ON A SEQUENCE INVOLVING SUMS OF PRIMES

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ABSTRACT. For \( n = 1, 2, 3, \ldots \) let \( S_n \) be the sum of the first \( n \) primes. We mainly show that the sequence \( a_n = \sqrt[n]{S_n} / n \) \((n = 1, 2, 3, \ldots)\) is strictly decreasing, and moreover the sequence \( a_{n+1}/a_n \) \((n = 10, 11, \ldots)\) is strictly increasing. We also formulate similar conjectures involving twin primes or partitions of integers.

1. Introduction

For \( n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \) let \( p_n \) denote the \( n \)th prime. The unsolved Firoozbakht conjecture (cf. [R, p. 185]) asserts that
\[
\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}} \quad \text{for all } n \in \mathbb{Z}^+,
\]
i.e., the sequence \((\sqrt[n]{p_n})_{n \geq 1}\) is strictly decreasing. This implies the inequality \( p_{n+1} - p_n < \log^2 p_n - \log p_n + 1 \) for large \( n \), which is even stronger than Cramér’s conjecture \( p_{n+1} - p_n = O(\log^2 p_n) \). Let \( P_n \) be the product of the first \( n \) primes. Then \( P_n < p_n^n \) and hence \( P_{n+1} < p_{n+1}^n \). So the sequence \((\sqrt[n]{P_n})_{n \geq 1}\) is strictly increasing.

Now let us look at a simple example not related to primes.

Example 1.1. Let \( a_n = \sqrt[n]{n} \) for \( n \in \mathbb{Z}^+ \). Then the sequence \((a_n)_{n \geq 3}\) is strictly decreasing, and the sequence \((a_{n+1}/a_n)_{n \geq 4}\) is strictly increasing. To see this we investigate the function \( f(x) = \log(x^{1/x}) = (\log x)/x \) with \( x \geq 3 \). As \( f'(x) = (1 - \log x)/x^2 < 0 \), we have \( f(n) > f(n+1) \) for \( n = 3, 4, \ldots \). Since
\[
f''(x) = \frac{2 \log x - 3}{x^3} > 0 \quad \text{for } x \geq 4.5,
\]

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the function $f(x)$ is strictly convex over the interval $(4.5, +\infty)$ and so

$$2f(n + 1) < f(n) + f(n + 2) \quad \text{(i.e., } a_{n+1}^2 < a_n a_{n+2} \text{) for } n = 5, 6, \ldots.$$ 

The inequality $a_2^2 < a_4 a_6$ can be verified directly.

A sequence $(a_n)_{n\geq 1}$ of nonnegative real numbers is said to be log-convex if $a_{n+1}^2 \leq a_n a_{n+2}$ for all $n = 1, 2, 3, \ldots$. Many combinatorial sequences (such as the sequence of the Catalan numbers) are log-convex, the reader may consult [LW] for some results on log-convex sequences.

For $n \in \mathbb{Z}^+$ let $S_n = \sum_{k=1}^n p_k$ be the sum of the first $n$ primes. For instance, $S_1 = 2$, $S_2 = 2 + 3 = 5$, $S_3 = 2 + 3 + 5 = 10$, $S_4 = 2 + 3 + 5 + 7 = 17$.

Recently the author [S] conjectured that for any positive integer $n$ the interval $(S_n, S_{n+1})$ contains a prime. As $S_n < n p_{n+1}$ for all $n \in \mathbb{Z}^+$, the sequence $(S_n/n)_{n\geq 1}$ is strictly increasing.

In the next section we will state our theorems involving the sequence $(a_n)_{n\geq 1}$ with $a_n = \sqrt[n]{S_n/n}$, and pose three related conjectures for further research. Section 3 is devoted to our proofs of the theorems.

2. Our results and conjectures

**Theorem 2.1.** The sequences $(\sqrt[n]{S_n})_{n\geq 2}$ and $(\sqrt[n]{S_n/n})_{n\geq 1}$ are strictly decreasing.

**Remark 2.2.** Note that $S_n/n$ is just the arithmetic mean of the first $n$ primes. It is interesting to compare Theorem 2.1 with Firoozbakht’s conjecture that $(\sqrt[n]{p_n})_{n\geq 1}$ is strictly decreasing.

For $\alpha > 0$ and $n \in \mathbb{Z}^+$ define

$$S_n^{(\alpha)} = \sum_{k=1}^n p_k^\alpha.$$

We actually obtain the following extension of Theorem 2.1.

