Illinois J. Math. 56(2012), no. 3, 967–979.

A REFINEMENT OF A CONGRUENCE RESULT BY VAN HAMME AND MORTENSON

Zhi-Wei Sun

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

Abstract. Let p be an odd prime. In 2008 E. Mortenson proved van Hamme's following conjecture:

$$\sum_{k=0}^{(p-1)/2}(4k+1)\binom{-1/2}{k}^3\equiv(-1)^{(p-1)/2}p \pmod{p^3}.$$

In this paper we show further that

$$\begin{aligned}\sum_{k=0}^{p-1}(4k+1)\binom{-1/2}{k}^3\equiv&\sum_{k=0}^{(p-1)/2}(4k+1)\binom{-1/2}{k}^3\\ \equiv&(-1)^{(p-1)/2}p+p^3E_{p-3} \pmod{p^4},\end{aligned}$$

where $E_0,E_1,E_2,\ldots$ are Euler numbers. We also prove that if $p>3$ then

$$\sum_{k=0}^{(p-1)/2}\frac{20k+3}{(-2^{10})^k}\binom{4k}{k,k,k,k}\equiv(-1)^{(p-1)/2}p(2^{p-1}+2-(2^{p-1}-1)^2) \pmod{p^4}.$$

2010 *Mathematics Subject Classification.* Primary 11B65; Secondary 05A10, 11A07, 11B68.

Keywords. Central binomial coefficients, congruences, Euler numbers.

Supported by the National Natural Science Foundation (grant 11171140) of China and and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

1. Introduction

In 1859 G. Bauer obtained the identity

$$\sum_{k=0}^{\infty}(4k+1)\binom{-1/2}{k}^3=\frac{2}{\pi}$$

which was later reproved by S. Ramanujan [R] in 1914. (Note that $\binom{-1/2}{k}=\binom{2k}{k}/(-4)^k$ for all $k=0,1,2,\ldots$.) In 1997 van Hamme [vH] conjectured that

$$\sum_{k=0}^{p-1}(4k+1)\binom{-1/2}{k}^3=\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv(-1)^{(p-1)/2}p \pmod{p^3}$$

for any odd prime p, which was first confirmed by E. Mortenson [Mo] in 2008 via a deep method involving the p-adic Γ-function and Gauss and Jacobi sums.

Throughout this paper, for an odd prime p, we use $(\frac{\cdot}{p})$ to denote the Legendre symbol. Recall that the Euler numbers $E_0,E_1,E_2,\ldots$ are integers given by

$$E_0=1 \quad\text{and}\quad \sum_{\substack{k=0\\ 2\mid k}}^{n}\binom{n}{k}E_{n-k}=0 \quad (n=1,2,3,\ldots).$$

It is well known that

$$\frac{2e^x}{e^{2x}+1}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}\qquad\text{for } |x|<\frac{\pi}{2}.$$

In this paper we obtain the following refinement of the congruence by van Hamme and Mortenson via an elementary approach.

Theorem 1.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3} \pmod{p^4}. \tag{1.1}$$

Remark 1.1. The only previously proved congruence mod p^4 of the same kind is the following one conjectured by van Hamme [vH] and confirmed by L. Long [Lo]:

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{256^k}\equiv p\left(\frac{-1}{p}\right) \pmod{p^4}\quad\text{for any prime } p>3.$$

For each nonnegative integer k, it is clear that

$$\binom{4k}{k,k,k,k} = \frac{(4k)!}{k!^4} = \binom{4k}{2k}\binom{2k}{k}^2.$$

In a way similar to the proof of Theorem 1.1, we also deduce the following result.

Theorem 1.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{20k+3}{(-2^{10})^k}\binom{4k}{k,k,k,k} \equiv p\left(\frac{-1}{p}\right)\left(2^{p-1}+2-(2^{p-1}-1)^2\right) \pmod{p^4}. \tag{1.2}$$

Remark. 1.2. (a) The congruence in Theorem 1.2 gives the mod p^4 analogy of the Ramanujan series

$$\sum_{k=0}^{\infty} \frac{20k+3}{(-2^{10})^k}\binom{4k}{k,k,k,k} = \frac{8}{\pi}.$$

See [BB], [BBC] and [Be, pp.353-354] for more such series. The mod p^3 analogy of the above series is known (cf. [Zu]).

