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ARITHMETIC THEORY OF HARMONIC NUMBERS (II)

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Abstract. For $k = 1, 2, \ldots$ let H_k denote the harmonic number $\sum_{j=1}^k 1/j$. In this paper we establish some new congruences involving harmonic numbers. For example, we show that for any prime $p > 3$ we have

$$\sum_{k=1}^{p-1}\frac{H_k}{k2^k} \equiv \frac{7}{24}pB_{p-3} \pmod{p^2},\quad \sum_{k=1}^{p-1}\frac{H_{k,2}}{k2^k} \equiv -\frac{3}{8}B_{p-3} \pmod{p},$$

and

$$\sum_{k=1}^{p-1}\frac{H_{k,2n}^2}{k^{2n}} \equiv \frac{\binom{6n+1}{2n-1}+n}{6n+1}pB_{p-1-6n} \pmod{p^2}$$

for any positive integer $n < (p-1)/6$, where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers, and $H_{k,m} := \sum_{j=1}^k 1/j^m$.

1. Introduction

Recall that harmonic numbers are those

$$H_n := \sum_{0<k\leqslant n}\frac{1}{k}\quad (n \in \mathbb{N} = \{0, 1, 2, \ldots\}),$$

where $H_0 := 0$ since we consider the value of an empty sum as zero. They play important roles in mathematics. In 1862 J. Wolstenholme [W] showed

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the congruence $H_{p-1} \equiv 0 \pmod{p^2}$ for any prime $p > 3$. Throughout this paper, for a prime p and two rational p-adic integers A and B, we write $A \equiv B \pmod{p^n}$ (with $n \in \mathbb{N}$) to mean that $A - B$ is divisible by p^n in the ring of p-adic integers.

In [Su] the first author investigated arithmetic properties of harmonic numbers systematically. For example, he proved that for any prime $p > 5$ we have

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}.$$

For $m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, harmonic numbers of order m are defined by

$$H_{n,m} := \sum_{0<k\leqslant n} \frac{1}{k^m} \quad (n \in \mathbb{N}).$$

It is known that

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{\pi^2}{12} \qquad \text{(S. W. Coffman [C], 1987)}$$

and

$$\sum_{k=1}^{\infty} \frac{H_{k,2}}{k2^k} = \frac{5}{8}\zeta(3) \qquad \text{(B. Cloitre, 2004)}.$$

Both identities can be found in [SW].

Our first theorem is as follows.

Theorem 1.1. *For any prime $p > 3$, we have*

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \frac{7}{24} pB_{p-3} \pmod{p^2} \tag{1.1}$$

and

$$\sum_{k=1}^{p-1} \frac{H_{k,2}}{k2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p}, \tag{1.2}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers.

Remark 1.1. (1.1) confirms the first part of [Su, Conjecture 1.1]. The second part of [Su, Conjecture 1.1] states that $\sum_{k=1}^{p-1} H_k^2/k^2 \equiv \frac{4}{5} pB_{p-5} \pmod{p^2}$ for any prime $p > 3$; this was confirmed by R. Meštrović [M] quite recently.

Our second theorem confirms the second conjecture of [Su].

Theorem 1.2 ([Su, Conjecture 1.2])**.** *Let p be an odd prime and let n be a positive integer with $p-1\nmid 6n$. Then*

$$\sum_{k=1}^{p-1}\frac{H_{k,2n}^2}{k^{2n}}\equiv 0 \pmod p. \tag{1.3}$$

Furthermore, when $p>6n+1$ we have

$$\sum_{k=1}^{p-1}\frac{H_{k,2n}^2}{k^{2n}}\equiv \frac{s(n)}{6n+1}pB_{p-1-6n} \pmod{p^2}, \tag{1.4}$$

where

$$s(n)=\binom{6n+1}{2n-1}+n.$$

Remark 1.2. We give here four initial values of the integer sequence $\{s(n)\}_{n\geqslant 1}$:

$$s(1)=8,\ s(2)=288,\ s(3)=11631,\ s(4)=480704.$$

We will show Theorems 1.1 and 1.2 in Sections 2 and 3 respectively.

