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CONJECTURES AND RESULTS ON x^2 mod p^2 WITH $4p = x^2 + dy^2$

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In honor of Prof. Keqin Feng on the occasion of his 70th birthday

Abstract. Given a squarefree positive integer d, we want to find *integers* (or rational numbers with denominators not divisible by large primes) $a_0, a_1, a_2, \ldots$ such that for sufficiently large primes p we have $\sum_{k=0}^{p-1} a_k \equiv x^2 - 2p \pmod{p^2}$ if $4p = x^2 + dy^2$ (and $4 \nmid x$ if $d = 1$), and $\sum_{k=0}^{p-1} a_k \equiv 0 \pmod{p^2}$ if $(\frac{-d}{p}) = -1$. We give a survey of conjectures and results on this topic and point out the connection between this problem and series for $1/\pi$.

1. Introduction

Let $d \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ be squarefree. If an odd prime p not dividing d can be written in the form $x^2 + dy^2$ with $x, y \in \mathbb{Z}$ then the Legendre symbol $(\frac{-d}{p})$ must be equal to 1. For $d = 1, 2, 3$, it is well known that any odd prime p with $(\frac{-d}{p}) = 1$ can be written uniquely in the form $x^2 + dy^2$ with $x, y \in \mathbb{Z}^+$ (and $2 \nmid x$ if $d = 1$). (See, e.g., Cox [C].)

Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. In 1828 Gauss determined x mod p by showing the congruence

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod{p}.$$

In 1986, S. Chowla, B. Dwork and R. J. Evans [CDE] used Gauss and Jacobi sums to prove that

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2},$$

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which was first conjectured by F. Beukers. This implies that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2-2p) \pmod{p^2}$$

since

$$\left(2x-\frac{p}{2x}\right)^2 \equiv 4x^2-2p \pmod{p^2}.$$

In 2010 J. B. Cosgrave and K. Dilcher [CD] determined $\binom{(p-1)/2}{(p-1)/4} \bmod p^3$. A result due to Cauchy and Whiteman (cf. [HW84]) asserts that if $p\equiv 1 \pmod{20}$ then

$$\binom{(p-1)/2}{(p-1)/20} \equiv (-1)^{[5|x]}\binom{(p-1)/2}{3(p-1)/20} \pmod{p}.$$

(Throughout this paper, for an assertion A we let $[A]$ takes 1 or 0 according as A holds or not.) The reader may consult [HW79] for more such congruences of the Cauchy type.

In 1989 K. M. Yeung [Ye] employed the Gross-Koblitz formula ([GK]) for the p-adic Gamma function to prove that if $p\equiv 1 \pmod 6$ is a prime and $p=x^2+3y^2$ with $x,y\in\mathbb{Z}$ and $3\mid x-1$ then

$$\binom{(p-1)/2}{(p-1)/6} \equiv \left(2x-\frac{p}{2x}\right)\left(1-\frac{2^p-2}{3}+\frac{3^p-3}{4}\right) \pmod{p^2}.$$

Let $p=mf+1$ be a prime, where $m,f\in\mathbb{Z}^+$ and $m>1$. It is interesting to determine $\binom{rf+sf}{rf} \bmod p^2$ in terms of parameters arising from representations of p by certain binary quadratic forms where r and s are positive integers with $r+s\leqslant m$. The reader may consult the survey [HW] and [BEW, Chapter 9] for results and methods on this problem with $m=3,\ldots,16,20,24$. See also [Yo] for related results obtained via the Gross-Koblitz formula.

Let d be a squarefree positive integer. It is known that the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number one if and only if $d\in\{1,2,3,7,11,19,43,67,163\}$.

Let $d\in\{7,11,19,43,67,163\}$. Then $K=\mathbb{Q}(\sqrt{-d})$ has class number one. Let p be an odd prime with $(\frac{-d}{p})=1$. By algebraic number theory, p splits in the ring O_K of algebraic integers in K. As O_K is a principal ideal domain, there are unique positive integers x and y with $x\equiv y \pmod 2$ such that

$$p=\frac{x+y\sqrt{-d}}{2}\times\frac{x-y\sqrt{-d}}{2}$$

and hence $4p=x^2+dy^2$.

Let $d\in\mathbb{Z}^+$ be squarefree. Suppose that p is an odd prime with $4p=x^2+dy^2$ for some $x,y\in\mathbb{Z}$. If both x and y are odd, then $d+1\equiv 4p\equiv 4 \pmod 8$ and hence $d \equiv 3 \pmod 8$. Thus, if $d \not\equiv 3 \pmod 8$, then x and y are both even, and $p = (x/2)^2 + d(y/2)^2$.

Let p be an odd prime. In 1977, A. R. Rajwade [Ra] proved that

$$\sum_{x=0}^{p-1}\left(\frac{x^3+21x^2+112x}{p}\right) = \begin{cases} -2x(\frac{x}{7}) & \text{if } (\frac{p}{7})=1 \ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 & \text{if } (\frac{p}{7})=-1, \text{ i.e., } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Using character sums, Jacobi (cf. [HW]) proved that if $p = 11f + 1$ with $f \in \mathbb{Z}^+$ and $4p = x^2 + 11y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 2 \pmod{11}$ then

$$x \equiv \binom{6f}{3f}\binom{3f}{f}\bigg/\binom{4f}{2f} \pmod p.$$

In 1982 J. C. Parnami and Rajwade [PR] showed that

$$\begin{aligned}&\sum_{x=0}^{p-1}\left(\frac{x^3-96\cdot 11x+112\cdot 11^2}{p}\right)\\ =&\begin{cases} (\frac{2}{p})(\frac{x}{11})x & \text{if } (\frac{p}{11})=1 \ \&\ 4p=x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0 & \text{if } (\frac{p}{11})=-1, \text{ i.e., } p\equiv 2,6,7,8,10 \pmod{11}.\end{cases}\end{aligned}$$

Via elliptic curves with complex multiplication, it is also known that (cf. [RPR], [JM] and [PV]) for $d = 19, 43, 67, 163$ we have

$$\sum_{x=0}^{p-1}\left(\frac{f_d(x)}{p}\right) = \begin{cases} (\frac{2}{p})(\frac{x}{d})x & \text{if } (\frac{p}{d})=1 \ \&\ 4p=x^2+dy^2\ (x,y\in\mathbb{Z}),\\ 0 & \text{if } (\frac{p}{d})=-1,\end{cases}$$

where

$$\begin{aligned} f_{19}(x) =& x^3 - 8\cdot 19x + 2\cdot 19^2,\\ f_{43}(x) =& x^3 - 80\cdot 43x + 42\cdot 43^2,\\ f_{67}(x) =& x^3 - 440\cdot 67x + 434\cdot 67^2,\\ f_{163}(x) =& x^3 - 80\cdot 23\cdot 29\cdot 163x + 14\cdot 11\cdot 19\cdot 127\cdot 163^2.\end{aligned}$$

See also Williams [W] for other similar results.

Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. The author [Su3, Conjecture 5.5] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k} \equiv \left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k} \equiv \left(\frac{2}{p}\right)\left(2x-\frac{p}{2x}\right) \pmod{p^2},$$

and this was completely proved by the author's twin brother Zhi-Hong Sun [S1] with the help of Legendre polynomials.

Let $n\in\mathbb{N}=\{0,1,2,\ldots\}$. In combinatorics, the central Delannoy number D_n denotes the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$ (cf. [S]). It is known that

$$D_n=\sum_{k=0}^n\binom nk\binom{n+k}{k}=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}=[x^n](x^2+3x+2)^n.$$

(Throughout this paper by $[x^n]P(x)$ we mean the coefficient of x^n in the power series of $P(x)$.)

Let $p=2n+1$ be an odd prime. For $k=n+1,\ldots,p-1$, we clearly have

$$\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\equiv 0\pmod p.$$

For $k=0,\ldots,n$, Z. H. Sun [S1] noted that

$$\begin{aligned}\binom{n+k}{2k}=&\frac{\prod_{0<j\leqslant k}(p^2-(2j-1)^2)}{4^k(2k)!}\\ \equiv&\frac{\prod_{0<j\leqslant k}(-(2j-1)^2)}{4^k(2k)!}=\frac{\binom{2k}{k}}{(-16)^k}\pmod{p^2}.\end{aligned}$$

and this implies van Hamme's observation ([vH])

$$\binom nk\binom{n+k}{k}=\binom{n+k}{2k}\binom{2k}{k}\equiv\frac{\binom{2k}{k}^2}{(-16)^k}\pmod{p^2}.$$

Therefore

$$D_{(p-1)/2}=D_n\equiv\sum_{k=0}^n\frac{\binom{2k}{k}^2}{(-16)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\pmod{p^2}.$$

Since

$$\binom{(p-1)/2}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod p\quad\text{for }k=0,1,\ldots,p-1,$$

we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv\sum_{k=0}^n(-1)^k\binom nk^2=[x^n](1+x)^n(1-x)^n=[x^n](1-x^2)^n\pmod p.$$

Thus, if $p\equiv 3\pmod 4$ (i.e., $2\nmid n$) then we have $\sum_{k=0}^{p-1}\binom{2k}{k}^2/(-16)^k\equiv 0\pmod p$. If $p\equiv 1\pmod 4$ and $p=x^2+y^2$ with $x\equiv 1\pmod 4$ and $y\equiv 0\pmod 2$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-1)/4}\binom{(p-1)/2}{(p-1)/4}\equiv(-1)^{(p-1)/4}2x\pmod p$$

by applying Gauss' congruence. This determines $\sum_{k=0}^{p-1}\binom{2k}{k}^2/(-16)^k$ mod p in a simple way. Similarly, in 2009 the author determined $\sum_{k=0}^{p-1}\binom{2k}{k}^2/8^k$ and $\sum_{k=0}^{p-1}\binom{2k}{k}^2/32^k$ modulo p by noting that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv&\sum_{k=0}^{n}2^k\binom{n}{k}\binom{n}{n-k}=[x^n](1+2x)^n(1+x)^n\\ =&[y^n](y^2+3y+2)^n=D_n\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\pmod p\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\equiv&\sum_{k=0}^{n}\frac{1}{2^k}\binom{n}{k}^2=[x^n]\left(1+\frac{x}{2}\right)^n(1+x)^n\\ =&2^{-n}[x^n](x^2+3x+2)^n=\frac{D_n}{2^n}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\pmod p.\end{aligned}$$

Recently the author established the following theorem.