**Theorem 2.3.** Let $\alpha \geq 1$ and $n \in \mathbb{Z}^+$ with $n \geq \max\{100, e^{2 \times 1.348 + 1}\}$. Then

$$\sqrt[n]{S_n^{(\alpha)}} > \sqrt[n+1]{S_{n+1}^{(\alpha)}}$$

and hence

$$\sqrt[n]{S_n^{(\alpha)}} > \sqrt[n+1]{S_{n+1}^{(\alpha)}}.$$  \hspace{1cm} (2.1)

**Remark 2.4.** In view of Example 1.1, (2.1) implies (2.2) if $n \geq 3$. We conjecture that (2.1) holds for any $\alpha > 0$ and $n \in \mathbb{Z}^+$.

Note that $\lfloor e^{2 \times 1.348 + 1} \rfloor = 40$ and we can easily verify that

$$\sqrt[n]{S_n^n} > \sqrt[n+1]{S_{n+1}^{(\alpha)}}$$

for every $n = 1, \ldots, 99$.

So Theorem 2.1 follows from Theorem 2.3 in the case $\alpha = 1$. 

Corollary 2.5. For each \( \alpha \in \{2, 3, 4\} \), the sequences

\[
\left( \sqrt[n+1]{\frac{S_{n+1}}{n+1}} \right)_{n \geq 1} \text{ and } \left( \sqrt[n]{\frac{S_n}{n}} \right)_{n \geq 1}
\]

are strictly decreasing.

Proof. Observe that

\[
\left\lfloor e^{2 \times 1.348^2 + 1} \right\rfloor = 102, \quad \left\lfloor e^{2 \times 1.348^3 + 1} \right\rfloor = 364, \quad \left\lfloor e^{2 \times 1.348^4 + 1} \right\rfloor = 2005.
\]

In light of Theorem 2.3 and Example 1.1, it suffices to verify that

\[
\frac{\sqrt[n]{S_n}}{n} > \frac{\sqrt[n+1]{S_{n+1}}}{n+1}
\]

whenever \( \alpha \in \{2, 3, 4\} \) and \( n \in \{1, \ldots, \left\lfloor e^{2 \times 1.348^\alpha + 1} \right\rfloor\} \). These can be easily done via computer. \( \square \)

Our following theorem is more sophisticated than Theorem 2.3.

Theorem 2.6. Let \( \alpha \geq 1 \). Then the sequence

\[
\left( \frac{n+1}{\sqrt[n+1]{S_{n+1}/(n+1)}} \right) \left( \frac{\sqrt[n]{S_n/n}}{\sqrt[n]{S_n/n}} \right)_{n \geq N(\alpha)}
\]

is strictly increasing, where

\[
N(\alpha) = \max \left\{ 350000, \left\lfloor e^{((\alpha+1)^2 \times 2^{\alpha+1} + (\alpha+1) \times 2^{\alpha+1})/\alpha} \right\rfloor \right\}.
\]

Corollary 2.7. All the sequences

\[
\left( \frac{n}{\sqrt[n+1]{S_{n+1}/(n+1)}} \right)_{n \geq 10}, \quad \left( \frac{n+1}{\sqrt[n+1]{S_{n+1}/n}} \right)_{n \geq 5},
\]

\[
\left( \frac{n+1}{\sqrt[n+1]{S_{n+1}^{(2)}/(n+1)}} \right)_{n \geq 13}, \quad \left( \frac{n}{\sqrt[n]{S_n/n}} \right)_{n \geq 5},
\]

\[
\left( \frac{n+1}{\sqrt[n+1]{S_{n+1}^{(3)}/(n+1)}} \right)_{n \geq 17}, \quad \left( \frac{n}{\sqrt[n]{S_n/n}} \right)_{n \geq 10},
\]

\[
\left( \frac{n+1}{\sqrt[n+1]{S_{n+1}^{(4)}/(n+1)}} \right)_{n \geq 35}, \quad \left( \frac{n}{\sqrt[n]{S_n/n}} \right)_{n \geq 17}
\]