2. Proof of Theorem 1.1

Lemma 2.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\equiv\frac{p}{2}B_{p-3}\pmod{p^2},\ \sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\equiv-\frac{B_{p-3}}{2}\pmod p, \tag{2.1}$$

and

$$\sum_{k=1}^{p-1}\frac{H_k}{k}\equiv\frac{p}{3}B_{p-3}\pmod{p^2}\ \text{ and }\ \sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}H_k\equiv-\frac{B_{p-3}}{4}\pmod p. \tag{2.2}$$

Proof. It is known that (cf. [S, Corollaries 5.1 and 5.2])

$$\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv\frac{2}{3}pB_{p-3}\pmod{p^2},\ \sum_{k=1}^{p-1}\frac{1}{k^3}\equiv\frac{3}{4}pB_{p-4}\equiv-p\delta_{p,5}\pmod{p^2},$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv\frac{7}{3}pB_{p-3}\pmod{p^2}\ \text{ and }\ \sum_{k=1}^{(p-1)/2}\frac{1}{k^3}\equiv-2B_{p-3}\pmod p.$$

Thus

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}=&\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k^2}-\sum_{k=1}^{p-1}\frac{1}{k^2}=\frac{1}{2}H_{(p-1)/2,2}-H_{p-1,2}\\ \equiv&\frac{7}{6}pB_{p-3}-\frac{2}{3}pB_{p-3}=\frac{p}{2}B_{p-3}\ (\mathrm{mod}\ p^2)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}=&\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k^3}-\sum_{k=1}^{p-1}\frac{1}{k^3}\\ =&\frac{1}{4}H_{(p-1)/2,3}-H_{p-1,3}\equiv\frac{-2B_{p-3}}{4}\ (\mathrm{mod}\ p).\end{aligned}$$

Therefore (2.1) holds.

By the proof of [S, Theorem 6.1],

$$\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{jk}\equiv-\frac{p}{3}B_{p-3}\pmod{p^2}.$$

So we have

$$\sum_{k=1}^{p-1}\frac{H_k}{k}=\sum_{k=1}^{p-1}\frac{1}{k^2}+\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{jk}\equiv\frac{2}{3}pB_{p-3}-\frac{p}{3}B_{p-3}=\frac{p}{3}B_{p-3}\pmod{p^2}.$$

This proves the first congruence in (2.2).

Now we prove the second congruence in (2.2). Since

$$H_{p-k}=H_{p-1}-\sum_{j=1}^{k-1}\frac{1}{p-j}\equiv H_{k-1}=H_k-\frac{1}{k}\pmod{p}$$

for all $k=1,\ldots,p-1$, we have

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}H_k=\sum_{k=1}^{p-1}\frac{(-1)^{p-k}}{(p-k)^2}H_{p-k}\equiv-\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\left(H_k-\frac{1}{k}\right)\pmod{p}$$

and hence

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}H_k\equiv\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\equiv-\frac{B_{p-3}}{4}\pmod{p}.$$

The proof of Lemma 2.1 is now complete. □

Lemma 2.2. (i) *For any positive integers k and m we have*

$$\sum_{n=1}^{m}\binom{n-1}{k-1}=\binom{m}{k}. \tag{2.3}$$

(ii) *For each $n=1,2,3,\ldots$ we have*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}H_k=H_{n,2}. \tag{2.4}$$

Proof. (2.3) is well known (cf. [G, (1.5)]) and it can be easily proved by induction on m.