Theorem 1.1. (Z. W. Sun [Su5]) *Let $p\equiv 1\pmod 4$ be a prime and write $p = x^2 + y^2$ with $x\equiv 1\pmod 4$ and $y\equiv 0\pmod 2$. Then we can determine x mod p^2 in the following way:*

$$(-1)^{(p-1)/4}x\equiv\sum_{k=0}^{(p-1)/2}\frac{k+1}{8^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2\pmod{p^2}.\quad(1.1)$$

Concerning the representations $p = x^2 + 3y^2$ and $p = x^2 + 7y^2$, we have formulated a conjecture similar to Theorem 1.1.

Conjecture 1.1. (Sun [Su4]) *Let p be an odd prime.*

(i) *If $p\equiv 1\pmod 3$ and $p = x^2 + 3y^2$ with $x, y\in\mathbb{Z}$ and $x\equiv 1\pmod 3$, then we can determine x mod p^2 in the following way:*

$$\sum_{k=0}^{p-1}\frac{k+1}{48^k}\binom{2k}{k}\binom{4k}{2k}\equiv x\pmod{p^2}.\quad(1.2)$$

(ii) *If $p\equiv 7\pmod{12}$ and $p = x^2 + 3y^2$ with $y\equiv 1\pmod 4$, then we can determine y mod p^2 via the congruence*

$$\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)\frac{k\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p+1)/4}y\pmod{p^2}.\quad(1.3)$$

(iii) *If* $(\frac{p}{7}) = 1$ *and* $p = x^2 + 7y^2$ *with* $x, y \in \mathbb{Z}$ *and* $(\frac{x}{7}) = 1$, *then we can determine* x *mod* p^2 *in the following way:*

$$\sum_{k=0}^{p-1} \frac{k+8}{63^k} \binom{2k}{k} \binom{4k}{2k} \equiv 8 \left(\frac{p}{3}\right) x \pmod{p^2}. \tag{1.4}$$

For congruences concerning the sum $\sum_{k=0}^{p-1} \binom{2k}{k}/m^k$ modulo powers of a prime p (where $m \in \mathbb{Z}$ and $m \not\equiv 0 \pmod p$), the reader may consult [PS], [ST1], [ST2], [Su1] and [Su2].

Let $d \in \mathbb{Z}^+$ be squarefree and let p be an odd prime with $(\frac{-d}{p}) = 1$. If p or $4p$ can be written in the form $x^2 + dy^2$ (in such a case we always assume $x, y \in \mathbb{Z}$ even though we may not mention it to avoid lines overfull), how to determine x^2 modulo p^2 in a simple and explicit form? We will focus on this question in this survey and present various conjectures and results on the author's following problems.

Problem 1.1. *Given a squarefree positive integer* d, *find integers* $a_0, a_1, \ldots$ *such that for sufficiently large primes* p *we have*

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \ (\text{and } 4 \nmid x \text{ if } d = 1), \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1. \end{cases}$$

If one thinks that the integral condition of $a_0, a_1, a_2, \ldots$ in Problem 1.1 is too strict, we may study the following easier problem.

Problem 1.2. *Given a squarefree* $d \in \mathbb{Z}^+$, *find rational numbers* $a_0, a_1, a_2, \ldots$ *with denominators not divisible by large primes such that for sufficiently large primes* p *we have*

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \ (\text{and } 4 \nmid x \text{ if } d = 1), \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1. \end{cases}$$

We find that Problems 1.1 and 1.2 have affirmative answers for most of those $d \in \mathbb{Z}^+$ with the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ having class number 1 or 2 or 4.

Now we give suggested solutions to the problems with $d = 7, 15, 427$ as examples.

Conjecture 1.2 (Sun [Su3])**.** *Let* p *be an odd prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1 \ \&\ p = x^2 + 7y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1, \ \text{i.e., } p \equiv 3, 5, 6 \pmod 7. \end{cases} \tag{1.5}$$

Remark 1.1. The congruence modulo p can be easily deduced from (6) in Ahlgren [A, Theorem 5]. Recently the author's twin brother Z. H. Sun [S2] confirmed the conjecture in the case $(\frac{p}{7}) = -1$ via Legendre polynomials.

The imaginary quadratic fields $\mathbb{Q}(\sqrt{-15})$ and $\mathbb{Q}(\sqrt{-427})$ have class number two. Our next two conjectures provide affirmative answers to Problem 1.1 with $d = 15$ and Problem 1.2 with $d = 427$.

Conjecture 1.3. *For $k = 0, 1, 2, \ldots$ let T_k denote the trinomial coefficient defined as the coefficient of x^k in $(x^2 + x + 1)^k$. For any prime $p > 3$, we have*

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2 T_k$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15})=-1,\ \text{i.e., } p\equiv 7,11,13,14 \pmod{15},\end{cases}$$

and

$$\sum_{k=0}^{p-1}(105k+44)(-1)^k\binom{2k}{k}^2 T_k \equiv p\left(20+24\left(\frac{p}{3}\right)(2-3^{p-1})\right) \pmod{p^3}.$$

Also,

$$\frac{1}{2n\binom{2n}{n}}\sum_{k=0}^{n-1}(-1)^{n-1-k}(105k+44)\binom{2k}{k}^2 T_k \in \mathbb{Z}^+ \quad \text{for all } n=1,2,3,\ldots.$$

Conjecture 1.4. *Let p be an odd prime with $p \neq 11$. Then*

$$\sum_{n=0}^{p-1}\frac{1}{(-22)^{3n}}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}$$
$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{61})=1\ \&\ 4p=x^2+427y^2\ (x,y\in\mathbb{Z}),\\ 2p-7x^2 \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{61})=-1\ \&\ 4p=7x^2+61y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{427})=-1.\end{cases}$$

We also have

$$\sum_{n=0}^{p-1}\frac{11895n+1286}{(-22)^{3n}}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n} \equiv 1286p\left(\frac{p}{7}\right) \pmod{p^2}.$$

L. van Hamme [vH] observed that certain hypergeometric series have their p-adic analogues, e.g., motivated by the identity (of Ramanujan's type)

$$\sum_{k=0}^{\infty}\frac{4k+1}{(-64)^k}\binom{2k}{k}^3=\frac{2}{\pi}$$

he conjectured that

$$\sum_{k=0}^{p-1}\frac{4k+1}{(-64)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)\pmod{p^3}$$

for any odd prime p (and this was confirmed by E. Mortenson [M]). The reader may consult [BBC], [Be], [BoBo] and [ChCh] for more Ramanujan-type series for $1/\pi$.

The author has the following viewpoint (or hypothesis) the initial version of which appeared in his message to Number Theory Mailing List (sent on March 30, 2010) available from the website `http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind1003&L=nmbrthry&T=0&P=1668`.

Philosophy about Series for $1/\pi$. (i) *Given a 'regular' identity of the form*

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k,b,c,m\in\mathbb{Z}$, bm is nonzero and C^2 is rational, we must have

$$\sum_{k=0}^{n-1}(bk+c)a_km^{n-1-k}\equiv 0\pmod{n}$$

for any positive integer n. Furthermore, there exist an integer m' (often $m'=m$) and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1,2,2^2,\ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d>1$ and for any prime $p>3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k}\equiv\begin{cases}(\frac{m'}{p})(x^2-2p)\pmod{p^2} & \text{if } 4p=x^2+dy^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-d}{p})=-1,\end{cases}$$

or $d=1$, $\gcd(15,m)>1$, and $\sum_{k=0}^{p-1}a_k/m^k\equiv 0\pmod{p^2}$ for any prime $p\equiv 3\pmod 4$ with $p\nmid 3m$.

(ii) *Let $b,c,m,a_0,a_1,\ldots$ be integers with bm nonzero and the series $\sum_{k=0}^{\infty}(bk+c)a_k/m^k$ convergent. Suppose that there are $d\in\mathbb{Z}^+$, $d'\in\mathbb{Z}$, and rational numbers c_0 and c_1 such that*

$$\sum_{k=0}^{p-1}(bk+c)\frac{a_k}{m^k}\equiv p\left(c_0\left(\frac{-d}{p}\right)+c_1\left(\frac{d'}{p}\right)\right)\pmod{p^2}$$

for all sufficiently large primes p. If $d' \geqslant 0$, then

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi}$$

for some C with C^2 rational (and with $C/\sqrt{d}$ rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{\lambda_0\sqrt{d}+\lambda_1\sqrt{d_1}}{\pi}.$$

Remark 1.2. We don't have a precise description of the meaning of the word *regular* in the above philosophy, but almost all identities of the stated form are regular.

Now we illustrate the philosophy by the author's next three conjectures.

Conjecture 1.5. *For any prime $p > 3$, we have*

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{256^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}12^{n-k}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,7 \pmod{24}\ \&\ p=x^2+6y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } p\equiv 5,11 \pmod{24}\ \&\ p=2x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p})=-1,\end{cases}$$

and

$$\sum_{n=0}^{p-1}\frac{6n-1}{256^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}12^{n-k}\equiv -p \pmod{p^2}.$$

Also,

$$\sum_{n=0}^{\infty}\frac{6n-1}{256^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}12^{n-k}=\frac{8\sqrt{3}}{\pi}.$$

For $b, c \in \mathbb{Z}$ the nth generalized central trinomial coefficient $T_n(b, c)$ is defined as $[x^n](x^2 + bx + c)^n$. In [Su8] the author conjectured that

$$\sum_{k=0}^{\infty}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi}.$$

Here are related conjectural congruences.

Conjecture 1.6. *Let $p>3$ be a prime with $p\neq 7$. Then*

$$\left(\frac{-42}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)}{(-168^2)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2,\\12x^2-2p\pmod{p^2}&\text{if }p\equiv 2,8\pmod{15}\ \&\ p=3x^2+5y^2,\\0\pmod{p^2}&\text{if }(\frac{p}{15})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)\equiv p\left(3\left(\frac{-42}{p}\right)+10\left(\frac{-210}{p}\right)\right)\pmod{p^2}.$$

The imaginary quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number eight, so our following conjecture is of particular interest.