are strictly increasing.
Proof. For \( N(\alpha) \) given by (2.3), via computation we find that
\[
N(1) = 350000, \quad N(2) = 974267, \quad N(3) = 3163983273
\]
and
\[
N(4) = 2271069361863763.
\]
Via computer we can verify that
\[
\frac{n+1}{\sqrt{n+1}} S_n^{\alpha} / (n+1) < \frac{n+2}{\sqrt{n+2}} S_n^{\alpha} / (n+1)
\]
for all \( \alpha \in \{1, 2, 3, 4\} \) and \( n = N_0(\alpha), \ldots, N(\alpha) - 1 \), where
\[
N_0(1) = 10, \quad N_0(2) = 13, \quad N_0(3) = 17, \quad N_0(4) = 35.
\]
Combining this with Theorem 2.3 we obtain that
\[
\left( \frac{n+1}{\sqrt{n+1}} S_n^{\alpha} / (n+1) \right)_{n \geq N_0(\alpha)}
\]
is strictly increasing for each \( \alpha = 1, 2, 3, 4 \). Recall that \( \frac{n+1}{\sqrt{n+1}} \sqrt{n+1} \) is strictly increasing by Example 1.1. So \( \left( \frac{n+1}{\sqrt{n+1}} S_n^{\alpha} / \sqrt{n} \right)_{n \geq N_0(\alpha)} \) is strictly increasing for any \( \alpha \in \{1, 2, 3, 4\} \). It remains to check that
\[
\frac{n+1}{\sqrt{n+1}} S_n^{\alpha} / \sqrt{n} < \frac{n+2}{\sqrt{n+2}} S_n^{\alpha} / \sqrt{n+1}
\]
for all \( \alpha \in \{1, 2, 3, 4\} \) and \( n = n_0(\alpha), \ldots, N_0(\alpha) - 1 \), where \( n_0(1) = 5, \ n_0(2) = n_0(3) = 10, \) and \( n_0(4) = 17 \). This can be easily done via computer. \( \square \)

We conclude this section by posing three conjectures.

**Conjecture 2.8.** The two constants
\[
s_1 = \sum_{n=1}^{\infty} \frac{1}{S_n} \quad \text{and} \quad s_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{S_n}
\]
are both transcendental numbers.

**Remark 2.9.** Our computation shows that \( s_1 \approx 1.023476 \) and \( s_2 \approx -0.3624545778 \).

If \( p \) and \( p+2 \) are both primes, then they are called twin primes. The famous twin prime conjecture states that there are infinitely many twin primes.
Conjecture 2.10. (i) If \( \{t_1, t_1 + 2\}, \ldots, \{t_n, t_n + 2\}\) are the first \( n \) pairs of twin primes, then the first prime \( t_{n+1} \) in the next pair of twin primes is smaller than \( t_n^{1+1/n} \), i.e., \( \sqrt[n]{T_n} > \sqrt[n]{t_{n+1}} \).

(ii) The sequence \( \sqrt[n]{T_{n+1}} / \sqrt[n]{T_n} \) is strictly increasing, and that the sequences \( (\sqrt[n]{T_{n+1}} / \sqrt[n]{T_n})_{n \geq 2} \) is strictly increasing with limit 1, where \( T_n = \sum_{k=1}^{n} t_k \).

Remark 2.11. Via Mathematica the author has verified that \( \sqrt[n]{T_n} > \sqrt[n]{t_{n+1}} \) for all \( n = 1, \ldots, 500000 \), and \( \sqrt[n]{T_{n+1}} / \sqrt[n]{T_n} < \sqrt[n]{T_{n+2}} / \sqrt[n]{T_{n+1}} \) for all \( n = 9, \ldots, 500000 \). Note that \( t_{500000} = 115438667 \).

Recall that a partition of a positive integer \( n \) is a way of writing \( n \) as a sum of positive integers with the order of addends ignored. Also, a strict partition of \( n \in \mathbb{Z}^+ \) is a way of writing \( n \) as a sum of distinct positive integers with the order of addends ignored. For \( n = 1, 2, 3, \ldots \) we denote by \( p(n) \) and \( p_\ast(n) \) the number of partitions of \( n \) and the number of strict partitions of \( n \) respectively. It is known that

\[
p(n) \sim \frac{e^{\sqrt{2n/3}}}{4\sqrt{3n^3}} \quad \text{and} \quad p_\ast(n) \sim \frac{e^{\sqrt{n/3}}}{4(3n^3)^{1/4}} \quad \text{as} \quad n \to +\infty
\]

(cf. [HR] and [AS, p. 826]) and hence \( \lim_{n \to +\infty} \sqrt[n]{p(n)} = \lim_{n \to +\infty} \sqrt[n]{p_\ast(n)} = 1 \).

