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ON A SEQUENCE INVOLVING SUMS OF PRIMES

Zhi-Wei Sun

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

Abstract. For $n = 1, 2, 3, \ldots$ let S_n be the sum of the first n primes. We mainly show that the sequence $a_n = \sqrt[n]{S_n/n}$ $(n = 1, 2, 3, \ldots)$ is strictly decreasing, and moreover the sequence a_{n+1}/a_n $(n = 10, 11, \ldots)$ is strictly increasing. We also formulate similar conjectures involving twin primes or partitions of integers.

1. Introduction

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ let p_n denote the nth prime. The unsolved Firoozbakht conjecture (cf. [R, p. 185]) asserts that

$$\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}} \quad \text{for all } n \in \mathbb{Z}^+,$$

i.e., the sequence $(\sqrt[n]{p_n})_{n\geqslant 1}$ is strictly decreasing. This implies the inequality $p_{n+1}-p_n < \log^2 p_n - \log p_n + 1$ for large n, which is even stronger than Cramér's conjecture $p_{n+1} - p_n = O(\log^2 p_n)$. Let P_n be the product of the first n primes. Then $P_n < p_{n+1}^n$ and hence $P_n^{n+1} < P_{n+1}^n$. So the sequence $(\sqrt[n]{P_n})_{n\geqslant 1}$ is strictly increasing.

Now let us look at a simple example not related to primes.

Example 1.1. Let $a_n = \sqrt[n]{n}$ for $n \in \mathbb{Z}^+$. Then the sequence $(a_n)_{n\geqslant 3}$ is strictly decreasing, and the sequence $(a_{n+1}/a_n)_{n\geqslant 4}$ is strictly increasing. To see this we investigate the function $f(x) = \log(x^{1/x}) = (\log x)/x$ with $x \geqslant 3$. As $f'(x) = (1 - \log x)/x^2 < 0$, we have $f(n) > f(n + 1)$ for $n = 3, 4, \ldots$. Since

$$f''(x) = \frac{2\log x - 3}{x^3} > 0 \quad \text{for } x \geqslant 4.5,$$

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the function $f(x)$ is strictly convex over the interval $(4.5, +\infty)$ and so

$$2f(n+1) < f(n) + f(n+2) \text{ (i.e., } a_{n+1}^2 < a_n a_{n+2}) \quad \text{for } n = 5, 6, \ldots .$$

The inequality $a_5^2 < a_4 a_6$ can be verified directly.

A sequence $(a_n)_{n\geqslant 1}$ of nonnegative real numbers is said to be *log-convex* if $a_{n+1}^2 \leqslant a_n a_{n+2}$ for all $n = 1, 2, 3, \ldots$. Many combinatorial sequences (such as the sequence of the Catalan numbers) are log-convex, the reader may consult [LW] for some results on log-convex sequences.

For $n \in \mathbb{Z}^+$ let $S_n = \sum_{k=1}^n p_k$ be the sum of the first n primes. For instance,

$$S_1 = 2,\ S_2 = 2 + 3 = 5,\ S_3 = 2 + 3 + 5 = 10,\ S_4 = 2 + 3 + 5 + 7 = 17.$$

Recently the author [S] conjectured that for any positive integer n the interval (S_n, S_{n+1}) contains a prime. As $S_n < np_{n+1}$ for all $n \in \mathbb{Z}^+$, the sequence $(S_n/n)_{n\geqslant 1}$ is strictly increasing.

In the next section we will state our theorems involving the sequence $(a_n)_{n\geqslant 1}$ with $a_n = \sqrt[n]{S_n/n}$, and pose three related conjectures for further research. Section 3 is devoted to our proofs of the theorems.

2. Our results and conjectures

Theorem 2.1. *The sequences* $(\sqrt[n]{S_n})_{n\geqslant 2}$ *and* $(\sqrt[n]{S_n/n})_{n\geqslant 1}$ *are strictly decreasing.*

Remark 2.2. Note that S_n/n is just the arithmetic mean of the first n primes. It is interesting to compare Theorem 2.1 with Firoozbakht's conjecture that $(\sqrt[n]{p_n})_{n\geqslant 1}$ is strictly decreasing.

For $\alpha > 0$ and $n \in \mathbb{Z}^+$ define

$$S_n^{(\alpha)} = \sum_{k=1}^n p_k^\alpha.$$

We actually obtain the following extension of Theorem 2.1.