Conjecture 1.7. *For any prime $p>5$, we have*

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if}(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=1,\ p=x^2+105y^2,\\2x^2-2p\pmod{p^2}&\text{if}(\frac{-1}{p})=(\frac{p}{7})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ 2p=x^2+105y^2,\\2p-12x^2\pmod{p^2}&\text{if}(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=-1,\ p=3x^2+35y^2,\\2p-6x^2\pmod{p^2}&\text{if}(\frac{-1}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1,\ 2p=3x^2+35y^2,\\20x^2-2p\pmod{p^2}&\text{if}(\frac{-1}{p})=(\frac{p}{5})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1,\ p=5x^2+21y^2,\\10x^2-2p\pmod{p^2}&\text{if}(\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1,\ 2p=5x^2+21y^2,\\28x^2-2p\pmod{p^2}&\text{if}(\frac{-1}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{7})=1,\ p=7x^2+15y^2,\\14x^2-2p\pmod{p^2}&\text{if}(\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{7})=1,\ 2p=7x^2+15y^2,\\0\pmod{p^2}&\text{if }(\frac{-105}{p})=-1,\end{cases}$$

and

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(54\left(\frac{-1}{p}\right)+49\left(\frac{15}{p}\right)\right)\pmod{p^2}.$$

Also,

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

We mention that series for $1/\pi$ are usually difficult to prove; most proofs of them involve modular functions or elliptic integrals.

From Section 2 we will give various conjectures and results on Problems 1.1 and 1.2. Problem 1.1 for $d = 1$ already has a positive answer. We suggest positive answers to Problem 1.1 for

$$d \in \{2, 3, 5, 6, 7, 10, 13, 15, 22, 30, 37, 58, 70, 85, 130, 190\}.$$

We also formulate many conjectures concerning Problem 1.2; in particular, we give explicit conjectural positive answers for those squarefree positive integers d with $\mathbb{Q}(\sqrt{-d})$ having class number at most two except for $d = 187, 403$. Note that $\mathbb{Q}(\sqrt{-d})$ has class number two if and only if

$$d \in \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}.$$

For each of

$$d \in \{21, 30, 33, 42, 57, 70, 78, 85, 93, 102, 130, 133, 177, 190\},$$

the quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number four; for these values of d we also provide explicit conjectural positive answers to Problem 1.2 or even Problem 1.1.

2. Using Apéry polynomials and products of three binomial coefficients

The well-known Apéry numbers (introduced by Apéry in his proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ (see [Ap] and [P]), are given by

$$A_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2 \ (n = 0, 1, 2, \dots),$$

They also have close connections to modular forms (cf. K. Ono [O]). As in [Su7] we define Apéry polynomials by

$$A_n(x) = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 x^k = \sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2 x^k \quad (n = 0, 1, 2, \dots).$$

Clearly $A_n(1) = A_n$. The author [Su7] proved that for all $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$ we have

$$\sum_{k=0}^{n-1}(2k+1)A_k(x) \equiv 0 \pmod{n}.$$

He also conjectured that $n \mid \sum_{k=0}^{n-1}(2k+1)(-1)^k A_k(x)$ for all $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$, which was later confirmed by V. J. W. Guo and J. Zeng [GZ].

The following theorem solves Problem 1.1 for $d = 1$.

Theorem 2.1 (Sun [Su7]). *Let p be an odd prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^kA_k(-2)\equiv\sum_{k=0}^{p-1}(-1)^kA_k\left(\frac14\right)\\ \equiv&\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1\ (\mathrm{mod}\ 4)\ \&\ p=x^2+y^2\ (2\nmid x),\\ 0\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 3\ (\mathrm{mod}\ 4).\end{cases}\end{aligned}\tag{2.1}$$

Remark 2.1. A lemma for the proof of Theorem 2.1 states that for any odd prime p we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1\ (\mathrm{mod}\ 4)\ \&\ p=x^2+y^2\ (2\nmid x),\\ 0\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 3\ (\mathrm{mod}\ 4).\end{cases}$$

This was first conjectured by the author [Su4] and later confirmed by his twin brother Z.-H. Sun [S2].

Concerning Problem 1.1 for $d=2,3$ the author made the following conjecture.

Conjecture 2.1 (Sun [Su7]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}A_k\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1,3\ (\mathrm{mod}\ 8)\ \&\ p=x^2+2y^2,\\ 0\ (\mathrm{mod}\ p^2) & \text{if } (\frac{-2}{p})=-1,\ \text{i.e.},\ p\equiv 5,7\ (\mathrm{mod}\ 8).\end{cases}\tag{2.2}$$

And

$$\sum_{k=0}^{p-1}(-1)^kA_k\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1\ (\mathrm{mod}\ 3)\ \text{and}\ p=x^2+3y^2,\\ 0\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 2\ (\mathrm{mod}\ 3).\end{cases}\tag{2.3}$$

Remark 2.2. The author [Su7] proved the mod p version of (2.2) and (2.3), and (2.3) in the case $p\equiv 2\ (\mathrm{mod}\ 3)$.

The following theorem relates sums of Apéry polynomials to sums of products of three binomial coefficients. Obviously, for each $k\in\mathbb{N}$ we have

$$\binom{4k}{k,k,k,k}=\frac{(4k)!}{(k!)^4}=\binom{2k}{k}^2\binom{4k}{2k}.$$

Theorem 2.2 (Sun [Su7]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2}. \tag{2.4}$$

Also, for any p-adic integer $x \not\equiv 0 \pmod p$ we have

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod p. \tag{2.5}$$

Recall that Conjecture 1.2 determines $\sum_{k=0}^{p-1}\binom{2k}{k}^3$ mod p^2 for any odd prime p. By Theorem 2.2, we have

$$\sum_{k=0}^{p-1}(-1)^k A_k(16) \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^3 \pmod{p^2}.$$

Conjecture 2.2. *Let $p \neq 2,5$ be a prime. Then*

$$\sum_{k=0}^{p-1} A_k(-4) \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,9 \pmod{20}\ \&\ p=x^2+5y^2, \\ 2x^2-2p \pmod{p^2} & \text{if } p\equiv 3,7 \pmod{20}\ \&\ 2p=x^2+5y^2, \\ 0 \pmod{p^2} & \text{if } p\equiv 11,13,17,19 \pmod{20}. \end{cases}$$

Remark 2.3. Let $p \neq 2,5$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $p \equiv 1,9 \pmod{20}$ then $p = x^2+5y^2$ for some $x,y\in\mathbb{Z}$; if $p\equiv 3,7 \pmod{20}$ then $2p = x^2+5y^2$ for some $x,y\in\mathbb{Z}$.

Conjecture 2.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} A_k(9) \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,7 \pmod{24}\ \&\ p=x^2+6y^2, \\ 2p-8x^2 \pmod{p^2} & \text{if } p\equiv 5,11 \pmod{24}\ \&\ p=2x^2+3y^2, \\ 0 \pmod{p^2} & \text{if } p\equiv 13,17,19,23 \pmod{24}. \end{cases}$$

Conjecture 2.4. *Let $p > 3$ be a prime with $p \neq 11$. Then*

$$\sum_{k=0}^{p-1} A_k(99^2) \equiv \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{1584^{2k}}$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{-11}{p})=(\frac{2}{p})=1\ \&\ p=x^2+22y^2, \\ 2p-8x^2 \pmod{p^2} & \text{if } (\frac{-11}{p})=(\frac{2}{p})=-1\ \&\ p=2x^2+11y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-11}{p})=-(\frac{2}{p}). \end{cases}$$

And

$$\begin{aligned}&\sum_{k=0}^{p-1}A_k(99^4)\equiv\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{396^{4k}}\\&\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{29}{p})=(\frac{-2}{p})=1\ \&\ p=x^2+58y^2,\\2p-8x^2\pmod{p^2}&\text{if }(\frac{29}{p})=(\frac{-2}{p})=-1\ \&\ p=2x^2+29y^2,\\0\pmod{p^2}&\text{if }(\frac{-58}{p})=-1.\end{cases}\end{aligned}$$

Furthermore, for $n=2,3,4,\ldots$ *we have*

$$a_n:=\frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(280k+19)\binom{4k}{k,k,k,k}1584^{2(n-1-k)}\in\mathbb{Z}$$

unless $2n+1$ *is a power of* 3 *in which case* $3a_n\in\mathbb{Z}\setminus3\mathbb{Z}$. *Also, for* $n=2,3,4,\ldots$ *we have*

$$b_n:=\frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(26390k+1103)\binom{4k}{k,k,k,k}396^{4(n-1-k)}\in\mathbb{Z}$$

unless $2n+1$ *is a power of* 3 *in which case* $3b_n\in\mathbb{Z}\setminus3\mathbb{Z}$.

Remark 2.4. Ramanujan (cf. [Be, p. 354]) found that

$$\sum_{k=0}^{\infty}\frac{280k+19}{1584^{2k}}\binom{4k}{k,k,k,k}=\frac{2\times99^2}{\pi\sqrt{11}}$$

and

$$\sum_{k=0}^{\infty}\frac{26390k+1103}{396^{4k}}\binom{4k}{k,k,k,k}=\frac{99^2}{2\pi\sqrt{2}}.$$

The reader may consult [BB], [BBC] and [CC] for other known Ramanujan-type series not mentioned in this paper.

Conjecture 2.5. *Let* $p>3$ *be a prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}A_k(-324)\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(-2^{10}3^4)^k}\\&\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{13}{p})=(\frac{-1}{p})=1\ \&\ p=x^2+13y^2,\\2x^2-2p\pmod{p^2}&\text{if }(\frac{13}{p})=(\frac{-1}{p})=-1\ \&\ 2p=x^2+13y^2,\\0\pmod{p^2}&\text{if }(\frac{13}{p})=-(\frac{-1}{p}).\end{cases}\end{aligned}$$

We also have

$$\sum_{k=0}^{p-1}\frac{260k+23}{(-2^{10}3^4)^k}\binom{4k}{k,k,k,k}\equiv 23p\left(\frac{-1}{p}\right)+\frac{5}{3}p^3E_{p-3}\pmod{p^4},$$

where $E_0, E_1, E_2, \ldots$ are Euler numbers. Furthermore, for $n = 2, 3, 4, \ldots$ we have

$$c_n := \frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(260k+23)\binom{4k}{k,k,k,k}(-82944)^{n-1-k}\in\mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3c_n \in \mathbb{Z}\setminus 3\mathbb{Z}$.

Conjecture 2.6. *Let $p > 3$ be a prime with $p \neq 7$. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}A_k(-882^2)&\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(-2^{10}21^4)^k}\\ &\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{37}{p})=(\frac{-1}{p})=1\ \&\ p=x^2+37y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{37}{p})=(\frac{-1}{p})=-1\ \&\ 2p=x^2+37y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-37}{p})=-1.\end{cases}\end{aligned}$$

Furthermore, for $n = 2, 3, 4, \ldots$ we have

$$d_n := \frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(21460k+1123)\binom{4k}{k,k,k,k}(-2^{10}21^4)^{n-1-k}\in\mathbb{Z}$$

unless $2n+1$ is a power of 3 in which case $3d_n \in \mathbb{Z}\setminus 3\mathbb{Z}$.

