one. If $\binom{kn}{n} \mid \binom{ln}{n}\binom{kln}{ln-1}$ for all $n \in \mathbb{Z}^+$, then $k = l$, or $l = 2$, or $\{k, l\} = \{3, 5\}$. If $\binom{kn}{n} \mid \binom{ln}{n-1}\binom{kln}{ln}$ for all $n \in \mathbb{Z}^+$, then $k = 2$, and $l + 1$ is a power of two.*

We will show Theorems 1.1-1.2 in the next section. Section 3 is devoted to the sophisticated proofs of Theorems 1.3-1.4. Throughout this paper, for a real number x we let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x.

2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Clearly (1.1) holds if and only if $(ln+1) \mid k\binom{kn+ln}{kn}$. For any prime p, we have

$$\nu_p\left(\frac{k\binom{kn+ln}{kn}}{ln+1}\right) = \nu_p\left(\frac{(kn+ln)!k!}{(kn)!(ln+1)!(k-1)!}\right)$$
$$= \sum_{j=1}^{\infty}\left(\left\lfloor\frac{kn+ln}{p^j}\right\rfloor - \left\lfloor\frac{kn}{p^j}\right\rfloor - \left\lfloor\frac{ln+1}{p^j}\right\rfloor + \left\lfloor\frac{k}{p^j}\right\rfloor - \left\lfloor\frac{k-1}{p^j}\right\rfloor\right).$$

So, it suffices to show for any $m \in \mathbb{Z}^+$ the inequality

$$\left\lfloor\frac{kn+ln}{m}\right\rfloor - \left\lfloor\frac{kn}{m}\right\rfloor - \left\lfloor\frac{ln+1}{m}\right\rfloor + \left\lfloor\frac{k}{m}\right\rfloor - \left\lfloor\frac{k-1}{m}\right\rfloor \geqslant 0. \tag{2.1}$$

If $m \nmid kn$, then

$$\left\lfloor\frac{kn}{m}\right\rfloor + \left\lfloor\frac{ln+1}{m}\right\rfloor = \left\lfloor\frac{kn-1}{m}\right\rfloor + \left\lfloor\frac{ln+1}{m}\right\rfloor \leqslant \left\lfloor\frac{(kn-1)+(ln+1)}{m}\right\rfloor.$$

If $m \nmid (ln+1)$, then

$$\left\lfloor\frac{kn}{m}\right\rfloor + \left\lfloor\frac{ln+1}{m}\right\rfloor = \left\lfloor\frac{kn}{m}\right\rfloor + \left\lfloor\frac{ln}{m}\right\rfloor \leqslant \left\lfloor\frac{kn+ln}{m}\right\rfloor.$$

When $m \mid kn$ and $m \mid (ln+1)$, clearly $(m,n) = 1$, $m \mid k$ and hence

$$\left\lfloor\frac{kn+ln}{m}\right\rfloor - \left\lfloor\frac{kn}{m}\right\rfloor - \left\lfloor\frac{ln+1}{m}\right\rfloor + \left\lfloor\frac{k}{m}\right\rfloor - \left\lfloor\frac{k-1}{m}\right\rfloor = 0.$$

Therefore (2.1) holds and this concludes the proof. $\square$

Lemma 2.1. *Let $m \in \mathbb{Z}^+$ and $k, n \in \mathbb{Z}$. Then we have*

$$\begin{aligned}&\left\lfloor\frac{2kn}{m}\right\rfloor - \left\lfloor\frac{kn}{m}\right\rfloor + \left\lfloor\frac{(k-1)n}{m}\right\rfloor - \left\lfloor\frac{2(k-1)n}{m}\right\rfloor\\ &\geqslant \left\lfloor\frac{n+1}{m}\right\rfloor - \left\lfloor\frac{2k-1}{m}\right\rfloor + \left\lfloor\frac{2k-2}{m}\right\rfloor,\end{aligned} \tag{2.2}$$

unless $2 \mid m$, $k \equiv m/2+1 \pmod m$ *and* $n \equiv -1 \pmod m$, *in which case the left-hand side of* (2.2) *minus the right-hand side equals* -1.