Theorem 2.3. *Let* $\alpha \geqslant 1$ *and* $n \in \mathbb{Z}^+$ *with* $n \geqslant \max\{100, e^{2\times 1.348^\alpha + 1}\}$. *Then*

$$\sqrt[n]{\frac{S_n^{(\alpha)}}{n}} > \sqrt[n+1]{\frac{S_{n+1}^{(\alpha)}}{n+1}} \tag{2.1}$$

and hence

$$\sqrt[n]{S_n^{(\alpha)}} > \sqrt[n+1]{S_{n+1}^{(\alpha)}}. \tag{2.2}$$

Remark 2.4. In view of Example 1.1, (2.1) implies (2.2) if $n \geqslant 3$. We conjecture that (2.1) holds for any $\alpha > 0$ and $n \in \mathbb{Z}^+$.

Note that $\lfloor e^{2\times 1.348+1} \rfloor = 40$ and we can easily verify that

$$\sqrt[n]{\frac{S_n}{n}} > \sqrt[n+1]{\frac{S_{n+1}}{n+1}} \quad \text{for every } n = 1, \ldots, 99.$$

So Theorem 2.1 follows from Theorem 2.3 in the case $\alpha = 1$.

Corollary 2.5. *For each $\alpha \in \{2,3,4\}$, the sequences*

$$\left(\sqrt[n]{\frac{S_n^{(\alpha)}}{n}}\right)_{n\geqslant 1} \quad \textit{and} \quad \left(\sqrt[n]{S_n^{(\alpha)}}\right)_{n\geqslant 1}$$

are strictly decreasing.

Proof. Observe that

$$\lfloor e^{2\times 1.348^2+1}\rfloor = 102,\ \lfloor e^{2\times 1.348^3+1}\rfloor = 364,\ \lfloor e^{2\times 1.348^4+1}\rfloor = 2005.$$

In light of Theorem 2.3 and Example 1.1, it suffices to verify that

$$\sqrt[n]{\frac{S_n^{(\alpha)}}{n}} > \sqrt[n+1]{\frac{S_{n+1}^{(2)}}{n+1}}$$

whenever $\alpha \in \{2,3,4\}$ and $n \in \{1,\ldots,\lfloor e^{2\times 1.348^\alpha+1}\rfloor\}$. These can be easily done via computer. $\square$

Our following theorem is more sophisticated than Theorem 2.3.

Theorem 2.6. *Let $\alpha \geqslant 1$. Then the sequence*

$$\left(\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)} \Big/ \sqrt[n]{S_n^{(\alpha)}/n}\right)_{n\geqslant N(\alpha)}$$

is strictly increasing, where

$$N(\alpha) = \max\left\{350000,\ \lceil e^{((\alpha+1)^2 1.2^{2\alpha+1}+(\alpha+1)1.2^{\alpha+1})/\alpha}\rceil\right\}. \tag{2.3}$$

Corollary 2.7. *All the sequences*

$$\left(\sqrt[n+1]{S_{n+1}/(n+1)}/\sqrt[n]{S_n/n}\right)_{n\geqslant 10},\ \left(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n}\right)_{n\geqslant 5},$$
$$\left(\sqrt[n+1]{S_{n+1}^{(2)}/(n+1)}/\sqrt[n]{S_n^{(2)}/n}\right)_{n\geqslant 13},\ \left(\sqrt[n+1]{S_{n+1}^{(2)}}/\sqrt[n]{S_n^{(2)}}\right)_{n\geqslant 10},$$
$$\left(\sqrt[n+1]{S_{n+1}^{(3)}/(n+1)}/\sqrt[n]{S_n^{(3)}/n}\right)_{n\geqslant 17},\ \left(\sqrt[n+1]{S_{n+1}^{(3)}}/\sqrt[n]{S_n^{(3)}}\right)_{n\geqslant 10},$$
$$\left(\sqrt[n+1]{S_{n+1}^{(4)}/(n+1)}/\sqrt[n]{S_n^{(4)}/n}\right)_{n\geqslant 35},\ \left(\sqrt[n+1]{S_{n+1}^{(4)}}/\sqrt[n]{S_n^{(4)}}\right)_{n\geqslant 17}$$

are strictly increasing.