We will not list our similar conjectures on $\sum_{k=0}^{p-1}A_k(x)$ mod p^2 with x among

$$-48,\ 81,\ 2401,\ -25920,\ -\frac{9}{16},\ \frac{81}{32},\ \frac{81}{256},\ -\frac{3969}{256}.$$

Recall that $\mathbb{Q}(\sqrt{-11})$ has class number one. The author's following conjecture provides a positive answer to Problem 1.2 with $d = 11$.

Conjecture 2.7 (Sun [Su3])**.** *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv\begin{cases}x^2-2p\pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{11})=-1.\end{cases}$$

Remark 2.5. Recently Z. H. Sun [S3] confirmed the mod p version of the congruence.

Note that [Su4] contains some of the author's conjectural congruences related to Ramanujan-type series. Below we present more such conjectures.

Conjecture 2.8. *Let $p>3$ be a prime. Then*
$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k,k,k}}{(-96)^{3k}}\equiv\begin{cases}(\frac{-6}{p})(x^2-2p)\pmod{p^2}&\text{if }(\frac{p}{19})=1\ \&\ 4p=x^2+19y^2,\\0\pmod{p^2}&\text{if }(\frac{p}{19})=-1.\end{cases}$$
Also,
$$\sum_{k=0}^{p-1}\frac{342k+25}{(-96)^{3k}}\binom{6k}{3k}\binom{3k}{k,k,k}\equiv 25p\left(\frac{-6}{p}\right)\pmod{p^3}.$$

Remark 2.6. The conjectural congruences modulo p have been confirmed by Z. H. Sun [S3]. D. V. Chudnovsky and G. V. Chunovsky [CC] obtained that
$$\sum_{k=0}^{\infty}\frac{342k+25}{(-96)^{3k}}\binom{6k}{3k}\binom{3k}{k,k,k}=\frac{32\sqrt{6}}{\pi}.$$

Conjecture 2.9. *Let $p>5$ be a prime. Then*
$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k,k,k}}{(-960)^{3k}}\equiv\begin{cases}(\frac{p}{15})(x^2-2p)\pmod{p^2}&\text{if }(\frac{p}{43})=1\ \&\ 4p=x^2+43y^2,\\0\pmod{p^2}&\text{if }(\frac{p}{43})=-1.\end{cases}$$
If $p\neq 11$, then
$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k,k,k}}{(-5280)^{3k}}\equiv\begin{cases}(\frac{-330}{p})(x^2-2p)\pmod{p^2}&\text{if }(\frac{p}{67})=1\ \&\ 4p=x^2+67y^2,\\0\pmod{p^2}&\text{if }(\frac{p}{67})=-1.\end{cases}$$

Remark 2.7. The conjectural congruences modulo p has been confirmed by Z. H. Sun [S3].

Conjecture 2.10. *Let $p>5$ be a prime with $p\neq 23,29$. Then*
$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k,k,k}}{(-640320)^{3k}}$$
$$\equiv\begin{cases}(\frac{-10005}{p})(x^2-2p)\pmod{p^2}&\text{if }(\frac{p}{163})=1\ \&\ 4p=x^2+163y^2,\\0\pmod{p^2}&\text{if }(\frac{p}{163})=-1.\end{cases}$$
Also,
$$\sum_{k=0}^{p-1}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k,k,k}\equiv 13591409p\left(\frac{-10005}{p}\right)\pmod{p^3}.$$

Remark 2.8. The conjectural congruence modulo p has been confirmed by Z. H. Sun [S3]. D. V. Chudnovsky and G. V. Chudnovsky [CC] got the formula
$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k,k,k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}},$$
which enabled them to hold the record for the calculation of π during 1989-1994.

Conjecture 2.11. *Let $p>3$ be a prime with $p\neq 17$. Then*

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{68^n}\sum_{k=0}^{n}\binom{n}{k}^3 64^k\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12^3)^k}$$
$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{17})=1\ \&\ 4p=x^2+51y^2,\\ 2p-3x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{17})=-1\ \&\ 4p=3x^2+17y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{51})=-1.\end{cases}$$

Also,

$$\sum_{n=0}^{p-1}\frac{36n+19}{68^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^3 64^k\equiv 19p\left(\frac{p}{17}\right) \pmod{p^2},$$

and

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{51k+7}{(-12^3)^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 7\left(\frac{p^a}{3}\right)+\left(\frac{p^{a-1}}{3}\right)\frac{5}{6}p^2B_{p-2}\left(\frac13\right) \pmod{p^3}$$

for every $a\in\mathbb{Z}^+$, where $B_n(x)$ denotes the Bernoulli polynomial of degree n.

Remark 2.9. Ramanujan [R] found that

$$\sum_{k=0}^{\infty}\frac{51k+7}{(-12^3)^k}\binom{2k}{k}^2\binom{3k}{k}=\frac{12\sqrt3}{\pi}.$$

Conjecture 2.12. *Let $p>3$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}}\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{41})=1\ \&\ 4p=x^2+123y^2,\\ 2p-3x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{41})=-1\ \&\ 4p=3x^2+41y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{123})=-1.\end{cases}$$

Also,

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{615k+53}{(-48)^{3k}}\binom{2k}{k}^2\binom{3k}{k}\equiv 53\left(\frac{p^a}{3}\right)+\left(\frac{p^{a-1}}{3}\right)\frac{5}{12}p^2B_{p-2}\left(\frac13\right) \pmod{p^3}$$

for any positive integer a.

(ii) *Suppose $p>5$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}}\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{89})=1\ \&\ 4p=x^2+267y^2,\\ 2p-3x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{89})=-1\ \&\ 4p=3x^2+89y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{267})=-1.\end{cases}$$

Also,

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{14151k+827}{(-300)^{3k}}\binom{2k}{k}^2\binom{3k}{k}$$
$$\equiv 827\left(\frac{p^a}{3}\right)+\left(\frac{p^{a-1}}{3}\right)\frac{13}{150}p^2B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}$$

for any positive integer a.

Remark 2.10. It is known (cf. [CC]) that

$$\sum_{k=0}^{\infty}\frac{615k+53}{(-48)^{3k}}\binom{2k}{k}^2\binom{3k}{k}=\frac{96\sqrt{3}}{\pi}$$

and

$$\sum_{k=0}^{\infty}\frac{14151k+827}{(-300)^{3k}}\binom{2k}{k}^2\binom{4k}{2k}=\frac{1500\sqrt{3}}{\pi}.$$

3. Using the polynomials $S_n(x)=\sum_{k=0}^n\binom{n}{k}^4x^k$

In this section we give many conjectures on Problem 1.1 involving the polynomials

$$S_n(x):=\sum_{k=0}^n\binom{n}{k}^4x^k\quad(n=0,1,2,\ldots).$$

Conjecture 3.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}S_k(-2)\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x\ \&\ 2\mid y),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(3k+2)S_k(-2)\equiv\frac{p}{2}\left(1+3\left(\frac{-1}{p}\right)\right)\pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(6k+4)S_k(-2)\in\mathbb{Z}\qquad\text{for all } n=1,2,3,\ldots.$$

Conjecture 3.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(-4) \equiv \sum_{k=0}^{p-1} S_k(-64)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \& \ p = x^2 + 5y^2 \ (x, y \in \mathbb{Z}), \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \ \& \ 2p = x^2 + 5y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{-5}{p}) = -1, \ \text{i.e., } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (8k+7) S_k(-64) \equiv p \left(\frac{p}{15}\right) \left(3 + 4\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (8k+7) S_k(-64) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Conjecture 3.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(4)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \ \& \ p = x^2 + 6y^2 \ (x, y \in \mathbb{Z}), \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \ \& \ p = 2x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p}) = -1, \ \text{i.e., } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

And

$$\sum_{k=0}^{p-1} (24k+17) S_k(4) \equiv p \left(5 + 12\left(\frac{2}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} (24k+17) S_k(4) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Conjecture 3.4. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(16)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9, 11, 19 \pmod{40} \ \& \ p = x^2 + 10y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 7, 13, 23, 37 \pmod{40} \ \& \ p = 2x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-10}{p}) = -1. \end{cases}$$

When $(\frac{-10}{p})=1$ *we have*

$$\sum_{k=0}^{p-1}(160k+129)S_k(16)\equiv 80p\left(\frac{p}{5}\right)\pmod{p^2}.$$

Conjecture 3.5. *Let* p *be an odd prime. Then*

$$\sum_{k=0}^{p-1}S_k(1)\equiv\sum_{k=0}^{p-1}S_k(-9)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15}\ \&\ p=3x^2+5y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{15})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(3k+2)S_k(1)\equiv 2p\left(\frac{p}{5}\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(5k+4)S_k(-9)\equiv\frac{p}{2}\left(\frac{p}{3}\right)\left(3+5\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

Moreover,

$$\frac{1}{2n}\sum_{k=0}^{n-1}(3k+2)S_k(1)\in\mathbb{Z}\quad\text{and}\quad\frac{1}{2n}\sum_{k=0}^{n-1}(5k+4)S_k(-9)\in\mathbb{Z}$$

for all $n=1,2,3,\ldots$.