(2.4) is also known (cf. [H]). Here we prove it by induction. Clearly (2.4) holds for $n=1$. Assume that (2.4) holds for a fixed positive integer n. Then

$$\begin{aligned}\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^{k-1}}{k}H_k=&\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}H_k+\sum_{k=1}^{n+1}\binom{n}{k-1}\frac{(-1)^{k-1}}{k}H_k\\ =&H_{n,2}+\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^{k-1}H_k.\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^{k-1}H_k\\ =&\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}H_k+\sum_{k=1}^{n+1}\binom{n}{k-1}(-1)^{k-1}\left(H_{k-1}+\frac{1}{k}\right)\\ =&\sum_{k=1}^{n+1}\binom{n}{k-1}\frac{(-1)^{k-1}}{k}=-\frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^k=\frac{1}{n+1}.\end{aligned}$$

So

$$\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^{k-1}}{k}H_k=H_{n,2}+\frac{1}{n+1}\cdot\frac{1}{n+1}=H_{n+1,2}$$

as desired. □

Lemma 2.3. *Let $p>3$ be a prime. Then*

$$\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{2^j(j+k)}{j^2k^2}\equiv\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\pmod p. \tag{2.5}$$

Proof. Observe that

$$\begin{aligned}&\sum_{1\leqslant i\leqslant j\leqslant k\leqslant p-1}\frac{2^i}{ijk}-\sum_{1\leqslant i<j<k\leqslant p-1}\frac{2^i}{ijk}\\ =&\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{2^j}{j^2k}+\sum_{1\leqslant i\leqslant j\leqslant p-1}\frac{2^i}{ij^2}-\sum_{k=1}^{p-1}\frac{2^k}{k^3}\\ =&\sum_{1\leqslant j\leqslant k\leqslant p-1}\left(\frac{2^j}{j^2k}+\frac{2^j}{jk^2}\right)-\sum_{k=1}^{p-1}\frac{2^k}{k^3}.\end{aligned}$$

Similarly,

$$\begin{aligned}&2\sum_{1\leqslant i\leqslant j\leqslant k\leqslant p-1}\frac{(-1)^i}{ijk}-2\sum_{1\leqslant i<j<k\leqslant p-1}\frac{(-1)^i}{ijk}-2\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\\ =&2\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{(-1)^j}{j^2k}+\frac{(-1)^j}{jk^2}\right)\\ \equiv&\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{(-1)^j}{j^2k}+\frac{(-1)^j}{jk^2}+\frac{(-1)^{p-j}}{(p-j)^2(p-k)}+\frac{(-1)^{p-j}}{(p-j)(p-k)^2}\right)\\ =&\sum_{1\leqslant j<k\leqslant p-1}\frac{(-1)^j}{j^2k}+\sum_{1\leqslant k<j\leqslant p-1}\frac{(-1)^j}{j^2k}\\ &+\sum_{1\leqslant j<k\leqslant p-1}\frac{(-1)^j}{jk^2}+\sum_{1\leqslant k<j\leqslant p-1}\frac{(-1)^j}{jk^2}\\ =&H_{p-1}\sum_{j=1}^{p-1}\frac{(-1)^j}{j^2}+H_{p-1,2}\sum_{j=1}^{p-1}\frac{(-1)^j}{j}-2\sum_{j=1}^{p-1}\frac{(-1)^j}{j^3}\pmod p.\end{aligned}$$

Thus, with the help of $H_{p-1}\equiv H_{p-1,2}\equiv 0 \pmod p$, we have

$$\sum_{1\leqslant i\leqslant j\leqslant k\leqslant p-1}\frac{(-1)^i}{ijk}\equiv\sum_{1\leqslant i<j<k\leqslant p-1}\frac{(-1)^i}{ijk}\pmod p.$$

By [ZS, Theorem 1.2],

$$\sum_{1\leqslant i<j<k\leqslant p-1}\frac{(1-x)^i}{ijk}\equiv\sum_{1\leqslant i<j<k\leqslant p-1}\frac{x^i}{ijk}\pmod p.$$