Here we formulate a conjecture similar to Conjecture 2.10.

Conjecture 2.12. For \( n \in \mathbb{Z}^+ \) let

\[
q(n) = \frac{p(n)}{n}, \quad q_\ast(n) = \frac{p_\ast(n)}{n}, \quad r(n) = \sqrt[n]{q(n)}, \quad \text{and} \quad r_\ast(n) = \sqrt[n]{q_\ast(n)}.
\]

Then the sequences \( (q(n+1)/q(n))_{n \geq 31} \) and \( (q_\ast(n+1)/q_\ast(n))_{n \geq 44} \) are strictly decreasing, and the sequences \( (r(n+1)/r(n))_{n \geq 60} \) and \( (r_\ast(n+1)/r_\ast(n))_{n \geq 120} \) are strictly increasing.

Remark 2.13. Via Mathematica we have verified the conjecture for \( n \) up to \( 10^5 \).

In light of Example 1.1, Conjecture 2.12 implies that all the sequences

\[
\left( \frac{p(n+1)}{p(n)} \right)_{n \geq 25}, \quad \left( \frac{p_\ast(n+1)}{p_\ast(n)} \right)_{n \geq 32}, \quad (\sqrt[n]{p(n)})_{n \geq 26}, \quad (\sqrt[n]{p_\ast(n)})_{n \geq 29}
\]

are strictly decreasing, and that the sequences \( (\sqrt[n]{p(n+1)}/\sqrt[n]{p(n)})_{n \geq 26} \) and \( (\sqrt[n]{p_\ast(n+1)}/\sqrt[n]{p_\ast(n)})_{n \geq 25} \) are strictly increasing. The fact that \( (p(n+1)/p(n))_{n \geq 25} \) is strictly decreasing was conjectured by W.Y.C. Chen [C] and proved by J.E. Janoski [J, pp. 7-23].
3. Proofs of Theorems 2.3 and 2.6

Lemma 3.1. Let $\alpha \geq 1$ and $n \in \{2, 3, \ldots\}$. Then

$$S_n^{(\alpha)} > 2^\alpha + \frac{n^{\alpha+1} \log^{\alpha} n}{\alpha + 1} \left(1 - \frac{\alpha}{(\alpha + 1) \log n}\right). \quad (3.1)$$

Proof. It is known that $p_k \geq k \log k$ for $k = 2, 3, \ldots$ (cf. [Ro] and [RS, (3.12)]). Thus

$$S_n^{(\alpha)} - 2^\alpha = \sum_{k=2}^{n} p_k^\alpha \geq \sum_{k=2}^{n} (k \log k) \alpha > \sum_{k=2}^{n} \int_{k-1}^{k} (x \log x)^\alpha dx = \int_{1}^{n} (x \log x)^\alpha dx$$

Using integration by parts, we find that

$$\int_{1}^{n} (x \log x)^\alpha dx = \left. \frac{x^{\alpha+1}}{\alpha + 1} \log^{\alpha} x \right|_{x=1}^{n} - \int_{1}^{n} \left(\frac{x^{\alpha+1}}{\alpha + 1} \cdot \frac{(\log x)^{\alpha-1}}{x}\right) dx$$

$$= \frac{n^{\alpha+1}}{\alpha + 1} \log^{\alpha} n - \frac{\alpha}{\alpha + 1} \int_{1}^{n} x^{\alpha} (\log x)^{\alpha-1} dx$$

$$\geq \frac{n^{\alpha+1}}{\alpha + 1} \log^{\alpha} n - \frac{\alpha}{\alpha + 1} \int_{1}^{n} x^{\alpha} (\log n)^{\alpha-1} dx$$

$$\geq \frac{n^{\alpha+1}}{\alpha + 1} \log^{\alpha} n - \frac{\alpha n^{\alpha+1}}{(\alpha + 1)^2} (\log n)^{\alpha-1}.$$ 