Conjecture 3.6. *Let* p *be an odd prime. Then*

$$\sum_{k=0}^{p-1}S_k(36)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=x^2+30y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=5x^2+6y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-30}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(8k+7)S_k(36)\equiv p\left(\frac{p}{15}\right)\left(3+4\left(\frac{-6}{p}\right)\right)\pmod{p^2}.$$

We also have

$$\frac{1}{n}\sum_{k=0}^{n-1}(8k+7)S_k(36)\in\mathbb{Z}\qquad\text{for all } n=1,2,3,\ldots.$$

Conjecture 3.7. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(196)$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{5}) = (\frac{p}{7}) = 1 \ \&\ p = x^2 + 70y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1,\ (\frac{2}{p}) = (\frac{p}{5}) = -1 \ \&\ p = 2x^2 + 35y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } (\frac{p}{5}) = 1,\ (\frac{2}{p}) = (\frac{p}{7}) = -1 \ \&\ p = 5x^2 + 14y^2, \\ 28x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1,\ (\frac{p}{5}) = (\frac{p}{7}) = -1 \ \&\ p = 7x^2 + 10y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-70}{p}) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(120k + 109)S_k(196) \equiv p\left(\frac{p}{7}\right)\left(49 + 60\left(\frac{-14}{p}\right)\right) \pmod{p^2}.$$

We also have

$$\frac{1}{n}\sum_{k=0}^{n-1}(120k + 109)S_k(196) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Conjecture 3.8. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(-324)$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{-1}{p}) = (\frac{p}{5}) = (\frac{p}{17}) = 1 \ \&\ p = x^2 + 85y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{17}) = 1,\ (\frac{-1}{p}) = (\frac{p}{5}) = -1 \ \&\ 2p = x^2 + 85y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } (\frac{-1}{p}) = 1,\ (\frac{p}{5}) = (\frac{p}{17}) = -1 \ \&\ p = 5x^2 + 17y^2, \\ 2p - 10x^2 \pmod{p^2} & \text{if } (\frac{p}{5}) = 1,\ (\frac{-1}{p}) = (\frac{p}{17}) = -1 \ \&\ 2p = 5x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-85}{p}) = -1. \end{cases}$$

Provided $p > 3$ we have

$$\sum_{k=0}^{p-1}(34k + 31)S_k(-324) \equiv p\left(\frac{p}{5}\right)\left(17 + 14\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(34k + 31)S_k(-324) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Conjecture 3.9. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(1296)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{5})=(\frac{p}{13})=1\ \&\ p=x^2+130y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=1,\ (\frac{p}{5})=(\frac{p}{13})=-1\ \&\ p=2x^2+65y^2,\\ 2p-20x^2 \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{-2}{p})=(\frac{p}{13})=-1\ \&\ p=5x^2+26y^2,\\ 2p-40x^2 \pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{-2}{p})=(\frac{p}{5})=-1\ \&\ p=10x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-130}{p})=-1.\end{cases}$$

Provided $p>3$ we have

$$\sum_{k=0}^{p-1}(130k+121)S_k(1296)\equiv p\left(\frac{-2}{p}\right)\left(56+65\left(\frac{-26}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(130k+121)S_k(1296)\in\mathbb{Z} \qquad \text{for all } n=1,2,3,\ldots.$$

Conjecture 3.10. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(5776)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{5})=(\frac{p}{19})=1\ \&\ p=x^2+190y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{5})=(\frac{p}{19})=-1\ \&\ p=2x^2+95y^2,\\ 2p-20x^2 \pmod{p^2} & \text{if } (\frac{p}{19})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=5x^2+38y^2,\\ 2p-40x^2 \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{19})=-1\ \&\ p=10x^2+19y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-190}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(816k+769)S_k(5776)\equiv p\left(\frac{p}{95}\right)\left(361+408\left(\frac{p}{19}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(816k+769)S_k(5776)\in\mathbb{Z} \qquad \text{for all } n=1,2,3,\ldots.$$

Conjecture 3.11. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(12) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \ \&\ p = x^2 + y^2\ (3 \nmid x), \\ (\frac{xy}{3})4xy \pmod{p^2} & \text{if } p \equiv 5 \pmod{12} \ \&\ p = x^2 + y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(4k+3)S_k(12) \equiv p\left(1 + 2\left(\frac{3}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)S_k(12) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots .$$

Conjecture 3.12. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k(-20)$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \&\ p = x^2 + y^2\ (5 \nmid x), \\ 4xy \pmod{p^2} & \text{if } p \equiv 13, 17 \pmod{20} \ \&\ p = x^2 + y^2\ (5 \mid x - y), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(6k+5)S_k(-20) \equiv p\left(\frac{-1}{p}\right)\left(2 + 3\left(\frac{-5}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(6k+5)S_k(-20) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots .$$

In 2005 Yifan Yang [Y] found the interesting identity

$$\sum_{k=0}^{\infty} \frac{4k+1}{36^k} \sum_{j=0}^{k} \binom{k}{j}^4 = \frac{18}{\pi\sqrt{15}}.$$

Motivated by this we give the following easy result.

Theorem 3.1. *Let $a,b\in\mathbb{Z}$ and $m\in\mathbb{Z}\setminus\{0\}$. Let p be an odd prime not dividing m. Then*

$$\sum_{k=0}^{p-1}\frac{ak+b}{m^k}\sum_{j=0}^{k}\binom{k}{j}^4\equiv-\sum_{k=0}^{p-1}(2ak+2a-b)S_k(m)\pmod p.$$

In particular,

$$\sum_{k=0}^{p-1}\frac{\sum_{j=0}^{k}\binom{k}{j}^4}{m^k}\equiv\sum_{k=0}^{p-1}S_k(m)\pmod p.$$

Proof. Let

$$S:=\sum_{n=0}^{p-1}(2an+2a-b)S_n(m).$$

Then

$$\begin{aligned}S&=\sum_{k=0}^{p-1}m^k\sum_{n=k}^{p-1}(2a(n+1)-b)\binom{n}{k}^4\\&=\sum_{k=0}^{p-1}m^k\sum_{j=0}^{p-1-k}(2a(k+j+1)-b)\binom{k+j}{j}^4\\&=\sum_{k=0}^{p-1}m^k\sum_{j=0}^{p-1-k}(2a(k+j+1)-b)\binom{-k-1}{j}^4\\&\equiv\sum_{k=0}^{p-1}m^k\sum_{j=0}^{p-1-k}(2a(k+j+1)-b)\binom{p-1-k}{j}^4\pmod p\end{aligned}$$

and hence

$$\begin{aligned}S&\equiv\sum_{k=0}^{p-1}m^{p-1-k}\sum_{j=0}^{k}(2a(p-k+j)-b)\binom{k}{j}^4\\&\equiv-\sum_{k=0}^{p-1}\frac{1}{m^k}\sum_{l=0}^{k}(2al+b)\binom{k}{l}^4\pmod p.\end{aligned}$$

Note that

$$2\sum_{l=0}^{k}l\binom{k}{l}^4=\sum_{j=0}^{k}\left(j\binom{k}{j}^4+(k-j)\binom{k}{k-j}^4\right)=k\sum_{j=0}^{k}\binom{k}{j}^4.$$

So we have

$$S\equiv-\sum_{k=0}^{p-1}\frac{ak+b}{m^k}\sum_{j=0}^{k}\binom{k}{j}^4\pmod p.$$

This completes the proof. □

Conjecture 3.13. *Let m be among*

$$1,\ -2,\ \pm 4,\ -9,\ 12,\ 16,\ -20,\ 36,\ -64,\ 196,\ -324,\ 1296,\ 5776.$$

For any odd prime p not dividing m we have

$$\sum_{k=0}^{p-1}\frac{\sum_{j=0}^{k}\binom{k}{j}^4}{m^k}\equiv\sum_{k=0}^{p-1}S_k(m)\pmod{p^2}.$$

Now we give two more conjectures involving $\sum_{k=0}^{n}\binom{n}{k}^3x^k$.

Conjecture 3.14. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1}\frac{\binom{4n}{2n}}{16^n}\sum_{k=0}^{n}\binom{n}{k}^3$$
$$\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } p\equiv 1,3\pmod 8\ \&\ p=x^2+2y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

When $p\equiv 5,7\pmod 8$, we have

$$\sum_{n=0}^{p-1}(16n+5)\frac{\binom{4n}{2n}}{16^n}\sum_{k=0}^{n}\binom{n}{k}^3\equiv 0\pmod{p^2}.$$

Conjecture 3.15. (i) *Let $p>5$ be a prime with $p\neq 11$. Then*

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{(-121)^n}\sum_{k=0}^{n}\binom{n}{k}^3(-5)^{3k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15}\ \&\ p=3x^2+15y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 7,11,13,14\pmod{15}.\end{cases}$$

(ii) *For any prime $p>5$ with $p\neq 41$, we have*

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{4100^n}\sum_{k=0}^{n}\binom{n}{k}^3 2^{12k}$$
$$\equiv\begin{cases}x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{41})=1\ \&\ 4p=x^2+123y^2,\\ 2p-3x^2\pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{41})=-1\ \&\ 4p=3x^2+41y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{123})=-1.\end{cases}$$

(iii) *Let $p>3$ be a prime with $p\neq 53,89$. Then*

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{1000004^n}\sum_{k=0}^{n}\binom{n}{k}^3 10^{6k}$$
$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{89})=1\ \&\ 4p=x^2+267y^2,\\ 2p-3x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=(\frac{p}{89})=-1\ \&\ 4p=3x^2+89y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{267})=-1.\end{cases}$$

4. Using the function $F_n(x)=\sum_{k=0}^n\binom{n}{k}^3\binom{2k}{k}x^{-k}$

For $n=0,1,2,\ldots$ we define

$$F_n(x):=\sum_{k=0}^{n}\binom{n}{k}^3\binom{2k}{k}x^{-k}.$$

Conjecture 4.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}(-1)^kF_k(-2)\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod{12}\ \&\ p=x^2+y^2\ (3\nmid x),\\ (\frac{xy}{3})4xy \pmod{p^2} & \text{if } p\equiv 5 \pmod{12}\ \&\ p=x^2+y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod{4}.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(3k+2)(-1)^kF_k(-2)\equiv\frac{p}{2}\left(\frac{-1}{p}\right)\left(3\left(\frac{p}{3}\right)+1\right)\pmod{p^2}.$$

Conjecture 4.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}(-1)^kF_k(2)\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,9 \pmod{20}\ \&\ p=x^2+5y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } p\equiv 3,7 \pmod{20}\ \&\ 2p=x^2+y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 11,13,17,19 \pmod{20}.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(10k+7)(-1)^kF_k(2)\equiv p\left(5\left(\frac{-1}{p}\right)+2\right)\pmod{p^2}.$$

Conjecture 4.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^kF_k(4)\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,7 \pmod{24}\ \&\ p=x^2+6y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } p\equiv 5,11 \pmod{24}\ \&\ p=2x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 13,17,19,23 \pmod{24}.\end{cases}$$

When $p\equiv 1,5,7,11 \pmod{24}$, we have

$$\sum_{k=0}^{p-1}(72k+47)(-1)^kF_k(4)\equiv 0\pmod{p^2}.$$

Conjecture 4.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k F_k(12) \equiv \begin{cases} (\frac{p}{3})(4x^2-2p) \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8 \ \&\ p=x^2+2y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(5k+3)(-1)^k F_k(12) \equiv \frac{p}{2}\left(\frac{-2}{p}\right)\left(5\left(\frac{p}{3}\right)+1\right) \pmod{p^2}.$$