So, in view of the above, we have

$$\begin{aligned}\sum_{1\leqslant i\leqslant j\leqslant k\leqslant p-1}\frac{(-1)^i}{ijk}&\equiv\sum_{1\leqslant i<j<k\leqslant p-1}\frac{2^i}{ijk}\\ &\equiv\sum_{1\leqslant i\leqslant j\leqslant k\leqslant p-1}\frac{2^i}{ijk}+\sum_{k=1}^{p-1}\frac{2^k}{k^3}-\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{2^j(j+k)}{j^2k^2}\pmod p.\end{aligned}$$

It remains to show that

$$\sum_{1\leqslant i\leqslant j\leqslant k\leqslant p-1}\frac{2^i-(-1)^i}{ijk}\equiv\sum_{k=1}^{p-1}\frac{(-1)^k-2^k}{k^3}\pmod p. \tag{2.6}$$

With the help of Lemma 2.2, we have

$$\begin{aligned}\sum_{1\leqslant i\leqslant j\leqslant k\leqslant p-1}\frac{2^i-(-1)^i}{ijk}&=\sum_{1\leqslant i\leqslant j\leqslant k\leqslant p-1}\frac{1}{ijk}\sum_{r=0}^{i}(1-(-2)^r)\binom{i}{r}\\&=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r}\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{jk}\sum_{i=1}^{j}\binom{i-1}{r-1}\\&=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r}\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{jk}\binom{j}{r}\\&=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r^2}\sum_{k=1}^{p-1}\frac{1}{k}\sum_{j=1}^{k}\binom{j-1}{r-1}\\&=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r^2}\sum_{k=1}^{p-1}\frac{1}{k}\binom{k}{r}=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r^3}\sum_{k=1}^{p-1}\binom{k-1}{r-1}\\&=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r^3}\binom{p-1}{r}\equiv\sum_{r=1}^{p-1}\frac{(-1)^r-2^r}{r^3}\pmod p.\end{aligned}$$

This proves the desired (2.6). □

Proof of Theorem 1.1. We prove (1.2) first. In view of (2.4), we have

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{H_{n,2}}{n2^n}&=\sum_{n=1}^{p-1}\frac{1}{n2^n}\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}H_k\\&=\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}H_k\sum_{n=k}^{p-1}\frac{1}{n2^n}\binom{n}{k}=\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^22^k}H_k\sum_{n=k}^{p-1}\binom{n-1}{k-1}\frac{1}{2^{n-k}}\\&=\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^22^k}H_k\sum_{j=0}^{p-1-k}\binom{k+j-1}{j}\frac{1}{2^j}\\&=\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^22^k}H_k\sum_{j=0}^{p-1-k}\binom{-k}{j}\frac{1}{(-2)^j}\end{aligned}$$

and hence

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{H_{n,2}}{n2^n}\equiv&\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^2 2^k}H_k\sum_{j=0}^{p-1-k}\binom{p-k}{j}\frac{1}{(-2)^j}\\=&\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^2 2^k}H_k\frac{1+(-1)^k}{2^{p-k}}\\\equiv&-\frac{1}{2}\sum_{k=1}^{p-1}\frac{H_k}{k^2}\left(1+(-1)^k\right)\pmod p.\end{aligned}$$

Note that

$$\sum_{k=1}^{p-1}\frac{H_k}{k^2}\equiv B_{p-3}\pmod p\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}H_k\equiv-\frac{B_{p-3}}{4}\pmod p$$

by [ST, (5.4)] and (2.2) respectively. So we get

$$\sum_{n=1}^{p-1}\frac{H_{n,2}}{n2^n}\equiv-\frac{1}{2}\left(B_{p-3}-\frac{B_{p-3}}{4}\right)=-\frac{3}{8}B_{p-3}\pmod p.$$

Now we show (1.1). Observe that

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_k}{k2^k}=&\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{jk2^k}=\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{(p-k)(p-j)2^{p-j}}\\=&\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{2^{j-p}(p+j)(p+k)}{(p^2-j^2)(p^2-k^2)}\\\equiv&\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{2^{j-p}(jk+p(j+k))}{j^2k^2}\\\equiv&2^{-p}\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{2^j}{jk}+\frac{p}{2}\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{2^j(j+k)}{j^2k^2}\pmod{p^2}.\end{aligned}$$

