Adv. in Appl. Math. 51(2013), no. 4, 524–535.

CONGRUENCES FOR FRANEL NUMBERS

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Abstract. The Franel numbers given by $f_n = \sum_{k=0}^{n}\binom{n}{k}^3$ $(n = 0, 1, 2, \ldots)$ play important roles in both combinatorics and number theory. In this paper we initiate the systematic investigation of fundamental congruences for the Franel numbers. We mainly establish for any prime $p > 3$ the following congruences:

$$\sum_{k=0}^{p-1}(-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2},\quad \sum_{k=0}^{p-1}(-1)^k k f_k \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_k \equiv 0 \pmod{p^2},\quad \sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}f_k \equiv 0 \pmod{p}.$$

1. Introduction

In 1894, Franel [F] noted that the numbers

$$f_n = \sum_{k=0}^{n}\binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots) \tag{1.1}$$

(cf. [Sl, A000172]) satisfy the recurrence relation:

$$(n+1)^2 f_{n+1} = (7n^2 + 7n + 2)f_n + 8n^2 f_{n-1} \quad (n = 1, 2, 3, \ldots). \tag{1.2}$$

Such numbers are now called Franel numbers. For a combinatorial interpretation of the Franel numbers, see Callan [C]. Recall that the Apéry numbers given by

$$A_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2 \ (n = 0, 1, 2, \ldots)$$

2010 *Mathematics Subject Classification.* Primary 11A07, 11B65; Secondary 05A10, 11B37, 11B75.

Keywords. Franel numbers, Apéry numbers, binomial coefficients, congruences.

Supported by the National Natural Science Foundation (grant 11171140) of China and the PAPD of Jiangsu Higher Education Institutions.

were introduced by Apéry [A], and they can be expressed in terms of Franel numbers as follows:

$$A_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} f_k \tag{1.3}$$

(see Strehl [St92]). The Franel numbers are also related to the theory of modular forms, see, e.g., Zagier [Z].

In this paper we study congruences for the Franel numbers systematically. As usual, for any odd prime p and integer a, $(\frac{a}{p})$ denotes the Legendre symbol, and $q_p(a)$ stands for the Fermat quotient $(a^{p-1}-1)/p$ if $p \nmid a$.

Now we state our main result.

Theorem 1.1. *Let $p>3$ be a prime. For any p-adic integer r we have*

$$\sum_{k=0}^{p-1}(-1)^k\binom{k+r}{k} f_k \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{k+r}{k}^2 \pmod{p^2}. \tag{1.4}$$

In particular,

$$\sum_{k=0}^{p-1}(-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \tag{1.5}$$

$$\sum_{k=0}^{p-1}(-1)^k k f_k \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2}, \tag{1.6}$$

$$\sum_{k=0}^{p-1}(-1)^k k^2 f_k \equiv \frac{10}{27}\left(\frac{p}{3}\right) \pmod{p^2}, \tag{1.7}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k} f_k}{(-4)^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k} \pmod{p^2}. \tag{1.8}$$

We also have

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k} f_k \equiv 0 \pmod{p^2}, \tag{1.9}$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2} f_k \equiv 0 \pmod{p}, \tag{1.10}$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k} f_{k-1} \equiv 3q_p(2) + 3p\,q_p(2)^2 \pmod{p^2}, \tag{1.11}$$

Remark 1.1. Fix a prime $p > 3$. In contrast with (1.5), we conjecture that

$$\sum_{n=0}^{p-1}(-1)^n\sum_{k=0}^{n}\binom{n}{k}^3(-8)^k \equiv \sum_{k=0}^{p-1}\frac{f_k}{8^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

As $f_k \equiv (-8)^k f_{p-1-k} \pmod p$ for all $k = 0, \ldots, p-1$ by [JV, Lemma 2.6], (1.11) implies that

$$\sum_{k=1}^{p-1}\frac{f_k}{k8^k} \equiv \sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_{p-1-k} = \sum_{k=1}^{p-1}\frac{(-1)^{p-k}}{p-k}f_{k-1} \equiv 3q_p(2) \pmod p.$$

Motivated by (1.5) and (1.6), we conjecture that both $(\sum_{k=0}^{n-1}(-1)^k f_k)/n^2$ and $(\sum_{k=0}^{n-1}(-1)^k k f_k)/n^2$ are 3-adic integers for any positive integer n. Concerning (1.8) the author [S11, Conj. 5.2(ii)] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p = x^2+3y^2\ (x, y \in \mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3.\end{cases}$$

See also [S13] for other connections between $p = x^2 + 3y^2$ and Franel numbers. (1.10) can be extended as

$$\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}f_k^{(r)} \equiv 0 \pmod p, \tag{1.12}$$

where r is any positive integer and $f_k^{(r)} := \sum_{j=0}^{k}\binom{k}{j}^r$. Note that $f_k^{(2)} = \binom{2k}{k}$ and $\sum_{k=1}^{p-1}\binom{2k}{k}/k \equiv 0 \pmod{p^2}$ by [ST10].