Proof. For $N(\alpha)$ given by (2.3), via computation we find that

$$N(1) = 350000,\ N(2) = 974267,\ N(3) = 3163983273$$

and

$$N(4) = 2271069361863763.$$

Via computer we can verify that

$$\frac{\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)}}{\sqrt[n]{S_n^{(\alpha)}/n}} < \frac{\sqrt[n+2]{S_{n+2}^{(\alpha)}/(n+2)}}{\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)}}$$

for all $\alpha \in \{1, 2, 3, 4\}$ and $n = N_0(\alpha), \ldots, N(\alpha) - 1$, where

$$N_0(1) = 10,\ N_0(2) = 13,\ N_0(3) = 17,\ N_0(4) = 35.$$

Combining this with Theorem 2.3 we obtain that

$$\left(\sqrt[n+1]{S_{n+1}/(n+1)}/\sqrt[n]{S_n/n}\right)_{n\geqslant N_0(\alpha)}$$

is strictly increasing for each $\alpha = 1, 2, 3, 4$. Recall that $(\sqrt[n+1]{n+1}/\sqrt[n]{n})_{n\geqslant 4}$ is strictly increasing by Example 1.1. So $\left(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n}\right)_{n\geqslant N_0(\alpha)}$ is strictly increasing for any $\alpha \in \{1, 2, 3, 4\}$. It remains to check that

$$\frac{\sqrt[n+1]{S_{n+1}^{(\alpha)}}}{\sqrt[n]{S_n^{(\alpha)}}} < \frac{\sqrt[n+2]{S_{n+2}^{(\alpha)}}}{\sqrt[n+1]{S_{n+1}^{(\alpha)}}}$$

for all $\alpha \in \{1, 2, 3, 4\}$ and $n = n_0(\alpha), \ldots, N_0(\alpha) - 1$, where $n_0(1) = 5$, $n_0(2) = n_0(3) = 10$, and $n_0(4) = 17$. This can be easily done via computer. $\square$

We conclude this section by posing three conjectures.

Conjecture 2.8. *The two constants*

$$s_1 = \sum_{n=1}^{\infty} \frac{1}{S_n} \quad \text{and} \quad s_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{S_n}$$

are both transcendental numbers.

Remark 2.9. Our computation shows that $s_1 \approx 1.023476$ and $s_2 \approx -0.3624545778$.

If p and $p+2$ are both primes, then they are called twin primes. The famous twin prime conjecture states that there are infinitely many twin primes.

Conjecture 2.10. (i) *If $\{t_1, t_1+2\}, \ldots, \{t_n, t_n+2\}$ are the first n pairs of twin primes, then the first prime t_{n+1} in the next pair of twin primes is smaller than $t_n^{1+1/n}$, i.e., $\sqrt[n]{t_n} > \sqrt[n+1]{t_{n+1}}$.*

(ii) *The sequence $(\sqrt[n+1]{T_{n+1}}/\sqrt[n]{T_n})_{n\geqslant 9}$ is strictly increasing with limit 1, where $T_n = \sum_{k=1}^n t_k$.*

Remark 2.11. Via `Mathematica` the author has verified that $\sqrt[n]{t_n} > \sqrt[n+1]{t_{n+1}}$ for all $n = 1, \ldots, 500000$, and $\sqrt[n+1]{T_{n+1}}/\sqrt[n]{T_n} < \sqrt[n+2]{T_{n+2}}/\sqrt[n+1]{T_{n+1}}$ for all $n = 9, \ldots, 500000$. Note that $t_{500000} = 115438667$.

Recall that a partition of a positive integer n is a way of writing n as a sum of positive integers with the order of addends ignored. Also, a *strict partition* of $n \in \mathbb{Z}^+$ is a way of writing n as a sum of *distinct* positive integers with the order of addends ignored. For $n = 1, 2, 3, \ldots$ we denote by $p(n)$ and $p_*(n)$ the number of partitions of n and the number of strict partitions of n respectively. It is known that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n} \quad \text{and} \quad p_*(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4(3n^3)^{1/4}} \qquad \text{as } n \to +\infty$$

(cf. [HR] and [AS, p. 826]) and hence $\lim_{n\to\infty} \sqrt[n]{p(n)} = \lim_{n\to\infty} \sqrt[n]{p_*(n)} = 1$. Here we formulate a conjecture similar to Conjecture 2.10.