(b) By the same method, the author ever proved that

$$\sum_{k=0}^{p-1} \frac{20k+3}{(-2^{10})^k}\binom{4k}{k,k,k,k} \equiv 3p\left(\frac{-1}{p}\right) + 3p^3E_{p-3} \pmod{p^4} \tag{1.3}$$

for any odd prime p; unfortunately he has lost the draft containing the complicated details.

Theorems 1.1 and 1.2 will be proved in Sections 2 and 3 respectively.

The author [Su2, Conjecture 5.1] raised several conjectures similar to (1.1). Here we pose a new conjecture motivated by the Ramanujan series

$$\sum_{k=0}^{\infty} \frac{7k+1}{648^k}\binom{4k}{k,k,k,k} = \frac{9}{2\pi}.$$

Conjecture 1.1. *For any prime $p>3$ we have*

$$\sum_{k=0}^{p-1}\frac{7k+1}{648^k}\binom{4k}{k,k,k,k}\equiv p\left(\frac{-1}{p}\right)-\frac{5}{3}p^3E_{p-3}\pmod{p^4}. \tag{1.4}$$

Also, for $n=2,3,\ldots$ we have

$$\frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(7k+1)\binom{4k}{k,k,k,k}648^{n-1-k}\in\mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case the quotient is a rational number with denominator 3.

Remark 1.3. It seems that the method for our proofs of (1.1) and (1.2) does not work for (1.4).

In 2010, L. L. Zhao, H. Pan and the author [ZPS] proved that

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv 0\pmod{p}$$

for any odd prime p. Here we raise a further conjecture.

Conjecture 1.2. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv-\frac{3}{p}(2^{p-1}-1)^2\pmod{p^2} \tag{1.5}$$

and

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2}\binom{3k}{k}\equiv 6\left(\frac{-1}{p}\right)E_{p-3}\pmod{p}. \tag{1.6}$$

Also,

$$p\sum_{k=1}^{p-1}\frac{1}{k2^k\binom{3k}{k}}\equiv\begin{cases}0\pmod{p^2} & \text{if } p\equiv 1\pmod 4,\\ -3/5\pmod{p^2} & \text{if } p\equiv 3\pmod 4,\end{cases} \tag{1.7}$$

$$p\sum_{k=1}^{p-1}\frac{1}{k^2 2^k\binom{3k}{k}}\equiv\frac{1-4^{p-1}}{4p}\pmod{p^2}\quad\text{if } p>3, \tag{1.8}$$

and

$$\sum_{k=1}^{p-1}2^k\binom{3k}{k}\sum_{j=1}^{k}\frac{1}{j^2}\equiv 0\pmod{p}\quad\text{if } p>5 \text{ and } p\equiv 1\pmod 4. \tag{1.9}$$

2. Proof of Theorem 1.1

We need some classical congruences.

Lemma 2.1. *Let $p>3$ be a prime.*

(i) (J. Wolstenholme [W]) *We have*

$$\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod{p^2},\quad \sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0 \pmod{p}, \tag{2.1}$$

and

$$\frac{1}{2}\binom{2p}{p}=\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}. \tag{2.2}$$

(ii) (F. Morley [M]) *We have*

$$\binom{p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}4^{p-1} \pmod{p^3}. \tag{2.3}$$

The most crucial lemma we need is the following sophisticated result.

Lemma 2.2 (Sun [Su1])**.** *Let p be an odd prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{(2k-1)\binom{2k}{k}}\equiv E_{p-3}-1+\left(\frac{-1}{p}\right) \pmod{p} \tag{2.4}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k(2k-1)\binom{2k}{k}}\equiv 2E_{p-3} \pmod{p}. \tag{2.5}$$

Remark 2.1. Actually (2.4) and (2.5) are equivalent since

$$\frac{1}{2}\sum_{k=1}^{n}\frac{4^k}{k\binom{2k}{k}}=\frac{4^n}{\binom{2n}{n}}-1;$$

they are (1.3) and (3.1) of Sun [Su1] respectively.