Let $p > 3$ be a prime. Similar to (1.5)-(1.7), we are also able to show that

$$\sum_{k=0}^{p-1}(-1)^k k^3 f_k \equiv -\frac{10}{81}\left(\frac{p}{3}\right) \pmod{p^2} \text{ and } \sum_{k=0}^{p-1}(-1)^k k^4 f_k \equiv -\frac{14}{243}\left(\frac{p}{3}\right) \pmod{p^2}.$$

In general, for any positive integer r and prime $p > \max\{r, 3\}$ there should be an odd integer a_r (not dependent on p) such that

$$\sum_{k=0}^{p-1}(-1)^k k^r f_k \equiv \frac{2a_r}{3^{2r-1}}\left(\frac{p}{3}\right) \pmod{p^2}.$$

2. Proof of Theorem 1.1

We first establish an auxiliary theorem on the polynomials

$$f_n(x) := \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}x^k = \sum_{k=0}^{n}\binom{n}{k}\binom{k}{n-k}\binom{2k}{k}x^k \quad (n=0,1,2,\ldots).$$

Theorem 2.1. *Let p be an odd prime and let r be any p-adic integer. Then*

$$\sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}f_l(x) \equiv \sum_{k=0}^{p-1}\binom{2k}{k}x^k\binom{k+r}{k}^2 \pmod{p^2}. \tag{2.1}$$

Proof. Observe that

$$\begin{aligned}\sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}f_l(x) =&\sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}\sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2k}{k}x^k\\ =&\sum_{k=0}^{p-1}\binom{2k}{k}x^k\sum_{l=k}^{\min\{2k,p-1\}}(-1)^l\binom{l}{k}\binom{k}{l-k}\binom{l+r}{l}.\end{aligned}$$

If $(p-1)/2 < k \leqslant p-1$ and $p \leqslant l \leqslant 2k$, then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod p \quad\text{and}\quad \binom{l}{k} = \frac{l!}{k!(l-k)!} \equiv 0 \pmod p.$$

Thus

$$\sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}f_l(x) \equiv \sum_{k=0}^{p-1}\binom{2k}{k}x^k\sum_{l=k}^{2k}(-1)^l\binom{l}{k}\binom{k}{l-k}\binom{l+r}{l} \pmod{p^2},$$

and hence it suffices to show the identity

$$\sum_{l=k}^{2k}(-1)^l\binom{l}{k}\binom{k}{l-k}\binom{x+l}{l} = \binom{x+k}{k}^2. \tag{2.2}$$

By the well-known Chu-Vandermonde identity (cf. (3.1) of [G, p.22]),

$$\sum_{j=0}^{k}\binom{y}{j}\binom{z}{k-j} = \binom{y+z}{k}.$$

Therefore

$$\begin{aligned}&\sum_{l=k}^{2k}(-1)^l\binom{l}{k}\binom{k}{l-k}\binom{x+l}{l}\\ =&\sum_{l=k}^{2k}\binom{l}{k}\binom{k}{l-k}\binom{-x-1}{l} = \binom{-x-1}{k}\sum_{l=k}^{2k}\binom{-x-1-k}{l-k}\binom{k}{l-k}\\ =&\binom{-x-1}{k}\sum_{j=0}^{k}\binom{-x-1-k}{j}\binom{k}{k-j} = \binom{-x-1}{k}^2 = \binom{x+k}{k}^2.\end{aligned}$$

This proves (2.2) and hence (2.1) follows. □

Lemma 2.1. *For any nonnegative integer n, the integer $f_n(1)$ coincides with the Franel number f_n.*

Proof. The identity $\sum_{k=0}^n \binom{n}{k}^2\binom{2k}{n} = f_n$ is a known result due to Strehl [St94]. □

Lemma 2.2. *For each positive integer m we have*

$$\sum_{k=0}^{n-1} P_m(k)\binom{2k}{k} = n^m\binom{2n}{n} \quad \text{for all } n = 1, 2, 3, \dots,$$

where $P_m(x) := 2(2x+1)(x+1)^{m-1} - x^m$.

Proof. The desired result follows immediately by induction on n. □

Lemma 2.3. *Let m be a positive integer. For $n = 0, 1, \dots, m$ we have*

$$\sum_{k=0}^{n}\binom{x}{k}\binom{-x}{m-k} = \frac{m-n}{m}\binom{x-1}{n}\binom{-x}{m-n}.$$

Remark 2.1. This is a known result due to Andersen, see, e.g., (3.14) of [G, p. 23].