Therefore (3.1) holds. \qed

Lemma 3.2. Let $\alpha \geq 1$ and $n \in \mathbb{Z}^+$ with $n \geq 55$. Then

$$\log S_n^{(\alpha)} > (\alpha + 1) \log n. \quad (3.2)$$

Proof. Note that $54 < e^4 < 55 \leq n$. As $\log^{\alpha} n > 4^{\alpha} = (2^\alpha)^2 \geq (\alpha + 1)^2$, by Lemma 3.1 we have

$$S_n^{(\alpha)} > \frac{n^{\alpha+1} \log^{\alpha} n}{\alpha + 1} \left(1 - \frac{\alpha}{\alpha + 1}\right) = \frac{n^{\alpha+1}}{(\alpha + 1)^2} \log^{\alpha} n \geq n^{\alpha+1}$$

and hence (3.2) follows. \qed

Proof of Theorem 2.3. It is known that

$$p_m < m \log m + \log \log m$$

for any $m \geq 6$ (cf. [RS, (3.13)] and [D, Lemma 1]). If $m \geq 101$, then

$$\frac{\log \log m}{\log m} \leq \frac{\log 101}{\log 101} < 0.3314$$
and hence $p_n < 1.3314m \log m$. As $n + 1 \leq 1.01n$, we have

$$\frac{\log(n + 1)}{\log n} = 1 + \frac{\log((n + 1)/n)}{\log n} \leq 1 + \frac{\log 1.01}{\log n} \leq 1 + \frac{\log 1.01}{\log 100} < 1.0022.$$ 

Therefore

$$p_{n+1} < 1.3314(n + 1) \log(n + 1) < 1.3314 \times 1.01n \times 1.0022 \log n < 1.348n \log n.$$ 

Combining Lemmas 3.1 and 3.2, we see that

$$S_n^{(\alpha)} \left(\frac{n + 1}{n^{1 + 1/n}} \sqrt[n]{S_n^{(\alpha)} - 1}\right) - S_n^{(\alpha)} \left(e^{(\alpha \log n)/n - 1}\right) \geq S_n^{(\alpha)} \left(e^{(\alpha \log n)/n - 1}\right) - \left(1 + \frac{\log((n + 1)/n)}{\log n}\right) \frac{\alpha \log n}{n} \geq S_n^{(\alpha)} \left(e^{(\alpha \log n)/n - 1}\right) > \frac{n^{\alpha + 1} \log^\alpha n}{\alpha + 1} \left(1 - \frac{\alpha}{\alpha + 1} \frac{\log n}{\log n}\right) - \frac{(n \log n)^\alpha}{\alpha + 1} \left(\log n - \frac{\alpha}{\alpha + 1}\right) > \frac{(n \log n)^\alpha}{2} \left(\log n - 1\right).$$

As $(\log n - 1)/2 \geq 1.348^\alpha$, from the above we get

$$(n + 1) \left(\frac{S_n^{(\alpha)}}{n}\right)^{1 + 1/n} - S_n^{(\alpha)} > (1.348n \log n)^\alpha > p_{n+1}^{\alpha}$$

and hence

$$\left(\frac{S_n^{(\alpha)}}{n}\right)^{(n+1)/n} > \frac{S_{n+1}^{(\alpha)}}{n + 1}$$

which yields (2.1). As mentioned in Remark 2.4, (2.2) follows from (2.1). This concludes the proof. □

Proof of Theorem 2.6. Fix an integer $n \geq N(\alpha)$. For any integer $m \geq 350001$, we have

$$\frac{\log \log m}{\log m} \leq \frac{\log \log 350001}{\log 350001} < 0.1996$$

and hence

$$p_m < m(\log m) \left(1 + \frac{\log \log m}{\log m}\right) < 1.1996m \log m.$$
As \( n \geq 350000 \), we have
\[
\frac{\log(n+1)}{\log n} = 1 + \frac{\log(1 + 1/n)}{\log n} \leq \frac{\log 350001}{\log 350000} < 1 + 10^{-6}.
\]
Therefore
\[
p_{n+1} < 1.1996(n + 1) \log(n + 1)
\]
\[
< 1.1996 \times \frac{350001}{350000} n \times (1 + 10^{-6}) \log n < 1.2n \log n.
\]
Since \( \log n \geq \log 350000 > 1/0.078335 \), Lemma 3.1 implies that
\[
S_n^{(\alpha)} > \frac{n^{\alpha+1} \log^\alpha n}{\alpha + 1} (1 - 0.078335) > \frac{n^{\alpha+1} \log^\alpha n}{1.085(\alpha + 1)}.
\]
Therefore
\[
q_n^{(\alpha)} := \frac{p_{n+1}^{(\alpha)}}{S_n^{(\alpha)}} < \frac{c_\alpha}{n},
\]
where \( c_\alpha = 1.085(\alpha + 1)1.2^\alpha \).