Conjecture 2.12. *For $n \in \mathbb{Z}^+$ let*

$$q(n) = \frac{p(n)}{n}, \ q_*(n) = \frac{p_*(n)}{n}, \ r(n) = \sqrt[n]{q(n)}, \ \textit{and } r_*(n) = \sqrt[n]{q_*(n)}.$$

Then the sequences $(q(n+1)/q(n))_{n\geqslant 31}$ and $(q_*(n+1)/q_*(n))_{n\geqslant 44}$ are strictly decreasing, and the sequences $(r(n+1)/r(n))_{n\geqslant 60}$ and $(r_*(n+1)/r_*(n))_{n\geqslant 120}$ are strictly increasing.

Remark 2.13. Via `Mathematica` we have verified the conjecture for n up to 10^5. In light of Example 1.1, Conjecture 2.12 implies that all the sequences

$$\left(\frac{p(n+1)}{p(n)}\right)_{n\geqslant 25}, \ \left(\frac{p_*(n+1)}{p_*(n)}\right)_{n\geqslant 32}, \ (\sqrt[n]{p(n)})_{n\geqslant 6}, \ (\sqrt[n]{p_*(n)})_{n\geqslant 9}$$

are strictly decreasing, and that the sequences $(\sqrt[n+1]{p(n+1)}/\sqrt[n]{p(n)})_{n\geqslant 26}$ and $(\sqrt[n+1]{p_*(n+1)}/\sqrt[n]{p_*(n)})_{n\geqslant 45}$ are strictly increasing. The fact that $(p(n+1)/p(n))_{n\geqslant 25}$ is strictly decreasing was conjectured by W.Y.C. Chen [C] and proved by J.E. Janoski [J, pp. 7-23].

3. Proofs of Theorems 2.3 and 2.6

Lemma 3.1. *Let* $\alpha \geqslant 1$ *and* $n \in \{2, 3, \dots\}$. *Then*

$$S_n^{(\alpha)} > 2^\alpha + \frac{n^{\alpha+1}\log^\alpha n}{\alpha+1}\left(1 - \frac{\alpha}{(\alpha+1)\log n}\right). \tag{3.1}$$

Proof. It is known that $p_k \geqslant k\log k$ for $k = 2, 3, \dots$ (cf. [Ro] and [RS, (3.12)]). Thus

$$S_n^{(\alpha)} - 2^\alpha = \sum_{k=2}^n p_k^\alpha \geqslant \sum_{k=2}^n (k\log k)^\alpha > \sum_{k=2}^n \int_{k-1}^k (x\log x)^\alpha dx = \int_1^n (x\log x)^\alpha dx$$

Using integration by parts, we find that

$$\begin{aligned}\int_1^n (x\log x)^\alpha dx =& \frac{x^{\alpha+1}}{\alpha+1}\log^\alpha x\Big|_{x=1}^n - \int_1^n \left(\frac{x^{\alpha+1}}{\alpha+1}\cdot\frac{\alpha(\log x)^{\alpha-1}}{x}\right)dx \\ =& \frac{n^{\alpha+1}}{\alpha+1}\log^\alpha n - \frac{\alpha}{\alpha+1}\int_1^n x^\alpha(\log x)^{\alpha-1}dx \\ \geqslant& \frac{n^{\alpha+1}}{\alpha+1}\log^\alpha n - \frac{\alpha}{\alpha+1}\int_1^n x^\alpha(\log n)^{\alpha-1}dx \\ \geqslant& \frac{n^{\alpha+1}}{\alpha+1}\log^\alpha n - \frac{\alpha n^{\alpha+1}}{(\alpha+1)^2}(\log n)^{\alpha-1}.\end{aligned}$$