Proof of Theorem 1.1. (i) Clearly the first congruence in (1.1) has the following equivalent form:

$$\sum_{p/2<k<p}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv 0 \pmod{p^4}.$$

For $k \in \{1, \ldots, (p-1)/2\}$, it is obvious that

$$\begin{aligned}\frac{1}{p}\binom{2(p-k)}{p-k} =& \frac{1}{p} \times \frac{p!\prod_{s=1}^{p-2k}(p+s)}{((p-1)!/\prod_{0<t<k}(p-t))^2}\\ \equiv& \frac{(k-1)!^2}{(p-1)!/(p-2k)!} \equiv -\frac{(k-1)!^2}{(2k-1)!} = -\frac{2}{k\binom{2k}{k}} \pmod p.\end{aligned}$$

(See also [Su2, Lemma 2.1].) Thus

$$\begin{aligned}&\frac{1}{p^3}\sum_{p/2<k<p}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\\ =&\sum_{k=1}^{(p-1)/2}\frac{4(p-k)+1}{(-64)^{p-k}}\left(\frac{\binom{2(p-k)}{p-k}}{p}\right)^3\\ \equiv&\sum_{k=1}^{(p-1)/2}(1-4k)(-64)^{k-1}\left(\frac{-2}{k\binom{2k}{k}}\right)^3\\ =&-\frac{1}{8}\sum_{k=1}^{(p-1)/2}\frac{4k-1}{k^3\binom{-1/2}{k}^3} = \sum_{k=1}^{(p-1)/2}\frac{4k-1}{\binom{-3/2}{k-1}^3}\\ \equiv&\sum_{k=0}^{(p-3)/2}\frac{4(k+1)-1}{\binom{(p-3)/2}{k}^3} = \frac{1}{2}\sum_{k=0}^{(p-3)/2}\frac{(4k+3)+4((p-3)/2-k)+3}{\binom{(p-3)/2}{k}^3}\\ \equiv&0 \pmod p\end{aligned}$$

and hence the first congruence in (1.1) follows.

(ii) Below we prove the second congruence in (1.1). For $k, n = 0, 1, 2, \ldots$ define

$$F(n,k) = \frac{(-1)^{n+k}(4n+1)}{4^{3n-k}}\binom{2n}{n}^2\frac{\binom{2n+2k}{n+k}\binom{n+k}{2k}}{\binom{2k}{k}}$$

and

$$G(n,k) = \frac{(-1)^{n+k}(2n-1)^2\binom{2n-2}{n-1}^2}{2(n-k)4^{3(n-1)-k}}\binom{2(n-1+k)}{n-1+k}\frac{\binom{n-1+k}{2k}}{\binom{2k}{k}}.$$

Clearly $F(n,k) = G(n,k) = 0$ if $n < k$. It can be easily verified that

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k)$$

for all nonnegative integers n and $k > 0$ as observed by S. B. Ekhad and D. Zeilberger [EZ].

Let $m = (p-1)/2$. In the spirit of the WZ (Wilf-Zeilberger) method (see the book of M. Petkovšek, H. S. Wilf and D. Zeilberger [PWZ], and [AZ] and [Z] for this method), we have

$$\begin{aligned}\sum_{n=0}^{m} F(n,0) - F(m,m) &= \sum_{n=0}^{m} F(n,0) - \sum_{n=0}^{m} F(n,m)\\ &= \sum_{k=1}^{m}\bigg(\sum_{n=0}^{m} F(n,k-1) - \sum_{n=0}^{m} F(n,k)\bigg)\\ &= \sum_{k=1}^{m}\sum_{n=0}^{m}(G(n+1,k)-G(n,k)) = \sum_{k=1}^{m} G(m+1,k),\end{aligned}$$

that is,

$$\begin{aligned}&\sum_{n=1}^{m}\frac{4n+1}{(-64)^n}\binom{2n}{n}^3 - \frac{4m+1}{4^{2m}}\binom{4m}{2m}\binom{2m}{m}\\ =&\sum_{k=1}^{m}\frac{(-1)^{m+k+1}(2m+1)^2\binom{2m}{m}^2}{2(m+1-k)4^{3m-k}}\binom{2m+2k}{m+k}\frac{\binom{m+k}{2k}}{\binom{2k}{k}}.\end{aligned}\tag{2.6}$$