Proof. As $2kn - kn + (k-1)n - 2(k-1)n + (2k-1) - (2k-2) = n+1$, and $\lfloor x \rfloor = x - \{x\}$ for any rational number x, (2.2) holds if and only if

$$\left\{\frac{2kn}{m}\right\}-\left\{\frac{kn}{m}\right\}+\left\{\frac{(k-1)n}{m}\right\}-\left\{\frac{2(k-1)n}{m}\right\}+\left\{\frac{2k-1}{m}\right\}-\left\{\frac{2k-2}{m}\right\}<1. \tag{2.3}$$

Clearly (2.3) holds when $m=1$. Below we assume that $m \geqslant 2$.

Case 1. $\{kn/m\} < 1/2$ & $\{(k-1)n/m\} < 1/2$, or ($\{kn/m\} \geqslant 1/2$ & $\{(k-1)n/m\} \geqslant 1/2$).

In this case, the left-hand side of (2.3) equals

$$C := \left\{\frac{kn}{m}\right\} - \left\{\frac{(k-1)n}{m}\right\} + \left\{\frac{2k-1}{m}\right\} - \left\{\frac{2k-2}{m}\right\}.$$

If $m \nmid (k-1)n$, then $C < \{kn/m\} + 1/m \leqslant 1$. If $m \mid (k-1)n$ and $n \not\equiv -1 \pmod m$, then $C \leqslant \{n/m\} + 1/m < 1$. If $m \mid (k-1)n$ and $n \equiv -1 \pmod m$, then $\{kn/m\} = (m-1)/m \geqslant 1/2 > \{(k-1)n/m\} = 0$ which leads a contradiction.

Case 2. $\{kn/m\} < 1/2 \leqslant \{(k-1)n/m\}$.

In this case, the left-hand side of (2.3) equals

$$D := \left\{\frac{kn}{m}\right\} - \left\{\frac{(k-1)n}{m}\right\} + 1 + \left\{\frac{2k-1}{m}\right\} - \left\{\frac{2k-2}{m}\right\}.$$

If $n \not\equiv -1 \pmod m$, then $\{(k-1)n/m\} - \{kn/m\} \neq 1/m$ and hence $D < -1/m + 1 + 1/m = 1$. If $n \equiv -1 \pmod m$ and $2k \equiv 1 \pmod m$, then $D = -1/m + 1 - (m-1)/m < 1$. If $n \equiv -1 \pmod m$ and $2k \not\equiv 1 \pmod m$, then we must have $2 \mid m$ and $k \equiv m/2+1 \pmod m$ since $\{-k/m\} < 1/2 \leqslant \{(1-k)/m\}$.

When $2 \mid m$, $k \equiv m/2+1 \pmod m$ and $n \equiv -1 \pmod m$, it is easy to verify that the right-hand side of (2.2) minus the left-hand side of (2.2) equals 1.

Case 3. $\{kn/m\} \geqslant 1/2 > \{(k-1)n/m\}$.

In this case, the left-hand side of (2.3) is

$$\left\{\frac{kn}{m}\right\} - 1 - \left\{\frac{(k-1)n}{m}\right\} + \left\{\frac{2k-1}{m}\right\} - \left\{\frac{2k-2}{m}\right\} \leqslant \left\{\frac{kn}{m}\right\} - 1 + \frac{1}{m} \leqslant 0.$$

Combining the above we have completed the proof of Lemma 2.1. $\square$

Lemma 2.2. *Let $m > 2$ be an integer. For any $k, n \in \mathbb{Z}$ we have*

$$\left\lfloor \frac{2kn}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{k+1}{m} \right\rfloor \geqslant \left\lfloor \frac{k}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor + \left\lfloor \frac{kn}{m} \right\rfloor + \left\lfloor \frac{(k-1)n+1}{m} \right\rfloor. \tag{2.4}$$

Proof. As $k + ((k-1)n+1) + kn - 2kn + 2n - n = k+1$, (2.4) is equivalent to the inequality $M \geqslant 0$, where

$$M := \left\{ \frac{k}{m} \right\} + \left\{ \frac{(k-1)n+1}{m} \right\} + \left\{ \frac{kn}{m} \right\} - \left\{ \frac{2kn}{m} \right\} + \left\{ \frac{2n}{m} \right\} - \left\{ \frac{n}{m} \right\}.$$

If $\{n/m\} < 1/2 \leqslant \{kn/m\}$, or $\{n/m\} < 1/2$ & $\{kn/m\} < 1/2$, or $(\{n/m\} \geqslant 1/2$ & $\{kn/m\} \geqslant 1/2)$, then one can easily show $M \geqslant 0$.