In view of Lemmas 2.2 and 2.1,

$$\begin{aligned}\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{2^j-1}{jk}=&\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{jk}\sum_{i=1}^{j}\binom{j}{i}=\sum_{i=1}^{p-1}\frac{1}{i}\sum_{k=1}^{p-1}\frac{1}{k}\sum_{j=1}^{k}\binom{j-1}{i-1}\\=&\sum_{i=1}^{p-1}\frac{1}{i}\sum_{k=1}^{p-1}\frac{1}{k}\binom{k}{i}=\sum_{i=1}^{p-1}\frac{1}{i^2}\sum_{k=1}^{p-1}\binom{k-1}{i-1}\\=&\sum_{i=1}^{p-1}\frac{1}{i^2}\binom{p-1}{i}=\sum_{i=1}^{p-1}\frac{(-1)^i}{i^2}\prod_{r=1}^{i}\left(1-\frac{p}{r}\right)\\\equiv&\sum_{i=1}^{p-1}\frac{(-1)^i(1-pH_i)}{i^2}\equiv\frac{p}{2}B_{p-3}-p\left(-\frac{B_{p-3}}{4}\right)\pmod{p^2}.\end{aligned}$$

Note that

$$\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{jk}=\sum_{k=1}^{p-1}\frac{H_k}{k}\equiv\frac{p}{3}B_{p-3}\pmod{p^2}$$

by (2.2). Combining the above with (2.5), we finally obtain that

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_k}{k2^k}\equiv&2^{-p}\left(\frac{3}{4}pB_{p-3}+\frac{p}{3}B_{p-3}\right)+\frac{p}{2}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\\ \equiv&\frac{13}{24}pB_{p-3}+\frac{p}{2}\left(-\frac{B_{p-3}}{2}\right)=\frac{7}{24}pB_{p-3}\pmod{p^2}\ \ (\text{by (2.1)}).\end{aligned}$$

This concludes the proof. □

3. Proof of Theorem 1.2

Lemma 3.1. *Let $p>3$ be a prime and let m be a positive integer with $p-1\nmid 3m$. Then*

$$\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^mk^{2m}}+\frac{1}{j^{2m}k^m}\right)\equiv 0\pmod{p}. \tag{3.1}$$

Moreover, if $p>3m+1$, then

$$\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^mk^{2m}}+\frac{1}{j^{2m}k^m}\right)\equiv -p\frac{3m}{3m+1}B_{p-1-3m}\pmod{p^2}. \tag{3.2}$$

Proof. It is well-known that

$$\sum_{k=1}^{p-1}\frac{1}{k^n}\equiv 0\pmod{p}\quad\text{for any integer } n\not\equiv 0\pmod{p-1}.$$

Also,

$$\sum_{k=1}^{p-1}\frac{1}{k^n}\equiv\frac{pn}{n+1}B_{p-1-n}\pmod{p^2}\quad\text{for } n=1,\dots,p-2$$

(see, e.g., [S, Corollary 5.1]). Thus

$$\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^mk^{2m}}+\frac{1}{j^{2m}k^m}\right)=\sum_{j=1}^{p-1}\frac{1}{j^m}\sum_{k=1}^{p-1}\frac{1}{k^{2m}}-\sum_{k=1}^{p-1}\frac{1}{k^{3m}}\equiv 0\pmod{p}.$$

Moreover, we have (3.2) if $p>3m+1$. □

Lemma 3.2. *Let $p>3$ be a prime and let m be a positive even integer. Then*

$$\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^mk^{2m}}-\frac{1}{j^{2m}k^m}\right)\equiv 0 \pmod p. \tag{3.3}$$

Moreover, if $p>3m+1$ then

$$\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^mk^{2m}}-\frac{1}{j^{2m}k^m}\right)\equiv \frac{pm\binom{3m}{m}B_{p-1-3m}}{(m+1)(2m+1)} \pmod{p^2}. \tag{3.4}$$