For $0 < k \leqslant m = (p-1)/2$, clearly

$$\begin{aligned}\frac{1}{p}\binom{2m+2k}{m+k} &= \frac{(p-1)!(p+1)\cdots(p+2k-1)}{m!^2\prod_{j=1}^{k}((p+2j-1)/2)^2}\\ \equiv&(-1)^{(p-1)/2}\frac{(p-1)!}{\prod_{k=1}^{(p-1)/2}k(p-k)}\cdot\frac{(2k-1)!}{((2k-1)!!/2^k)^2}\\ \equiv&\left(\frac{-1}{p}\right)\frac{(2k-1)!}{((2k)!/(k!4^k))^2} = \left(\frac{-1}{p}\right)\frac{4^{2k}}{2k\binom{2k}{k}} \pmod{p}\end{aligned}$$

and

$$\begin{aligned}\binom{m+k}{2k} \equiv&\binom{k-1/2}{2k} = \frac{\prod_{j=1}^{k}(-(2j-1)/2)(2j-1)/2}{(2k)!}\\ =&\frac{(-1)^k((2k-1)!!)^2}{4^k(2k)!} = \frac{((2k)!/\prod_{j=1}^{k}(2j))^2}{(-4)^k(2k)!} = \frac{\binom{2k}{k}}{(-16)^k} \pmod{p}.\end{aligned}$$

Note also that

$$(4m+1)\binom{4m}{2m} = (2p-1)\binom{2p-2}{p-1} = p\binom{2p-1}{p} \equiv p \pmod{p^4}$$

by the Wolstenholme congruence (2.2). Thus, in view of the above and Morley's congruence (2.3), we obtain from (2.6) that

$$\begin{aligned}&\sum_{k=0}^{m}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}-p(-1)^{(p-1)/2}\\ \equiv& p^3\sum_{k=1}^{m}\frac{(-1)^{k-1}4^{2k}}{2((p+1)/2-k)2^{3(p-1)-2k}2k\binom{2k}{k}(-16)^k}\\ \equiv&\frac{p^3}{2}\sum_{k=1}^{(p-1)/2}\frac{4^k}{k(2k-1)\binom{2k}{k}}\pmod{p^4}\end{aligned}$$

Combining this with (2.5) we get the second congruence in (1.1).

The proof of Theorem 1.1 is now complete. $\square$

3. Proof of Theorem 1.2

Lemma 3.1. *Let p be an odd prime. Then*

$$\binom{(p-1)/2+k}{2k}\equiv\frac{\binom{2k}{k}}{(-16)^k}\pmod{p^2}. \tag{3.1}$$

Remark 3.1. (3.1) is easy, see [S, Lemma 2.2] for a proof.

Recall that the harmonic numbers are those rational numbers

$$H_n:=\sum_{k=1}^{n}\frac{1}{k}\quad(n=1,2,\ldots),$$

together with $H_0=0$. For an odd prime p we write $q_p(2)$ for the Fermat quotient $(2^{p-1}-1)/p$.