Below we suppose that $\{kn/m\} < 1/2 \leqslant \{n/m\}$. Clearly $m \nmid n$ and

$$M = \left\{ \frac{k}{m} \right\} + \left\{ \frac{(k-1)n+1}{m} \right\} - \left\{ \frac{kn}{m} \right\} + \left\{ \frac{n}{m} \right\} - 1.$$

If $(k-1)n+1 \equiv 0 \pmod{m}$, then $\{(n-1)/m\} = \{kn/m\} < 1/2 \leqslant \{n/m\}$, hence m is odd (otherwise $n \equiv m/2 \pmod{m}$ and thus $1 \equiv 0 \pmod{m/2}$ which is impossible) and $n \equiv (m+1)/2 \pmod{m}$, therefore $k - 1 \equiv (k-1)2n \equiv -2 \pmod{m}$ and

$$M = \left\{ \frac{k}{m} \right\} - \left\{ \frac{n-1}{m} \right\} + \left\{ \frac{n}{m} \right\} - 1 = \left\{ \frac{k}{m} \right\} - \frac{m-1}{m} = 0.$$

When $(k-1)n + 1 \not\equiv 0 \pmod{m}$, we have $\{kn/m\} < \{(n-1)/m\}$ and hence

$$M = \left\{ \frac{k}{m} \right\} + \left(\left\{ \frac{kn}{m} \right\} - \left\{ \frac{n-1}{m} \right\} + 1 \right) - \left\{ \frac{kn}{m} \right\} + \left\{ \frac{n}{m} \right\} - 1 \geqslant \frac{1}{m}.$$

This concludes the proof. □

Proof of Theorem 1.2. (i) Observe that

$$Q_1 := \frac{(2k-1)C_n \binom{2kn}{kn}}{\binom{kn}{n}} = \frac{(2kn)!((k-1)n)!(2k-1)!}{(n+1)!(kn)!(2(k-1)n)!(2k-2)!}.$$

So, for any prime p we have $\nu_p(Q_1) = \sum_{i=1}^{\infty} A_{p^i}(k,n)$, where $A_m(k,n)$ denotes the left-hand side of (2.2) minus the right-hand side of (2.2). By Lemma 2.1, $A_{p^i}(k,n) \geqslant 0$ unless $p = 2$, $k \equiv 2^{i-1} + 1 \pmod{2^i}$ and $n \equiv -1 \pmod{2^i}$, in which case $A_{p^i}(k,n) = -1$. Therefore $2Q_1 \in \mathbb{Z}$.

Note that

$$Q_1 = \frac{2^n(2k-1)}{(n+1)!}\prod_{j=1}^{n}((2k-2)n+2j-1)$$

and thus $\nu_2(Q_1) = n - \nu_2((n+1)!)$. In view of Lemma 1.1(iii), we obtain that $Q_1 \in \mathbb{Z}$ and that Q_1 is odd if and only if $n+1$ is a power of two.

(ii) Obviously

$$Q_2 := \frac{(k+1)C_n^{(k-1)}\binom{2kn}{kn}}{\binom{2n}{n}} = \frac{(k+1)!(2kn)!n!}{k!(kn)!((k-1)n+1)!(2n)!}.$$

As in (i), by Lemma 2.2 we have $\nu_p(Q_2) \geqslant 0$ for any odd prime p.