Proof. As m is even, we have

$$\begin{aligned}\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^mk^{2m}}&=\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{(p-k)^m(p-j)^{2m}}\\&\equiv\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^{2m}k^m}\pmod p.\end{aligned}$$

Now suppose that $p>3m+1$. Then

$$\begin{aligned}\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^mk^{2m}}&=\sum_{1\leqslant j<k\leqslant p-1}\frac{(p+k)^m(p+j)^{2m}}{(p^2-k^2)^m(p^2-j^2)^{2m}}\\ \equiv&\sum_{1\leqslant j<k\leqslant p-1}\frac{(k^m+pmk^{m-1})(j^{2m}+p2mj^{2m-1})}{j^{4m}k^{2m}}\\ \equiv&\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^{2m}k^m}+pm\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^{2m}k^{m+1}}+\frac{2}{j^{2m+1}k^m}\right)\pmod{p^2}.\end{aligned}$$

So, (3.4) is reduced to

$$\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^{2m}k^{m+1}}+\frac{2}{j^{2m+1}k^m}\right)\equiv\frac{\binom{3m}{m}B_{p-1-3m}}{(m+1)(2m+1)}\pmod p. \tag{3.5}$$

Recall that for any integer n we have

$$\sum_{k=1}^{p-1}k^n\equiv\begin{cases}p-1 \pmod p & \text{if } p-1\mid n,\\ 0 \pmod p & \text{if } p-1\nmid n.\end{cases}$$

(See, e.g., [IR, p.235].) Also,

$$\sum_{j=0}^{k-1}j^n=\frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}B_jk^{n+1-j}$$

for any $k = 1, 2, 3\ldots$ and $n = 0, 1, 2, \ldots$. (See, e.g., [IR, p. 230].) Therefore

$$\begin{aligned}
&\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^{2m}k^{m+1}}\\
\equiv&\sum_{k=1}^{p-1}\frac{1}{k^{m+1}}\sum_{j=0}^{k-1}j^{p-1-2m}=\sum_{k=1}^{p-1}\frac{1}{k^{m+1}(p-2m)}\sum_{j=0}^{p-1-2m}\binom{p-2m}{j}B_jk^{p-2m-j}\\
\equiv&-\frac{1}{2m}\sum_{j=0}^{p-1-2m}\binom{p-2m}{j}B_j\sum_{k=1}^{p-1}k^{p-1-3m-j}\\
\equiv&\frac{1}{2m}\sum_{\substack{j=0\\ p-1\mid j+3m}}^{p-1-2m}\binom{p-2m}{j}B_j=\frac{1}{2m}\binom{p-2m}{m+1}B_{p-1-3m}\\
\equiv&\frac{1}{2m}\binom{-2m}{m+1}B_{p-1-3m}=\frac{(-1)^{m+1}}{2m}\binom{3m}{m+1}B_{p-1-3m}\pmod p.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^{2m+1}k^{m}}\\
\equiv&\sum_{k=1}^{p-1}\frac{1}{k^{m}}\sum_{j=0}^{k-1}j^{p-2-2m}=\sum_{k=1}^{p-1}\frac{1}{k^{m}(p-1-2m)}\sum_{j=0}^{p-2-2m}\binom{p-1-2m}{j}B_jk^{p-1-2m-j}\\
\equiv&-\frac{1}{2m+1}\sum_{j=0}^{p-2-2m}\binom{p-1-2m}{j}B_j\sum_{k=1}^{p-1}k^{p-1-3m-j}\\
\equiv&\frac{1}{2m+1}\sum_{\substack{j=0\\ p-1\mid j+3m}}^{p-2-2m}\binom{p-1-2m}{j}B_j=\frac{1}{2m+1}\binom{p-1-2m}{m}B_{p-1-3m}\\
\equiv&\frac{1}{2m+1}\binom{-1-2m}{m}B_{p-1-3m}=\frac{(-1)^{m}}{2m+1}\binom{3m}{m}B_{p-1-3m}\pmod p.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^{2m}k^{m+1}}+\frac{2}{j^{2m+1}k^{m}}\right)\\
\equiv&\left(\frac{(-1)^{m+1}}{2m}\binom{3m}{m+1}+2\frac{(-1)^m}{2m+1}\binom{3m}{m}\right)B_{p-1-3m}\\
=&\frac{(-1)^m}{(m+1)(2m+1)}\binom{3m}{m}B_{p-1-3m}\pmod p.
\end{aligned}$$