Lemma 3.2 (E. Lehmer [L]). *For any odd prime p we have*

$$H_{(p-1)/2}\equiv-2q_p(2)+p\,q_p(2)^2\pmod{p^2}. \tag{3.2}$$

Lemma 3.3. *Let p be an odd prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{H_{k-1}}{k}\equiv 2q_p(2)^2\pmod{p}. \tag{3.3}$$

Proof. For $k = 1, \ldots, p-1$ we have

$$\frac{\binom{p}{k}}{p} = \frac{\binom{p-1}{k-1}}{k} = \frac{(-1)^{k-1}}{k} \prod_{0<j<k} \left(1 - \frac{p}{j}\right) \equiv \frac{(-1)^{k-1}}{k}(1 - pH_{k-1}) \pmod{p^2}.$$

Thus

$$\sum_{k=1}^{(p-1)/2} \frac{pH_{k-1} - 1}{k} \equiv \frac{1}{p} \sum_{k=1}^{(p-1)/2} (-1)^k \binom{p}{k} \pmod{p^2}.$$

As $\sum_{k=0}^{(p-1)/2} (-1)^k \binom{p}{k}$ is the coefficient of $x^{(p-1)/2}$ in $(1-x)^p(1-x)^{-1}$, we have

$$\frac{1}{p} \sum_{k=1}^{(p-1)/2} (-1)^k \binom{p}{k} = \frac{\binom{p-1}{(p-1)/2}(-1)^{(p-1)/2} - 1}{p} \equiv \frac{4^{p-1} - 1}{p} \pmod{p^2}$$

with the help of Morley's congruence (2.3). Therefore, in view of Lehmer's congruence (3.2), we have

$$\begin{aligned} p \sum_{k=1}^{(p-1)/2} \frac{H_{k-1}}{k} \equiv& H_{(p-1)/2} + \frac{2^{p-1} - 1}{p}(2^{p-1} + 1) \\ \equiv& -2q_p(2) + p\, q_p(2)^2 + q_p(2)(2 + p\, q_p(2)) \\ =& 2p\, q_p(2)^2 \pmod{p^2} \end{aligned}$$

and hence (3.3) holds. $\square$

Lemma 3.4. *Let $p = 2m + 1$ be an odd prime. Then*

$$\frac{6m+1}{2^{8m}} \binom{6m}{3m} \binom{3m}{m} \equiv p \left(\frac{-1}{p}\right) \pmod{p^4}. \tag{3.4}$$

Proof. Observe that

$$\begin{aligned} &(6m+1)\binom{6m}{3m}\binom{3m}{m} = \frac{(3m+1)\cdots(6m+1)}{m!(2m)!} \\ =& \frac{(p + (p-1)/2)\cdots 2p \cdots (3p-2)}{(p-1)!((p-1)/2)!} = \frac{(p + (p+1)/2)\cdots 2p \cdots (3p-1)}{2 \times (p-1)!((p-1)/2)!} \\ =& p \prod_{k=1}^{(p-1)/2} \frac{(2p-k)(2p+k)}{k^2} \times \prod_{p/2<j<p} \frac{2p+j}{j} \\ =& p(-1)^{(p-1)/2} \prod_{k=1}^{(p-1)/2} \left(1 - \frac{4p^2}{k^2}\right) \prod_{p/2<j<p} \left(1 + \frac{2p}{j}\right). \end{aligned}$$

Clearly
$$\prod_{k=1}^{(p-1)/2}\left(1-\frac{4p^2}{k^2}\right)\equiv 1-4p^2\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv 1 \pmod{p^3}$$
since
$$2\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)\equiv\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0 \pmod{p}.$$
So it suffices to prove that
$$\prod_{p/2<j<p}\left(1+\frac{2p}{j}\right)\equiv 2^{4(p-1)} \pmod{p^3}. \tag{3.5}$$

Observe that
$$\begin{aligned}&\prod_{p/2<j<p}\left(1+\frac{2p}{j}\right)\\
\equiv&1+2p\sum_{p/2<j<p}\frac{1}{j}+4p^2\sum_{p/2<i<j<p}\frac{1}{ij}\\
\equiv&1+2p(H_{p-1}-H_{(p-1)/2})+2p^2\left(\left(\sum_{p/2<k<p}\frac{1}{k}\right)^2-\sum_{p/2<k<p}\frac{1}{k^2}\right)\\
\equiv&1-2pH_{(p-1)/2}+2p^2(-H_{(p-1)/2})^2 \qquad \text{(by (2.1))}\\
\equiv&1-2p(p\,q_p(2)^2-2q_p(2))+2p^2 4q_p(2)^2 \qquad \text{(by (3.2))}\\
=&1+4p\,q_p(2)+6p^2q_p(2)^2\equiv(1+p\,q_p(2))^4=2^{4(p-1)} \pmod{p^3}.\end{aligned}$$
This proves (3.5) and hence (3.4) follows. □