Now we consider $\nu_2(Q_2)$. Set $m = (k-1)n$. Then

$$Q_2 = \frac{2^m(k+1)}{(m+1)!}\prod_{j=1}^{m}(2j+2n-1)$$

and therefore $\nu_2(Q_2) = \nu_2(k+1) + m - \nu_2((m+1)!)$. Applying Lemma 1.1(iii) we see that $\nu_2(Q_2) \geqslant \nu_2(k+1)$. So $Q_2/2^{\nu_2(k+1)}$ is an integer. With the help of Lemma 1.1(iii), we also have

$$\begin{aligned}
&\frac{Q_2}{2^{\nu_2(k+1)}} = \frac{(k+1)'C_n^{(k-1)}\binom{2kn}{kn}}{\binom{2n}{n}} \text{ is odd}\\
\iff& \nu_2((m+1)!) = m\\
\iff& m+1 = (k-1)n+1 \text{ is a power of two.}
\end{aligned}$$

This concludes the proof of Theorem 1.2(ii). $\square$

3. Proofs of Theorems 1.3 and 1.4

Lemma 3.1. *Let p be a prime and let $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. Then*

$$\frac{\rho_p((p^k-1)n)}{p-1} = \sum_{j=1}^{\infty}\left\{\frac{(p^k-1)n}{p^j}\right\} \geqslant k \tag{3.1}$$

and hence the expansion of $(p^k-1)n$ in base p has at least k nonzero digits.

Proof. For any $m \in \mathbb{Z}^+$, by Lemma 1.1(ii) we have

$$\frac{\rho_p(m)}{p-1} = \frac{m}{p-1} - \nu_p(m!) = \sum_{j=1}^{\infty}\frac{m}{p^j} - \sum_{j=1}^{\infty}\left\lfloor\frac{m}{p^j}\right\rfloor = \sum_{j=1}^{\infty}\left\{\frac{m}{p^j}\right\}.$$

If the expansion of m in base p has less than k nonzero digits, then $\rho_p(m) < k(p-1)$. So it remains to show the inequality in (3.1).

Observe that

$$p^k\binom{p^kn-1}{n-1} = \binom{p^kn}{n} = \frac{(p^kn)!}{n!((p^k-1)n)!}$$

and

$$\begin{aligned}&\nu_p((p^kn)!) - \nu_p(n!) - \nu_p(((p^k-1)n)!)\\ =&\sum_{j=1}^{\infty}\left\lfloor\frac{p^kn}{p^j}\right\rfloor - \sum_{j=1}^{\infty}\left\lfloor\frac{n}{p^j}\right\rfloor - \sum_{j=1}^{\infty}\left\lfloor\frac{(p^k-1)n}{p^j}\right\rfloor\\ =&\sum_{j=1}^{k}p^{k-j}n - \sum_{j=1}^{\infty}\left\lfloor\frac{(p^k-1)n}{p^j}\right\rfloor = \sum_{j=1}^{\infty}\left\{\frac{(p^k-1)n}{p^j}\right\}.\end{aligned}$$

So the inequality in (3.1) follows. We are done. $\square$

Proof of Theorem 1.3. Since the odd part of $(2^k-1)+1$ is 1, by Theorem 1.2(ii) and its proof,

$$Q_3 := \frac{\binom{2(2^k-1)n}{(2^k-1)n}C_n^{(2^k-2)}}{\binom{2n}{n}} \in \mathbb{Z}$$

and also $\nu_2(Q_3) = m - \nu_2((m+1)!)$, where $m = ((2^k-1)-1)n$ is even. Applying Lemma 1.1(ii) and Lemma 3.1 with $p=2$, we get

$$\nu_2(Q_3) = m! - \nu_2(m!) = \rho_2(m) = \rho_2((2^{k-1}-1)n) \geqslant k-1.$$

Therefore $2^{k-1} \mid Q_3$ and hence (1.4) holds. $\square$

Lemma 3.2. (i) *For any real number x we have*

$$\{12x\} + \{5x\} + \{2x\} \geqslant \{4x\} + \{15x\}. \tag{3.2}$$

(ii) *Let x be a real number with $\{5x\} \geqslant \{2x\} \geqslant 1/2$. Then $\{5x\} \geqslant 2/3$.*

Proof. (i) Since $12x + 5x + 2x - 4x = 15x$, (3.2) reduces to

$$\{12x\} + \{5x\} + \{2x\} - \{4x\} \geqslant 0,$$

which can be easily checked.