Therefore (3.1) holds. □

Lemma 3.2. *Let* $\alpha \geqslant 1$ *and* $n \in \mathbb{Z}^+$ *with* $n \geqslant 55$. *Then*

$$\log S_n^{(\alpha)} > (\alpha+1)\log n. \tag{3.2}$$

Proof. Note that $54 < e^4 < 55 \leqslant n$. As $\log^\alpha n > 4^\alpha = (2^\alpha)^2 \geqslant (\alpha+1)^2$, by Lemma 3.1 we have

$$S_n^{(\alpha)} > \frac{n^{\alpha+1}\log^\alpha n}{\alpha+1}\left(1 - \frac{\alpha}{\alpha+1}\right) = \frac{n^{\alpha+1}}{(\alpha+1)^2}\log^\alpha n \geqslant n^{\alpha+1}$$

and hence (3.2) follows. □

Proof of Theorem 2.3. It is known that

$$p_m < m(\log m + \log\log m)$$

for any $m \geqslant 6$ (cf. [RS, (3.13)] and [D, Lemma 1]). If $m \geqslant 101$, then

$$\frac{\log\log m}{\log m} \leqslant \frac{\log\log 101}{\log 101} < 0.3314$$

and hence $p_m < 1.3314m\log m$. As $n+1 \leqslant 1.01n$, we have

$$\frac{\log(n+1)}{\log n} = 1 + \frac{\log((n+1)/n)}{\log n} \leqslant 1 + \frac{\log 1.01}{\log n} \leqslant 1 + \frac{\log 1.01}{\log 100} < 1.0022.$$

Therefore

$$p_{n+1} < 1.3314(n+1)\log(n+1) < 1.3314 \times 1.01n \times 1.0022\log n < 1.348n\log n.$$

Combining Lemmas 3.1 and 3.2, we see that

$$\begin{aligned}
&S_n^{(\alpha)}\left(\frac{n+1}{n^{1+1/n}}\sqrt[n]{S_n^{(\alpha)}}-1\right)\\
=&S_n^{(\alpha)}\left(e^{(\log S_n^{(\alpha)})/n+\log(n+1)-(1+1/n)\log n}-1\right)\\
\geqslant&S_n^{(\alpha)}\left(e^{(\log S_n^{(\alpha)}-\log n)/n}-1\right) \geqslant S_n^{(\alpha)}\left(e^{(\alpha\log n)/n}-1\right)\\
>&\frac{n^{\alpha+1}\log^{\alpha} n}{\alpha+1}\left(1-\frac{\alpha}{(\alpha+1)\log n}\right)\frac{\alpha\log n}{n}\\
=&\frac{\alpha}{\alpha+1}(n\log n)^{\alpha}\left(\log n-\frac{\alpha}{\alpha+1}\right)\\
>&\frac{(n\log n)^{\alpha}}{2}(\log n-1).
\end{aligned}$$

As $(\log n-1)/2 \geqslant 1.348^{\alpha}$, from the above we get

$$(n+1)\left(\frac{S_n^{(\alpha)}}{n}\right)^{1+1/n} - S_n^{(\alpha)} > (1.348n\log n)^{\alpha} > p_{n+1}^{\alpha}$$

and hence
$$\left(\frac{S_n^{(\alpha)}}{n}\right)^{(n+1)/n} > \frac{S_{n+1}^{(\alpha)}}{n+1}$$

which yields (2.1). As mentioned in Remark 2.4, (2.2) follows from (2.1). This concludes the proof. □

Proof of Theorem 2.6. Fix an integer $n \geqslant N(\alpha)$. For any integer $m \geqslant 350001$, we have
$$\frac{\log\log m}{\log m} \leqslant \frac{\log\log 350001}{\log 350001} < 0.1996$$

and hence

$$p_m < m(\log m)\left(1+\frac{\log\log m}{\log m}\right) < 1.1996m\log m.$$

As $n \geqslant 350000$, we have

$$\frac{\log(n+1)}{\log n} = 1 + \frac{\log(1+1/n)}{\log n} \leqslant \frac{\log 350001}{\log 350000} < 1 + 10^{-6}.$$

Therefore

$$\begin{aligned} p_{n+1} <& 1.1996(n+1)\log(n+1) \\ <& 1.1996 \times \frac{350001}{350000} n \times (1+10^{-6}) \log n < 1.2 n \log n. \end{aligned}$$

Since $\log n \geqslant \log 350000 > 1/0.078335$, Lemma 3.1 implies that

$$S_n^{(\alpha)} > \frac{n^{\alpha+1}\log^\alpha n}{\alpha+1}(1-0.078335) > \frac{n^{\alpha+1}\log^\alpha n}{1.085(\alpha+1)}.$$

Therefore

$$q_n^{(\alpha)} := \frac{p_{n+1}^\alpha}{S_n^{(\alpha)}} < \frac{c_\alpha}{n}, \tag{3.3}$$

where $c_\alpha = 1.085(\alpha+1)1.2^\alpha$.