Proof of Theorem 1.2. (i) For $n,k\in\mathbb{N}$, define
$$F(n,k):=\frac{(-1)^{n+k}(20n-2k+3)}{4^{5n-k}}\cdot\frac{\binom{2n}{n}\binom{4n+2k}{2n+k}\binom{2n+k}{2k}\binom{2n-k}{n}}{\binom{2k}{k}}.$$
and
$$G(n,k):=\frac{(-1)^{n+k}}{4^{5n-4-k}}\cdot\frac{n\binom{2n-1}{n-1}\binom{4n+2k-2}{2n+k-1}\binom{2n+k-1}{2k}\binom{2n-k-1}{n-1}}{\binom{2k}{k}}.$$
Clearly $F(n,k)=0$ if $n<k$. It can be easily verified that
$$F(n,k-1)-F(n,k)=G(n+1,k)-G(n,k)$$

for all nonnegative integers n and $k > 0$; the WZ-pair F and G stated in [Zu] was found in the spirit of [EZ] and [PWZ].

As in the proof of Theorem 1.1, for any positive integer N we have

$$\sum_{n=0}^{N} F(n,0) - F(N,N) = \sum_{k=1}^{N} G(N+1,k),$$

that is,

$$\begin{aligned}&\sum_{n=0}^{N} \frac{20n+3}{(-2^{10})^n}\binom{2n}{n}^2\binom{4n}{2n} - \frac{18N+3}{2^{8N}}\binom{6N}{3N}\binom{3N}{N}\\ =&(N+1)\binom{2N+1}{N}\sum_{k=1}^{N}\frac{(-1)^{N+k+1}\binom{4N+2k+2}{2N+k+1}\binom{2N+k+1}{2k}\binom{2N-k+1}{N}}{4^{5(N+1)-4-k}\binom{2k}{k}}.\end{aligned} \tag{3.6}$$

For $1 \leqslant k \leqslant N$, clearly

$$\begin{aligned}&\binom{4N+2k+2}{2N+k+1}\binom{2N+k+1}{2k}\binom{2N-k+1}{N}\\ =&\binom{4N+2k+2}{2k}\binom{4N+2}{2N-k+1}\binom{2N-k+1}{N}\\ =&\binom{4N+2k+2}{2k}\binom{4N+2}{N}\binom{3N+2}{N-k+1}.\end{aligned}$$

So we also have

$$\begin{aligned}&\sum_{n=0}^{N} \frac{20n+3}{(-2^{10})^n}\binom{2n}{n}^2\binom{4n}{2n} - \frac{18N+3}{2^{8N}}\binom{6N}{3N}\binom{3N}{N}\\ =&(N+1)\binom{2N+1}{N}\binom{4N+2}{N}\sum_{k=1}^{N}\frac{(-1)^{N+k+1}\binom{4N+2k+2}{2k}\binom{3N+2}{N-k+1}}{4^{5N+1-k}\binom{2k}{k}}.\end{aligned} \tag{3.7}$$

(ii) Let $m = (p-1)/2$. Observe that

$$(m+1)\binom{2m+1}{m} = p\binom{p-1}{(p-1)/2} \equiv p(-1)^m 4^{p-1} \pmod{p^4}$$

by Morley's congruence (2.3). Also,

$$\begin{aligned}\binom{4m+2}{m} =&\binom{2p}{(p-1)/2} = \frac{4p}{p+1}\binom{2p-1}{p}\binom{p-1}{(p-1)/2}\prod_{k=1}^{(p+1)/2}\left(1+\frac{p}{k}\right)^{-1}\\ \equiv&\frac{4p}{p+1}(-1)^{(p-1)/2}4^{p-1}\prod_{k=1}^{(p+1)/2}\left(1-\frac{p}{k}\right)\\ \equiv&p4^p(-1)^m(1-p)(1-pH_{(p+1)/2})\\ \equiv&p4^p(-1)^m(1-p)(1-2p+2p\,q_p(2))\\ \equiv&p4^p(-1)^m(1-3p+2p\,q_p(2)) \pmod{p^3}\end{aligned}$$