Conjecture 4.5. *Let $p \neq 2, 7$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k F_k(-14) \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=1 \ \&\ p=x^2+21y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1 \ \&\ 2p=x^2+21y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{7})=-1 \ \&\ p=3x^2+7y^2,\\ 2p-6x^2 \pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1 \ \&\ 2p=3x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-21}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(18k+11)(-1)^k F_k(-14) \equiv p\left(\frac{p}{7}\right)\left(2+9\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

Conjecture 4.6. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k F_k(40) \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1 \ \&\ p=x^2+30y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1 \ \&\ p=2x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1 \ \&\ p=3x^2+10y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1 \ \&\ p=5x^2+6y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(12k+7)(-1)^k F_k(40) \equiv p\left(\frac{10}{p}\right)\left(6+\left(\frac{-6}{p}\right)\right) \pmod{p^2}.$$

Conjecture 4.7. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1}(-1)^kF_k(-50)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{11})=1\ \&\ p=x^2+33y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{11})=-1\ \&\ 2p=x^2+33y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{11})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+11y^2,\\ 2p-6x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{11})=-1\ \&\ 2p=3x^2+11y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-33}{p})=-1.\end{cases}$$

If $p\neq 5$, then

$$\sum_{k=0}^{p-1}(99k+58)(-1)^kF_k(-50)\equiv\frac{p}{2}\left(\frac{-1}{p}\right)\left(99\left(\frac{p}{3}\right)+17\right) \pmod{p^2}.$$

Conjecture 4.8. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1}(-1)^kF_k(112)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+42y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{-2}{p})=(\frac{p}{3})=-1\ \&\ p=2x^2+21y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{-2}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ p=3x^2+14y^2,\\ 2p-24x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-2}{p})=(\frac{p}{7})=-1\ \&\ p=6x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-42}{p})=-1.\end{cases}$$

If $p\neq 7$, then

$$\sum_{k=0}^{p-1}(180k+103)(-1)^kF_k(112)\equiv p\left(\frac{p}{7}\right)\left(90\left(\frac{p}{3}\right)+13\right) \pmod{p^2}.$$

Conjecture 4.9. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1}(-1)^kF_k(-338)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{19})=1\ \&\ p=x^2+57y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{19})=-1\ \&\ 2p=x^2+57y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{19})=-1\ \&\ p=3x^2+19y^2,\\ 2p-6x^2 \pmod{p^2} & \text{if } (\frac{p}{19})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ 2p=3x^2+19y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-57}{p})=-1.\end{cases}$$

If $p \neq 13$, then

$$\sum_{k=0}^{p-1}(855k+482)(-1)^k F_k(-338) \equiv \frac{p}{2}\left(\frac{-1}{p}\right)\left(855\left(\frac{p}{19}\right)+109\right) \pmod{p^2}.$$

Conjecture 4.10. *Let $p > 5$ be a prime with $p \neq 13$. Then*

$$\sum_{k=0}^{p-1}(-1)^k F_k(1300)$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1\ \&\ p=x^2+78y^2, \\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{13})=-1\ \&\ p=2x^2+39y^2, \\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+26y^2, \\ 2p-24x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{13})=-1\ \&\ p=6x^2+13y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-78}{p})=-1. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(204k+113)(-1)^k F_k(1300) \equiv p\left(\frac{p}{39}\right)\left(102\left(\frac{p}{3}\right)+11\right) \pmod{p^2}.$$

Conjecture 4.11. *Let $p > 3$ be a prime with $p \neq 31$. We have*

$$\sum_{k=0}^{p-1}(-1)^k F_k(-3038)$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{31})=1\ \&\ p=x^2+93y^2, \\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{p}{31})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ 2p=x^2+93y^2, \\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{31})=-1\ \&\ p=3x^2+31y^2, \\ 2p-6x^2 \pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{31})=-1\ \&\ 2p=3x^2+31y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-93}{p})=-1. \end{cases}$$

If $p \neq 7$, then

$$\sum_{k=0}^{p-1}(1170k+643)(-1)^k F_k(-3038) \equiv p\left(\frac{p}{31}\right)\left(585\left(\frac{-1}{p}\right)+58\right) \pmod{p^2}.$$

Conjecture 4.12. *Let $p>3$ be a prime. We have*

$$\sum_{k=0}^{p-1}(-1)^kF_k(4900)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p3)=(\frac p{17})=1\ \&\ p=x^2+102y^2,\\ 8x^2-2p\pmod{p^2}&\text{if }(\frac p{17})=1,\ (\frac 2p)=(\frac p3)=-1\ \&\ p=2x^2+51y^2,\\ 2p-12x^2\pmod{p^2}&\text{if }(\frac p3)=1,\ (\frac 2p)=(\frac p{17})=-1\ \&\ p=3x^2+34y^2,\\ 2p-24x^2\pmod{p^2}&\text{if }(\frac 2p)=1,\ (\frac p3)=(\frac p{17})=-1\ \&\ p=6x^2+17y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-102}p)=-1.\end{cases}$$

If $p\neq 7$, then

$$\sum_{k=0}^{p-1}(561k+307)(-1)^kF_k(4900)\equiv\frac p2\left(\frac{-6}p\right)\left(561\left(\frac p{51}\right)+53\right)\pmod{p^2}.$$

5. Using the function $G_n(x)=\sum_{k=0}^n\binom nk^2\binom{2k}k\binom{2n-2k}{n-k}x^{-k}$

For $n\in\mathbb N$ we define

$$G_n(x):=\sum_{k=0}^n\binom nk^2\binom{2k}k\binom{2n-2k}{n-k}x^{-k}.$$

Those integers $G_n(1)$ $(n=0,1,2,\ldots)$ are called Domb numbers. In 2004 H. H. Chan, S. H. Chan and Z. Liu [CCL] proved that

$$\sum_{k=0}^\infty\frac{5k+1}{64^k}G_k(1)=\frac{8}{\sqrt3\,\pi}.$$

Motivated by his work with Mahler measures nad new transformation formulas for ${}_5F_4$ series, M. D. Rogers [Ro] discovered that

$$\sum_{k=0}^\infty\frac{3k+1}{32^k}(-1)^kG_k(1)=\frac2\pi,$$

which was independently showed by H. H. Chan and H. Verrill [CV].

Conjecture 5.1. *Let $p>3$ be a prime. We have*

$$\sum_{k=0}^{p-1}G_k(1)\equiv\sum_{k=0}^{p-1}G_k(-5)\equiv\sum_{k=0}^{p-1}\frac{G_k(1)}{64^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1,4\pmod{15}\ \&\ p=x^2+15y^2,\\ 2p-12x^2\pmod{p^2}&\text{if }p\equiv2,8\pmod{15}\ \&\ p=3x^2+5y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-15}p)=-1.\end{cases}$$

If $p \neq 5$, *then*

$$\sum_{k=0}^{p-1}(5k+4)G_k(1) \equiv 4p\left(\frac{p}{3}\right) \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}(21k+16)G_k(-5) \equiv 16p\left(\frac{p}{5}\right) \pmod{p^2}.$$

Also,

$$\frac{1}{4n}\sum_{k=0}^{n-1}(5k+4)G_k(1) \in \mathbb{Z} \qquad \text{for all } n = 1,2,3,\ldots.$$

Conjecture 5.2. *Let* $p > 3$ *be a prime. We have*

$$\sum_{k=0}^{p-1}\frac{G_k(1)}{(-2)^k} \equiv \sum_{k=0}^{p-1}\frac{G_k(1)}{4^k} \equiv \sum_{k=0}^{p-1}\frac{G_k(1)}{16^k} \equiv \sum_{k=0}^{p-1}\frac{G_k(1)}{(-32)^k} \equiv \left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}G_k(6)$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \ \&\ p = x^2 + 3y^2\ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Also,

$$\begin{aligned} \sum_{k=0}^{p-1}(3k+2)\frac{G_k(1)}{(-2)^k} &\equiv 2p\left(\frac{-1}{p}\right) \pmod{p^3}, \\ \sum_{k=0}^{p-1}(3k+1)\frac{G_k(1)}{(-32)^k} &\equiv p\left(\frac{-1}{p}\right) + p^3E_{p-3} \pmod{p^4}, \\ \sum_{k=0}^{p-1}(15k+11)G_k(6) &\equiv p\left(10\left(\frac{3}{p}\right)+1\right) \pmod{p^2}. \end{aligned}$$

If $p \equiv 1 \pmod 3$, *then*

$$\sum_{k=0}^{p-1}(3k+2)\frac{G_k(1)}{4^k} \equiv \sum_{k=0}^{p-1}(3k+1)\frac{G_k(1)}{16^k} \equiv 0 \pmod{p^2}.$$

Conjecture 5.3. *Let* $p > 3$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{G_k(1)}{8^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1,3 \pmod 8 \ \&\ p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{G_k(1)}{(-8)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1,7\pmod{24}\ \&\ p=x^2+6y^2,\\8x^2-2p\pmod{p^2}&\text{if }p\equiv5,11\pmod{24}\ \&\ p=2x^2+3y^2,\\0\pmod{p^2}&\text{if }(\frac{-6}{p})=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(2k+1)\frac{G_k(1)}{8^k}\equiv p\pmod{p^4}\quad\text{and}\quad\sum_{k=0}^{p-1}(2k+1)\frac{G_k(1)}{(-8)^k}\equiv p\left(\frac{p}{3}\right)\pmod{p^3}.$$

Conjecture 5.4. *Let p be an odd prime. We have*

$$\begin{aligned}\sum_{k=0}^{p-1}G_k(4)&\equiv\sum_{k=0}^{p-1}(-1)^kF_k(4)\pmod{p^2},\\\sum_{k=0}^{p-1}G_k(-12)&\equiv\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}(-1)^kF_k(-14)\pmod{p^2}\quad(p\neq3,7),\\\sum_{k=0}^{p-1}G_k(36)&\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}(-1)^kF_k(40)\pmod{p^2}\quad(p>5),\\\sum_{k=0}^{P-1}G_k(-44)&\equiv\sum_{k=0}^{p-1}(-1)^kF_k(-50)\pmod{p^2}\quad(p\neq5,11),\\\sum_{k=0}^{p-1}G_k(100)&\equiv\sum_{k=0}^{p-1}(-1)^kF_k(112)\pmod{p^2}\quad(p\neq5,7),\\\sum_{k=0}^{p-1}G_k(-300)&\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}(-1)^kF_k(-338)\pmod{p^2}\quad(p\neq3,5,13),\\\sum_{k=0}^{p-1}G_k(1156)&\equiv\sum_{k=0}^{p-1}(-1)^kF_k(1300)\pmod{p^2}\quad(p\neq5,13,17),\\\sum_{k=0}^{p-1}G_k(-2700)&\equiv\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}(-1)^kF_k(-3038)\pmod{p^2}\quad(p>7,\ p\neq31),\\\sum_{k=0}^{p-1}G_k(4356)&\equiv\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1}(-1)^kF_k(4900)\pmod{p^2}\quad(p>11).\end{aligned}$$