By calculus,

$$x - \frac{x^2}{2} < \log(1+x) < x \quad \text{for } x > 0$$

and

$$-x - x^2 < \log(1-x) < -x \quad \text{for } 0 < x < 0.5.$$

Thus

$$\log \frac{S_{n+1}^{(\alpha)}/(n+1)}{S_n^{(\alpha)}/n} = \log\left(1 - \frac{1}{n+1}\right) + \log(1+q_n^{(\alpha)}) < -\frac{1}{n+1} + q_n^{(\alpha)}$$

and

$$\begin{aligned} \log \frac{S_{n+2}^{(\alpha)}/(n+2)}{S_n^{(\alpha)}/n} >& \log\left(1 - \frac{2}{n+2}\right) + \log(1+2q_n^{(\alpha)}) \\ >& -\frac{2}{n+2} - \frac{4}{(n+2)^2} + 2q_n^{(\alpha)} - 2(q_n^{(\alpha)})^2. \end{aligned}$$

Hence

$$\begin{aligned} &D_n^{(\alpha)} := \frac{2}{n+1}\log\frac{S_{n+1}^{(\alpha)}}{n+1} - \frac{1}{n}\log\frac{S_n^{(\alpha)}}{n} - \frac{1}{n+2}\log\frac{S_{n+2}^{(\alpha)}}{n+2} \\ <& \frac{2}{n+1}\left(\log\frac{S_n^{(\alpha)}}{n} - \frac{1}{n+1} + q_n^{(\alpha)}\right) - \frac{1}{n}\log\frac{S_n^{(\alpha)}}{n} \\ &- \frac{1}{n+2}\left(\log\frac{S_n^{(\alpha)}}{n} - \frac{2}{n+2} - \frac{4}{(n+2)^2} + 2q_n^{(\alpha)} - 2(q_n^{(\alpha)})^2\right) \\ =& \frac{-2\log(S_n^{(\alpha)}/n)}{n(n+1)(n+2)} - \frac{2}{(n+1)^2} + \frac{2}{(n+2)^2} + \frac{4}{(n+2)^3} + \frac{2q_n^{(\alpha)}}{(n+1)(n+2)} + \frac{2(q_n^{(\alpha)})^2}{n+2}. \end{aligned}$$

Combining this with (3.2) and (3.3) and noting that $(350001/350000)n^2 \geqslant n(n+1)$, we obtain

$$\begin{aligned} D_n^{(\alpha)} <& \frac{-2\alpha\log n}{n(n+1)(n+2)} - \frac{2(2n+3)}{(n+1)^2(n+2)^2} + \frac{4}{(n+2)^3} \\ &+ \frac{2c_\alpha}{n(n+1)(n+2)} + \frac{2c_\alpha^2}{n^2(n+2)} \\ <& \frac{-2\alpha\log n}{n(n+1)(n+2)} - \frac{4}{(n+1)(n+2)^2} + \frac{4}{(n+1)(n+2)^2} \\ &+ \frac{2c_\alpha + 2(350001/350000)c_\alpha^2}{n(n+1)(n+2)} \\ =& \frac{2((350001/350000)c_\alpha^2 + c_\alpha - \alpha\log n)}{n(n+1)(n+2)}. \end{aligned}$$

Note that

$$\begin{aligned} &\frac{350001}{350000}c_\alpha^2 + c_\alpha \\ =& \frac{350001}{350000} \times 1.085^2(\alpha+1)^2 1.2^{2\alpha} + 1.085(\alpha+1)1.2^\alpha \\ <& 1.2(\alpha+1)^2 1.2^{2\alpha} + 1.2(\alpha+1)1.2^\alpha \leqslant \alpha\log N(\alpha) \leqslant \alpha\log n. \end{aligned}$$

So we have $D_n^{(\alpha)} < 0$ and hence

$$\frac{\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)}}{\sqrt[n]{S_n^{(\alpha)}/n}} < \frac{\sqrt[n+2]{S_{n+2}^{(\alpha)}/(n+2)}}{\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)}}$$

as desired. □

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