(ii) As $\{5x\} \geqslant \{2x\} \geqslant 1/2$, we can easily see that $\{x\} \in [1/3, 2/5) \cup [3/4, 4/5)$. It follows that $\{5x\} \geqslant 2/3$. $\square$

Lemma 3.3. *Let $m>1$ and n be integers.*

(i) *If $3\nmid m$, then*

$$\left\lfloor\frac{15n-1}{m}\right\rfloor+\left\lfloor\frac{2}{m}\right\rfloor+\left\lfloor\frac{4n}{m}\right\rfloor\geqslant\left\lfloor\frac{12n+2}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor+\left\lfloor\frac{5n-1}{m}\right\rfloor. \tag{3.3}$$

(ii) *If $5\nmid m$, then*

$$\left\lfloor\frac{15n-1}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor\geqslant\left\lfloor\frac{10n+1}{m}\right\rfloor+\left\lfloor\frac{4n}{m}\right\rfloor+\left\lfloor\frac{3n-1}{m}\right\rfloor. \tag{3.4}$$

Proof. (i) Clearly (3.3) holds when $m=2$. Below we assume that $m>2$ and $3\nmid m$.

Since $m\mid 15n$ if and only if $m\mid 5n$, we have

$$\left\{\frac{5n-1}{m}\right\}-\left\{\frac{15n-1}{m}\right\}=\left\{\frac{5n}{m}\right\}-\left\{\frac{15n}{m}\right\}$$

and thus (3.3) has the following equivalent form:

$$\left\{\frac{12n+2}{m}\right\}+\left\{\frac{5n}{m}\right\}+\left\{\frac{2n}{m}\right\}-\left\{\frac{4n}{m}\right\}\geqslant\left\{\frac{15n}{m}\right\}+\frac{2}{m}. \tag{3.5}$$

If $12n+1,12n+2\not\equiv 0\pmod{m}$, then (3.5) is equivalent to the inequality

$$\{12x\}+\{5x\}+\{2x\}-\{4x\}\geqslant\{15x\}$$

with $x=n/m$, which follows from Lemma 3.2(i).

Below we assume that $12n+\delta\equiv 0\pmod{m}$ for some $\delta\in\{1,2\}$. Clearly m does not divide $3n$ and (3.5) can be rewritten as

$$\left\{\frac{5n}{m}\right\}+\left\{\frac{2n}{m}\right\}-\left\{\frac{4n}{m}\right\}\geqslant\left\{\frac{3n-\delta}{m}\right\}+\frac{\delta}{m}=\left\{\frac{3n}{m}\right\}.$$

(Note that if $m\mid(12n+2)$ and $m\mid(3n-1)$ then m divides $12n+2-4(3n-1)=6$ which contradicts that $m>2$ and $3\nmid m$.)

Now it suffices to show $f(x):=\{5x\}+\{2x\}-\{4x\}-\{3x\}\geqslant 0$, where $x=\{n/m\}$. Clearly

$$\begin{aligned}f(x)=&\lfloor 3x\rfloor+\lfloor 4x\rfloor-\lfloor 2x\rfloor-\lfloor 5x\rfloor\\=&\left|\left\{\frac13,\frac23,\frac14,\frac24,\frac34\right\}\cap(0,x]\right|-\left|\left\{\frac12,\frac15,\frac25,\frac35,\frac45\right\}\cap(0,x]\right|.\end{aligned}$$

It follows that $f(x) < 0$ if and only if $x \in [1/5, 1/4) \cup [3/5, 2/3)$. Clearly $a := 12x + \delta/m \in \{1, \dots, 11\}$ and

$$x = \frac{a}{12} - \frac{\delta/m}{12} \in \left(\frac{a-1}{12}, \frac{a}{12}\right).$$

Note that

$$\left[\frac{1}{5}, \frac{1}{4}\right) \subseteq \left(\frac{2}{12}, \frac{3}{12}\right) \quad \text{and} \quad \left[\frac{3}{5}, \frac{2}{3}\right) \subseteq \left(\frac{7}{12}, \frac{8}{12}\right).$$

Also, $a \neq 3, 8$ since 12 divides neither $3m - \delta$ nor $8m - \delta$.