by Lehmer's congruence (3.2). Therefore

$$\begin{aligned}\frac{(m+1)\binom{2m+1}{m}\binom{4m+2}{m}}{4^{5m+1}} \equiv&p^2\frac{4^{2(p-1)}(1-3p+2p\,q_p(2))}{4^{4m}(1+p\,q_p(2))}\\ \equiv&p^2(1-p\,q_p(2))(1-3p+2p\,q_p(2))\\ \equiv&p^2(1-3p+p\,q_p(2)) \pmod{p^4}.\end{aligned}$$

Observe that

$$\begin{aligned}&\sum_{k=1}^{m}(-1)^k\frac{\binom{4m+2k+2}{2k}\binom{3m+2}{m-k+1}}{4^{-k}\binom{2k}{k}}\\ \equiv&\sum_{k=1}^{m}(-1)^k\frac{\binom{2p+2k}{2k}\binom{p+(p+1)/2}{(p+1)/2-k}}{4^{-k}\binom{(p-1)/2+k}{2k}(-16)^k}\\ =&\sum_{k=1}^{m}\frac{(2p+1)\cdots(2p+2k)(p+k+1)\cdots(p+(p+1)/2)}{((p+1)/2-k)!4^k((p-1)/2+k)!/((p-1)/2-k)!}\\ =&\frac{(p+1)\cdots(p+(p+1)/2)}{((p-1)/2)!}\sum_{k=1}^{m}\frac{\prod_{j=1}^k(2p+2j-1)}{((p+1)/2-k)2^k\prod_{j=1}^k((p-1)/2+j)}\\ =&\frac{3p+1}{2}\prod_{j=1}^{(p-1)/2}\left(1+\frac{p}{j}\right)\sum_{k=1}^{m}\frac{\prod_{j=1}^k(1+p/(p+2j-1))}{(p+1)/2-k} \pmod{p^2}\end{aligned}$$

and hence

$$\begin{aligned}
&\sum_{k=1}^{m}(-1)^k\frac{\binom{4m+2k+2}{2k}\binom{3m+2}{m-k+1}}{4^{-k}\binom{2k}{k}}\\
\equiv&\frac{3p+1}{2}(1+pH_{(p-1)/2})\sum_{s=1}^{m}\frac{1+p\sum_{j=1}^{(p+1)/2-s}1/(2j-1)}{s}\\
\equiv&\frac{1+3p-2p\,q_p(2)}{2}\left(H_m+\sum_{s=1}^{m}\frac{p}{s}\sum_{t=s}^{(p-1)/2}\frac{1}{2((p+1)/2-t)-1}\right)\\
\equiv&\frac{1+3p-2p\,q_p(2)}{2}\left(H_m-\frac{p}{2}\sum_{s=1}^{m}\frac{H_m-H_{s-1}}{s}\right)\\
\equiv&\frac{1+3p-2p\,q_p(2)}{2}\left(H_m-\frac{p}{2}H_m^2+\frac{p}{2}\sum_{k=1}^{m}\frac{H_{k-1}}{k}\right)\pmod{p^2}.
\end{aligned}$$

Applying Lemmas 3.2 and 3.3 we get

$$\begin{aligned}
&\sum_{k=1}^{m}(-1)^k\frac{\binom{4m+2k+2}{2k}\binom{3m+2}{m-k+1}}{4^{-k}\binom{2k}{k}}\\
\equiv&\frac{1+3p-2p\,q_p(2)}{2}\left(-2q_p(2)+p\,q_p(2)^2-\frac{p}{2}\cdot 4q_p(2)^2+\frac{p}{2}\cdot 2q_p(2)^2\right)\\
\equiv&-q_p(2)(1+3p-2p\,q_p(2))\pmod{p^2}.
\end{aligned}$$