So (3.5) holds as m is even. $\square$

Proof of Theorem 1.2. Let $m = 2n$. Clearly

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_{k,m}^2}{k^m} =&\sum_{k=1}^{p-1}\frac{1}{k^m}\bigg(\sum_{j=1}^{k}\frac{1}{j^m}\bigg)^2\\ =&\sum_{k=1}^{p-1}\frac{1}{k^m}\bigg(\sum_{j=1}^{k}\frac{1}{j^{2m}}+2\sum_{1\leqslant i<j\leqslant k}\frac{1}{i^mj^m}\bigg)\\ =&H_{p-1,3m}+\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^{2m}k^m}+2\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^mj^{2m}}\\ &+2\sum_{1\leqslant i<j<k\leqslant p-1}\frac{1}{i^mj^mk^m}\end{aligned}$$

and

$$\begin{aligned}H_{p-1,m}^3 =&\sum_{i=1}^{p-1}\frac{1}{i^m}\bigg(\sum_{k=1}^{p-1}\frac{1}{k^{2m}}+2\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^mk^m}\bigg)\\ =&H_{p-1,3m}+3\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^{2m}k^m}+\frac{1}{j^mk^{2m}}\right)+6\sum_{1\leqslant i<j<k\leqslant p-1}\frac{1}{i^mj^mk^m}.\end{aligned}$$

As $H_{p-1,m}\equiv 0 \pmod p$, from the above we obtain

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_{k,m}^2}{k^m} \equiv&H_{p-1,3m}+\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^{2m}k^m}+\frac{2}{j^mk^{2m}}\right)\\ &-\frac{H_{p-1,3m}}{3}-\sum_{1\leqslant j<k\leqslant p-1}\left(\frac{1}{j^{2m}k^m}+\frac{1}{j^mk^{2m}}\right)\\ =&\frac{2}{3}H_{p-1,3m}+\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^mk^{2m}} \pmod{p^2}.\end{aligned}$$

Thus, by (3.1), (3.3) and the congruence $H_{p-1,3m}\equiv 0 \pmod p$, we immediately get (1.3).

Below we assume that $p > 3m + 1$. Adding (3.2) and (3.4) we obtain

$$\begin{aligned}2\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^mk^{2m}} \equiv&pmB_{p-1-3m}\left(-\frac{3}{3m+1}+\frac{\binom{3m}{m}}{(m+1)(2m+1)}\right)\\ =&\frac{pm}{3m+1}\left(\frac{\binom{3m+1}{m}}{m+1}-3\right)B_{p-1-3m} \pmod{p^2}.\end{aligned}$$

Note also that

$$H_{p-1-3m} \equiv p\frac{3m}{3m+1}B_{p-1-3m} \pmod{p^2}.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_{k,m}^2}{k^m} \equiv &\frac{2}{3}\cdot p\frac{3m}{3m+1}B_{p-1-3m} + \left(\frac{\binom{3m+1}{m}}{m+1} - 3\right)\frac{pm/2}{3m+1}B_{p-1-3m} \\ =&\left(\frac{\binom{3m+1}{m}}{m+1} + 1\right)\frac{pm/2}{3m+1}B_{p-1-3m} \\ =&\left(\binom{3m+1}{m-1} + \frac{m}{2}\right)\frac{pB_{p-1-3m}}{3m+1} \pmod{p^2}.\end{aligned}$$

This proves (1.4).

So far we have completed the proof of Theorem 1.2. □

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