6. Using $a_n(x) = \sum_{k=0}^n \binom{n}{k}^2\binom{n+k}{k}x^k$

For $n \in \mathbb{N}$ define

$$a_n(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} x^k.$$

Those numbers $a_n(1)$ $(n = 0, 1, 2, \dots)$ first appeared in Apéry's proof of the irrationality of $\zeta(2)$ (see [Ap] and [P]). We observe the new identity

$$\binom{2n}{n} a_n(1) = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n}\binom{n+2k}{n}. \tag{6.1}$$

If we set $u_n = \sum_{k=0}^n \binom{n}{k}^2\binom{2k}{n}\binom{n+2k}{n}/\binom{2n}{n}$ or $u_n = a_n(1)$ for $n \in \mathbb{N}$, then $u_0 = 1$ and $u_1 = 3$, and by applying the Zeilberger algorithm (cf. [PWZ]) via `Mathematica` (version 7) we get the recurrence relation

$$(n+2)^2 u_{n+2} = (11n^2 + 33n + 25)u_{n+1} + (n+1)^2 u_n \quad (n = 0, 1, 2, \dots).$$

Thus (6.1) holds by induction.

We find $\sum_{k=0}^{p-1}\binom{2k}{k}a_k(1)/m^k$ mod p^2 related to the representation $4p = x^2 + dy^2$ with

$$(m, d) = (-3, 15), (4, 11), (18, 1), (-28, 35), (36, 19), (72, 10), (147, 15), (-828, 115), (-15228, 235).$$

Conjecture 6.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}a_k(1)}{4^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{11}) = 1 \ \&\ 4p = x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} \frac{22k+9}{4^k}\binom{2k}{k} a_k(1) \equiv 9p \pmod{p^2}.$$

Conjecture 6.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}a_k(1)}{36^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{19}) = 1 \ \&\ 4p = x^2 + 19y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{19}) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1} \frac{38k+13}{36^k}\binom{2k}{k} a_k(1) \equiv 13p \pmod{p^2}.$$

Conjecture 6.3. *Let $p \ne 2,7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k(1)}{(-28)^k}\equiv\begin{cases}x^2-2p\pmod{p^2}&\text{if }(\frac p5)=(\frac p7)=1\ \&\ 4p=x^2+35y^2,\\2p-5x^2\pmod{p^2}&\text{if }(\frac p5)=(\frac p7)=-1\ \&\ 4p=5x^2+7y^2,\\0\pmod{p^2}&\text{if }(\frac p{35})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{10k+3}{(-28)^k}\binom{2k}{k}a_k(1)\equiv 3p\left(\frac p7\right)\pmod{p^2}.$$

Conjecture 6.4. *Let $p \ne 2,3,23$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k(1)}{(-828)^k}\equiv\begin{cases}x^2-2p\pmod{p^2}&\text{if }(\frac p5)=(\frac p{23})=1\ \&\ 4p=x^2+115y^2,\\2p-5x^2\pmod{p^2}&\text{if }(\frac p5)=(\frac p{23})=-1\ \&\ 4p=5x^2+23y^2,\\0\pmod{p^2}&\text{if }(\frac p{115})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{190k+29}{(-828)^k}\binom{2k}{k}a_k(1)\equiv 29p\left(\frac p{23}\right)\pmod{p^2}.$$

Conjecture 6.5. *Let $p \ne 5,47$ be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k(1)}{(-15228)^k}\equiv\begin{cases}x^2-2p\pmod{p^2}&\text{if }(\frac p5)=(\frac p{47})=1\ \&\ 4p=x^2+235y^2,\\2p-5x^2\pmod{p^2}&\text{if }(\frac p5)=(\frac p{47})=-1\ \&\ 4p=5x^2+47y^2,\\0\pmod{p^2}&\text{if }(\frac p{235})=-1.\end{cases}$$

When $p \ne 3$ we have

$$\sum_{k=0}^{p-1}\frac{682k+71}{(-15228)^k}\binom{2k}{k}a_k(1)\equiv 71p\left(\frac p{47}\right)\pmod{p^2}.$$

Conjecture 6.6. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k(-3)}{4^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k(9)}{100^k}$$
$$\equiv\begin{cases}x^2-2p\pmod{p^2}&\text{if }p\equiv1\pmod 3\ \&\ 4p=x^2+27y^2,\\0\pmod{p^2}&\text{if }p\equiv 2\pmod 3.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(30k+13)\frac{\binom{2k}{k}a_k(-3)}{4^k}\equiv p\left(3+10\left(\frac p3\right)\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(66k+23)\frac{\binom{2k}{k}a_k(9)}{100^k}\equiv 23p\pmod{p^2}.$$

Conjecture 6.7. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k(-27)}{26^{2k}} \equiv \begin{cases} (\frac{p}{3})(x^2-2p) \pmod{p^2} & \text{if } (\frac{p}{11}) = 1\ \&\ 4p = x^2+11y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1. \end{cases}$$

When $p \neq 7$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k(27)}{28^{2k}} \equiv \begin{cases} (\frac{p}{3})(4x^2-2p) \pmod{p^2} & \text{if } p \equiv 1,3 \pmod 8\ \&\ p = x^2+2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Also,

$$\sum_{k=0}^{\infty}(114k+31)\frac{\binom{2k}{k}a_k(-27)}{26^{2k}} = \frac{338\sqrt{3}}{11\pi} \quad \text{and} \quad \sum_{k=0}^{\infty}(930k+143)\frac{\binom{2k}{k}a_k(27)}{28^{2k}} = \frac{980\sqrt{3}}{\pi}.$$

For $n \in \mathbb{N}$ we define

$$a_n^*(x) := \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}x^{n-k} = x^n a_n\left(\frac{1}{x}\right).$$

Conjecture 6.8. *Let $p > 3$ be a prime. Then*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k^*(-8)}{96^k}$$
$$\equiv \begin{cases} 4x^2-2p & \text{if } p \equiv 1,7 \pmod{24}\ \&\ p = x^2+6y^2, \\ 8x^2-2p \pmod{p^2} & \text{if } p \equiv 5,11 \pmod{24}\ \&\ p = 2x^2+3y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p}) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(13k+4)\frac{\binom{2k}{k}a_k^*(-8)}{96^k} \equiv 4p\left(\frac{-2}{p}\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{\infty}(13k+4)\frac{\binom{2k}{k}a_k^*(-8)}{96^k} = \frac{9\sqrt{2}}{2\pi}.$$

Conjecture 6.9. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k^*(-32)}{1152^k} \equiv \begin{cases} (\frac{2}{p})(4x^2-2p) & \text{if } p \equiv 1 \pmod 3\ \&\ p = x^2+3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(290k+61)\frac{\binom{2k}{k}a_k^*(-32)}{1152^k} \equiv p\left(\frac{2}{p}\right)\left(6+55\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{\infty}(290k+61)\frac{\binom{2k}{k}a_k^*(-32)}{1152^k} = \frac{99\sqrt{2}}{\pi}.$$

Conjecture 6.10. *Let $p>5$ be a prime. Then*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k^*(64)}{3840^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15}\ \&\ p=3x^2+5y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{15})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(962k+137)\frac{\binom{2k}{k}a_k^*(64)}{3840^k}\equiv p\left(\frac{-5}{p}\right)\left(147-10\left(\frac{p}{3}\right)\right)\pmod{p^2}.$$

We also have

$$\sum_{k=0}^{\infty}(962k+137)\frac{\binom{2k}{k}a_k^*(64)}{3840^k}=\frac{252\sqrt5}{\pi}.$$

7. Miscellaneous things

In this section we give some miscellaneous conjectures related to Problem 1.2.

Conjecture 7.1. *Let $p>3$ be a prime. Then*

$$\sum_{n=0}^{p-1}\frac{1}{9^n}\sum_{k=0}^{n}\binom{-1/3}{k}^2\binom{-2/3}{n-k}^2$$
$$\equiv\begin{cases}x^2-2p\pmod{p^2} & \text{if } p\equiv 1\pmod 3\ \&\ 4p=x^2+27y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

And

$$\sum_{n=0}^{p-1}\frac{8n+1}{9^n}\sum_{k=0}^{n}\binom{-1/3}{k}^2\binom{-2/3}{n-k}^2\equiv p\left(\frac{p}{3}\right)\pmod{p^3}.$$

We also have

$$\sum_{n=0}^{\infty}\frac{8n+1}{9^n}\sum_{k=0}^{n}\binom{-1/3}{k}^2\binom{-2/3}{n-k}^2=\frac{3\sqrt3}{\pi}.$$

Conjecture 7.2. *Let $p>3$ be a prime. Then*

$$\sum_{n=0}^{p-1}\frac{1}{2^n}\sum_{k=0}^{n}\binom{-1/3}{k}\binom{-2/3}{n-k}\binom{-1/6}{k}\binom{-5/6}{n-k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24}\ \&\ p=x^2+6y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24}\ \&\ p=2x^2+3y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1.\end{cases}$$

And

$$\sum_{n=0}^{p-1}\frac{3n-1}{2^n}\sum_{k=0}^{n}\binom{-1/3}{k}\binom{-2/3}{n-k}\binom{-1/6}{k}\binom{-5/6}{n-k}\equiv -p\left(\frac{-6}{p}\right)\pmod{p^2}.$$

We also have

$$\sum_{n=0}^{\infty}\frac{3n-1}{2^n}\sum_{k=0}^{n}\binom{-1/3}{k}\binom{-2/3}{n-k}\binom{-1/6}{k}\binom{-5/6}{n-k}=\frac{3\sqrt{6}}{2\pi}.$$

Conjecture 7.3. *Let p be an odd prime. Then*

$$\left(\frac{-1}{p}\right)\sum_{n=0}^{p-1}\frac{1}{64^n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{n}\binom{2k}{k}\binom{2n-2k}{n-k}3^{2k-n}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15})=-1.\end{cases}$$