Let L and R denote the left-hand side and the right-hand side of (3.7) with $N=m$ respectively. By the above,

$$\begin{aligned}
R\equiv&p^2(1-3p+p\,q_p(2))(-1)^{m+1}(-q_p(2))(1+3p-2pq_p(2))\\
\equiv&p^2(-1)^m q_p(2)(1-p\,q_p(2))\\
=&p\left(\frac{-1}{p}\right)(2^{p-1}-1)(1-(2^{p-1}-1))\pmod{p^4}.
\end{aligned}$$

On the other hand, with the help of Lemma 3.4 we have

$$L=\sum_{k=0}^{(p-1)/2}\frac{20k+3}{(-2^{10})^k}\binom{4k}{k,k,k,k}-3p\left(\frac{-1}{p}\right)\pmod{p^4}.$$

So (3.7) with $N=m$ yields the desired (1.2). We are done. □

Acknowledgment. The author is grateful to the referee for helpful comments.

References

[AZ] T. Amdeberhan and D. Zeilberger, *Hypergeometric series acceleration via the WZ method*, Electron. J. Combin. **4** (1997), no. 2, #R3.

[BB] N. D. Baruah and B. C. Berndt, *Eisenstein series and Ramanujan-type series for* $1/\pi$, Ramanujan J. **23** (2010), 17–44.

[BBC] N. D. Baruah, B. C. Berndt and H. H. Chan, *Ramanujan's series for* $1/\pi$*: a survey*, Amer. Math. Monthly **116** (2009), 567–587.

[Be] B. C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer, New York, 1994.

[EZ] S. B. Ekhad and D. Zeilberger, *A WZ proof of Ramanujan's formula for* π, in: Geometry, Analysis, and Mechanics (J. M. Rassias, ed.), World Sci. Publ., Singapore, 1994, 107–108.

[L] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. of Math. (2) **39** (1938), 350–360.

[Lo] L. Long, *Hypergeometric evaluation identities and supercongruences*, Pacific J. Math. **249** (2011), 405–418.

[M] F. Morley, *Note on the congruence* $2^{4n} \equiv (-1)^n(2n)!/(n!)^2$*, where* $2n+1$ *is a prime*, Ann. of Math. **9** (1895), 168–170.

[Mo] E. Mortenson, *A* p*-adic supercongruence conjecture of van Hamme*, Proc. Amer. Math. Soc. **136** (2008), 4321–4328.

[PWZ] M. Petkovšek, H. S. Wilf and D. Zeilberger, $A = B$, A K Peters, Wellesley, 1996.

[R] S. Ramanujan, *Modular equations and approximations to* π, Quart. J. Math. (Oxford) (2) **45** (1914), 350–372.

[S] Z. H. Sun, *Congruences concerning Legendre polynomials*, Proc. Amer. Math. Soc. **139** (2011), 1915–1929.

[Su1] Z. W. Sun, *On congruences related to central binomial coefficients*, J. Number Theory **131** (2011), 2219-2238.

[Su2] Z. W. Sun, *Supper congruences and Euler numbers*, Sci. China Math. **54** (2011), 2509–2535.

[vH] L. van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, in: p-adic Functional Analysis (Nijmegen, 1996), pp. 223–236, Lecture Notes in Pure and Appl. Math., Vol., 192, Dekker, 1997.

[W] J. Wolstenholme, *On certain properties of prime numbers*, Quart. J. Appl. Math. **5** (1862), 35–39.

[Z] D. Zeilberger, *Closed form (pun intended!)*, Contemp. Math. **143** (1993), 579–607.

[ZPS] L. L. Zhao, H. Pan and Z. W. Sun, *Some congruences for the second-order Catalan numbers*, Proc. Amer. Math. Soc. **138** (2010), 37–46.

[Zu] W. Zudilin, *Ramanujan-type supercongruences*, J. Number Theory **129** (2009), 1848–1857.