By calculus,
\[
x - \frac{x^2}{2} < \log(1 + x) < x \quad \text{for } x > 0
\]
and
\[
-x - \frac{x^2}{2} < \log(1 - x) < -x \quad \text{for } 0 < x < 0.5.
\]
Thus
\[
\log \frac{S_n^{(\alpha)}(n + 1)}{S_n^{(\alpha)}} = \log \left(1 - \frac{1}{n + 1}\right) + \log(1 + q_n^{(\alpha)}) < -\frac{1}{n + 1} + q_n^{(\alpha)}
\]
and
\[
\log \frac{S_n^{(\alpha)}(n + 2)}{S_n^{(\alpha)}} > \log \left(1 - \frac{2}{n + 2}\right) + \log(1 + 2q_n^{(\alpha)})
\]
\[
> -\frac{2}{n + 2} - \frac{4}{(n + 2)^2} + 2q_n^{(\alpha)} - 2(q_n^{(\alpha)})^2.
\]
Hence
\[
D_n^{(\alpha)} := \frac{2}{n + 1} \log \frac{S_n^{(\alpha)}}{n + 1} - \frac{1}{n} \log \frac{S_n^{(\alpha)}}{n} - \frac{1}{n + 2} \log \frac{S_n^{(\alpha+2)}}{n + 2}
\]
\[
< \frac{2}{n + 1} \left( \log \frac{S_n^{(\alpha)}}{n} - \frac{1}{n + 1} + q_n^{(\alpha)} \right) - \frac{1}{n} \log \frac{S_n^{(\alpha)}}{n}
\]
\[
- \frac{1}{n + 2} \left( \log \frac{S_n^{(\alpha)}}{n} - \frac{2}{n + 2} - \frac{4}{(n + 2)^2} + 2q_n^{(\alpha)} - 2(q_n^{(\alpha)})^2 \right)
\]
\[
= -2 \log(S_n^{(\alpha)}/n) \frac{2}{n(n + 1)(n + 2)} + \frac{2}{(n + 2)^2} + \frac{4}{(n + 2)^3} + \frac{2q_n^{(\alpha)}}{(n + 1)(n + 2)} + \frac{2(q_n^{(\alpha)})^2}{n + 2}.
\]
Combining this with (3.2) and (3.3) and noting that \((350001/350000)n^2 \geq n(n+1)\), we obtain

\[
D_n^{(\alpha)} \leq \frac{-2\alpha \log n}{n(n+1)(n+2)} - \frac{2(2n+3)}{(n+1)^2(n+2)^2} + \frac{4}{(n+2)^3} \\
+ \frac{2c_\alpha}{n(n+1)(n+2)} + \frac{2c_\alpha^2}{n^2(n+2)}
\]

\[
\leq \frac{-2\alpha \log n}{n(n+1)(n+2)} - \frac{4}{(n+1)(n+2)^2} + \frac{4}{(n+1)(n+2)^2} \\
+ \frac{2c_\alpha + 2(350001/350000)c_\alpha^2}{n(n+1)(n+2)}
\]

\[
= \frac{2((350001/350000)c_\alpha^2 + c_\alpha - \alpha \log n)}{n(n+1)(n+2)}
\]

Note that

\[
\frac{350001}{350000}c_\alpha^2 + c_\alpha \\
= \frac{350001}{350000} \times 1.085^2(\alpha+1)^21.2^{2\alpha} + 1.085(\alpha+1)1.2^\alpha \\
< 1.2(\alpha+1)^21.2^{2\alpha} + 1.2(\alpha+1)1.2^\alpha \leq \alpha \log N(\alpha) \leq \alpha \log n.
\]

So we have \(D_n^{(\alpha)} < 0\) and hence

\[
\frac{\sqrt[n+1]{S^{(\alpha)}_{n+1}/(n+1)}}{\sqrt[n]{S^{(\alpha)}_n/n}} < \frac{\sqrt[n+2]{S^{(\alpha)}_{n+2}/(n+2)}}{\sqrt[n+1]{S^{(\alpha)}_{n+1}/(n+1)}} \]

as desired. \(\square\)

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**References**