And

$$\sum_{n=0}^{p-1}\frac{21n+1}{64^n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{n}\binom{2k}{k}\binom{2n-2k}{n-k}3^{2k-n}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(4-3\left(\frac{p}{3}\right)\right)\pmod{p^2}.$$

We also have

$$\sum_{n=0}^{\infty}\frac{21n+1}{64^n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{n}\binom{2k}{k}\binom{2n-2k}{n-k}3^{2k-n}=\frac{64}{\pi}.$$

Conjecture 7.4. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1}\frac{1}{(-64)^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}$$
$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=1\ \&\ 4p=x^2+91y^2,\\ 2p-7x^2 \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=-1\ \&\ 4p=7x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{91})=-1.\end{cases}$$

And

$$\sum_{n=0}^{p-1}(39n+10)\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}\equiv 10p\left(\frac{p}{7}\right)\pmod{p^2}.$$

Conjecture 7.5. *Let $p>3$ be a prime. Then*

$$\sum_{n=0}^{p-1}\frac{1}{216^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}14^{2k-n}$$
$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{67})=1 \ \&\ 4p=x^2+67y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{67})=-1.\end{cases}$$

And

$$\sum_{n=0}^{p-1}(33165n+11546)\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}14^{2k-n}\equiv 11546p \pmod{p^2}.$$

Conjecture 7.6. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1}\frac{1}{(-8)^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}26^{2k-n}$$
$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{23})=1 \ \&\ 4p=x^2+115y^2,\\ 5x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{23})=-1 \ \&\ 4p=5x^2+23y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{115})=-1.\end{cases}$$

And

$$\sum_{n=0}^{p-1}\frac{6555n+3062}{(-8)^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}26^{2k-n}$$
$$\equiv 2p\left(220+1311\left(\frac{p}{23}\right)\right) \pmod{p^2}.$$

Conjecture 7.7. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1}\frac{1}{(-8)^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{3k}{n}\binom{3(n-k)}{n}$$
$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=1 \ \&\ 4p=x^2+35y^2,\\ 2p-5x^2 \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=-1 \ \&\ 4p=5x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{35})=-1.\end{cases}$$

And

$$\sum_{n=0}^{p-1}\frac{35n+18}{(-8)^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{3k}{n}\binom{3(n-k)}{n}\equiv 9p\left(7-5\left(\frac{p}{5}\right)\right) \pmod{p^2}.$$

Conjecture 7.8. *Let $p > 3$ be a prime. When $p \neq 11, 17$, we have*

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{198^{2n}} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{3}) = (\frac{p}{17}) = 1 \ \&\ p = x^2 + 102y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } (\frac{p}{17}) = 1,\ (\frac{2}{p}) = (\frac{p}{3}) = -1 \ \&\ p = 2x^2 + 51y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{3}) = 1,\ (\frac{2}{p}) = (\frac{p}{17}) = -1 \ \&\ p = 3x^2 + 34y^2, \\ 2p - 24x^2 \pmod{p^2} & \text{if } (\frac{2}{p}) = 1,\ (\frac{p}{3}) = (\frac{p}{17}) = -1 \ \&\ p = 6x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-102}{p}) = -1. \end{cases}$$

When $p \neq 23, 59$, we have

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{(-1123596)^n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{-1}{p}) = (\frac{p}{3}) = (\frac{p}{59}) = 1 \ \&\ p = x^2 + 177y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } (\frac{-1}{p}) = 1,\ (\frac{p}{3}) = (\frac{p}{59}) = -1 \ \&\ 2p = x^2 + 177y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{59}) = 1,\ (\frac{-1}{p}) = (\frac{p}{3}) = -1 \ \&\ p = 3x^2 + 59y^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } (\frac{p}{3}) = 1,\ (\frac{-1}{p}) = (\frac{p}{59}) = -1 \ \&\ 2p = 3x^2 + 59y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-177}{p}) = -1. \end{cases}$$

Conjecture 7.9. *Let $p > 5$ be a prime. Then*

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{324^n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (-20)^k$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \&\ p = x^2 + y^2\ (5 \nmid x), \\ 4xy \pmod{p^2} & \text{if } p \equiv 13, 17 \pmod{20} \ \&\ p = x^2 + y^2\ (5 \mid x - y), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

And

$$\sum_{n=0}^{p-1} \frac{16n+5}{324^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (-20)^k \equiv \frac{p}{25} \left(112 \left(\frac{-1}{p}\right) + 13\right) \pmod{p^2}.$$

We also have

$$\sum_{n=0}^{\infty} \frac{16n+5}{324^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (-20)^k = \frac{189}{25\pi}.$$

Conjecture 7.10. *Let $p>5$ be a prime. Then*

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{400^n}\sum_{k=0}^n\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}196^{n-k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=1\ \&\ p=x^2+22y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=-1\ \&\ p=2x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-22}{p})=-1.\end{cases}$$

And

$$\sum_{n=0}^{p-1}\frac{33n+19}{400^n}\binom{2n}{n}\sum_{k=0}^n\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}196^{n-k}\equiv 19p\pmod{p^2}.$$

For $n\in\mathbb{N}$ we define

$$f_n^-(x):=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{n}(-1)^kx^{2k-n}.$$

Conjecture 7.11. *Let $p>5$ be a prime. Then*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k^-(8)}{480^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,9\pmod{20}\ \&\ p=x^2+5y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } p\equiv 3,7\pmod{20}\ \&\ 2p=x^2+5y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 11,13,17,19\pmod{20},\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k^-(18)}{5760^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=(\frac{p}{17})=1\ \&\ p=x^2+85y^2,\\ 2p-2x^2\pmod{p^2} & \text{if } (\frac{p}{17})=1,\ (\frac{-1}{p})=(\frac{p}{5})=-1\ \&\ 2p=x^2+85y^2,\\ 20x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{5})=(\frac{p}{17})=-1\ \&\ p=5x^2+17y^2,\\ 2p-10x^2\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{-1}{p})=(\frac{p}{17})=-1\ \&\ 2p=5x^2+17y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-85}{p})=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}\frac{1054k+233}{480^k}\binom{2k}{k}f_k^-(8)\equiv p\left(\frac{-1}{p}\right)\left(221+12\left(\frac{p}{15}\right)\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{224434k+32849}{5760^k}\binom{2k}{k}f_k^-(18)\equiv p\left(32305\left(\frac{-1}{p}\right)+544\left(\frac{p}{5}\right)\right)\pmod{p^2}.$$

We also have

$$\sum_{k=0}^{\infty}\frac{1054k+233}{480^k}\binom{2k}{k}f_k^-(8)=\frac{520}{\pi}$$

and

$$\sum_{k=0}^{\infty}\frac{224434k+32849}{5760^k}\binom{2k}{k}f_k^-(18)=\frac{93600}{\pi}.$$

For $n = 0, 1, 2, \ldots$, we define

$$f_n^+(x) := \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}x^{2k-n}.$$

Recall that $f_n^+(1) = \sum_{k=0}^n \binom{n}{k}^3$ by an identity of V. Strehl (see also D. Zagier [Z]).

Conjecture 7.12. (i) *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k^+(9)}{144^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{13})=1\ \&\ p=x^2+13y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{13})=-1\ \&\ 2p=x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-13}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{65k+22}{144^k}\binom{2k}{k}f_k^+(9)\equiv 22p\pmod{p^2}.$$

(ii) *Let $p \neq 5$ be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(13,13)}{1300^k}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{13}{p})=1\ \&\ p=x^2+13y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{13}{p})=-1\ \&\ 2p=x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-13}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{312k+91}{1300^k}\binom{2k}{k}T_k^2(13,13)\equiv 91p\pmod{p^2}.$$

Conjecture 7.13. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k^+(-7)}{16^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(3,-3)}{(-108)^k}\equiv\left(\frac{p}{3}\right)\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{(-108)^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+21y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=1\ \&\ p=3x^2+7y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{7})=1\ \&\ 2p=x^2+21y^2,\\ 6x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1,\ \&\ 2p=3x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-21}{p})=-1.\end{cases}$$

We also have

$$\sum_{k=0}^{p-1}\frac{63k+26}{16^k}\binom{2k}{k}f_k^+(-7)\equiv p\left(5+21\left(\frac{p}{7}\right)\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{56k+19}{(-108)^k}\binom{2k}{k}T_k^2(3,-3)\equiv\frac{p}{2}\left(21\left(\frac{p}{7}\right)+17\right)\pmod{p^2}.$$

Conjecture 7.14. *Let $p\neq 43$ be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k^+(175)}{(-29584)^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{7})=(\frac{p}{19})=1\ \&\ p=x^2+133y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{-1}{p})=(\frac{p}{19})=-1\ \&\ 2p=x^2+133y^2,\\ 2p-28x^2\pmod{p^2} & \text{if } (\frac{p}{19})=1,\ (\frac{-1}{p})=(\frac{p}{7})=-1\ \&\ p=7x^2+19y^2,\\ 2p-14x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{7})=(\frac{p}{19})=-1\ \&\ 2p=7x^2+19y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-133}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{8851815k+1356374}{(-29584)^k}\binom{2k}{k}f_k^+(175)\equiv p\left(1300495\left(\frac{p}{7}\right)+55879\right)\pmod{p^2}.$$

Moreover,

$$\sum_{k=0}^{\infty}\frac{8851815k+1356374}{(-29584)^k}\binom{2k}{k}f_k^+(175)=\frac{1349770\sqrt{7}}{\pi}.$$

We also find $\sum_{k=0}^{p-1}\binom{2k}{k}f_k^+(x)/m^k$ mod p^2 (with $p > 3$ a prime not dividing $m \in \mathbb{Z}\setminus\{0\}$) related to the representation $4p = x^2 + dy^2$ with

$$\begin{aligned}(m,x,d) =&(2,16,6),(4,-1,7),(4,80,10),(5,64,10),(6,16,2),(6,240,30),\\&(7,-16,21),(14,289,7),(14,2800,70),(15,96,30),(20,16,30),\\&(21,576,42),(24,400,2),(25,-16,33),(25,384,2),\\&(36,336,42),(45,2304,70),(49,800,22),(50,784,22),\\&(56,16,42),(144,720,70),(169,-16,57),(441,7056,37).\end{aligned}$$

Actually we have many other conjectural congruences and series for $1/\pi$, which cannot be listed here due to the limitation of the length of this survey.

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