So far we have proved part (i) of Lemma 3.3.

(ii) Suppose that $5 \nmid m$. Then $m \mid 15n$ if and only if $m \mid 3n$. Note also that $(10n + 1) - 1 + 3n + 4n - 2n = 15n$. Thus (3.4) has the following equivalent form:

$$W := \left\{\frac{10n+1}{m}\right\} - \frac{1}{m} + \left\{\frac{3n}{m}\right\} + \left\{\frac{4n}{m}\right\} - \left\{\frac{2n}{m}\right\} \geqslant 0 \tag{3.6}$$

In the case $m \mid 3n$, (3.6) reduces to $\{(n+1)/m\} + \{n/m\} \geqslant \{2n/m\} + 1/m$, which holds despite that m divides $2n + 1$ or not.

Below we assume that $m \nmid 3n$. Then

$$W := \left\{\frac{10n+1}{m}\right\} + \left\{\frac{3n-1}{m}\right\} + \left\{\frac{4n}{m}\right\} - \left\{\frac{2n}{m}\right\}.$$

If $\{2n/m\} < 1/2$, then $\{4n/m\} - \{2n/m\} = \{2n/m\} \geqslant 0$. If $\{2n/m\} \geqslant 1/2$ and $\{(5n-1)/m\} < \{2n/m\}$, then

$$W = \left\{\frac{10n+1}{m}\right\} + \left\{\frac{3n-1}{m}\right\} + \left\{\frac{2n}{m}\right\} - 1 \geqslant \left\{\frac{5n-1}{m}\right\} \geqslant 0.$$

Now we consider the remaining case $\{(5n-1)/m\} \geqslant \{2n/m\} \geqslant 1/2$. Note that

$$W = \left\{\frac{10n+1}{m}\right\} + \left\{\frac{3n-1}{m}\right\} + \left\{\frac{2n}{m}\right\} - 1 = \left\{\frac{10n+1}{m}\right\} + \left\{\frac{5n-1}{m}\right\} - 1.$$

Clearly $W = 0$ if $m \mid 5n$. If $m \mid (10n+1)$, then $2 \nmid m$, $5n \equiv (m-1)/2 \pmod{m}$ and hence $\{(5n-1)/m\} < 1/2$.

Below we simply assume that $m \nmid 5n$ and $m \nmid (10n+1)$. Then $\{5x\} \geqslant \{2x\} \geqslant 1/2$, where $x = n/m$. Thus $W = 2\{5x\} - 1 + \{5x\} - 1 \geqslant 0$ with the help of Lemma 3.2(ii). This concludes the proof. $\square$

Proof of Theorem 1.4. Observe that

$$A := \frac{\binom{3n-1}{n-1}C_{3n}^{(4)}}{(6n+1)\binom{5n}{n}} = \frac{(15n-1)!2!(4n)!}{(12n+2)!(2n)!(5n-1)!}$$

and

$$B := \frac{\binom{5n-1}{n-1}C_{5n}^{(2)}}{\binom{3n}{n}} = \frac{(15n-1)!(2n)!}{(10n+1)!(4n)!(3n-1)!}.$$

By Lemma 3.3, $\nu_p(A) \geqslant 0$ for any prime $p \neq 3$, and $\nu_p(B) \geqslant 0$ for any prime $p \neq 5$. Thus, it suffices to show that $\nu_3(A) \geqslant 0$ and $\nu_5(B) \geqslant 0$. In fact,

$$\frac{C_{3n}^{(4)}}{(6n+1)\binom{5n}{n}} = \frac{1}{(6n+1)(12n+1)} \prod_{\substack{j=1 \\ 3\nmid j}}^{3n} \frac{12n+j}{j}$$

is a 3-adic integer, and

$$\frac{C_{5n}^{(2)}}{\binom{3n}{n}} = \frac{1}{10n+1} \prod_{\substack{j=1 \\ 5\nmid j}}^{5n} \frac{10n+j}{j}$$

is a 5-adic integer. We are done. □

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