Lemma 2.4 ([S11, Lemma 2.1])**.** *Let p be an odd prime. For any $k = 1, \dots, p-1$ we have*

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p\rfloor - 1}2p \pmod{p^2}.$$

Recall that the harmonic numbers and the second-order harmonic numbers are given by

$$H_n = \sum_{0<k\leqslant n}\frac{1}{k} \text{ and } H_n^{(2)} = \sum_{0<k\leqslant n}\frac{1}{k^2} \quad (n = 0, 1, 2, \dots)$$

respectively. Let $p > 3$ be a prime. In 1862, Wolstenholme [W] proved that

$$H_{p-1} \equiv 0 \pmod{p^2} \text{ and } H_{p-1}^{(2)} \equiv 0 \pmod{p}.$$

Note that

$$H_{(p-1)/2}^{(2)} \equiv \frac{1}{2}\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2} + \frac{1}{(p-k)^2}\right) = \frac{1}{2}H_{p-1}^{(2)} \equiv 0 \pmod{p}.$$

In 1938, Lehmer [L] showed that

$$H_{(p-1)/2} \equiv -2q_p(2) + p\,q_p(2)^2 \pmod{p^2}. \tag{2.3}$$

Lemma 2.5. *Let $p>3$ be a prime. Then*

$$f_{p-1}\equiv 1+3p\,q_p(2)+3p^2q_p(2)^2 \pmod{p^3}. \tag{2.4}$$

Proof. For any $k=1,\ldots,p-1$, we obviously have

$$\begin{aligned}(-1)^k\binom{p-1}{k}&=\prod_{j=1}^{k}\left(1-\frac{p}{j}\right)\\ \equiv&1-pH_k+\frac{p^2}{2}\sum_{1\leqslant i<j\leqslant k}\frac{2}{ij}=1-pH_k+\frac{p^2}{2}\left(H_k^2-H_k^{(2)}\right)\pmod{p^3}.\end{aligned}$$

Thus

$$\begin{aligned}f_{p-1}-1=&\sum_{k=1}^{p-1}\binom{p-1}{k}^3\equiv\sum_{k=1}^{p-1}(-1)^k\left(1-pH_k+\frac{p^2}{2}\left(H_k^2-H_k^{(2)}\right)\right)^3\\ \equiv&-3p\sum_{k=1}^{p-1}(-1)^kH_k+\frac{9}{2}p^2\sum_{k=1}^{p-1}(-1)^kH_k^2-\frac{3}{2}p^2\sum_{k=1}^{p-1}(-1)^kH_k^{(2)}\pmod{p^3}.\end{aligned}$$

Clearly

$$\begin{aligned}\sum_{k=1}^{p-1}(-1)^kH_k=&\sum_{k=1}^{p-1}\sum_{j=1}^{k}\frac{(-1)^k}{j}=\sum_{j=1}^{p-1}\frac{\sum_{k=j}^{p-1}(-1)^k}{j}=\sum_{\substack{j=1\\ 2\mid j}}^{p-1}\frac{1}{j}\\ =&\frac{1}{2}H_{(p-1)/2}\equiv -q_p(2)+\frac{p}{2}q_p(2)^2\pmod{p^2}\quad(\text{by }(2.3))\end{aligned}$$

and

$$\sum_{k=1}^{p-1}(-1)^kH_k^{(2)}=\sum_{j=1}^{p-1}\frac{\sum_{k=j}^{p-1}(-1)^k}{j^2}=\sum_{i=1}^{(p-1)/2}\frac{1}{(2i)^2}=\frac{H_{(p-1)/2}^{(2)}}{4}\equiv 0\pmod{p}.$$

Observe that

$$\begin{aligned}\sum_{k=1}^{p-1}(-1)^kH_k^2=&\sum_{k=1}^{p-1}(-1)^{p-k}H_{p-k}^2=\sum_{k=1}^{p-1}(-1)^{k-1}\Bigg(H_{p-1}-\sum_{0<j<k}\frac{1}{p-j}\Bigg)^2\\ \equiv&-\sum_{k=1}^{p-1}(-1)^k\left(H_k-\frac{1}{k}\right)^2\\ =&-\sum_{k=1}^{p-1}(-1)^kH_k^2+2\sum_{k=1}^{p-1}\frac{(-1)^k}{k}H_k-\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\pmod{p}.\end{aligned}$$

Clearly,

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\equiv\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k^2}=\sum_{j=1}^{(p-1)/2}\frac{2}{(2j)^2}\equiv 0 \pmod p,$$

and

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}H_k=\sum_{\substack{k=1\\ 2\mid k}}^{p-1}\frac{H_k}{k}-\sum_{\substack{k=1\\ 2\nmid k}}^{p-1}\frac{H_k}{k}\equiv\frac{q_p(2)^2}{2}-\left(-\frac{q_p(2)^2}{2}\right) \pmod p$$

by [S12a, Lemma 2.3]. Therefore

$$\sum_{k=1}^{p-1}(-1)^kH_k^2\equiv\sum_{k=1}^{p-1}\frac{(-1)^k}{k}H_k\equiv q_p(2)^2 \pmod p.$$

Combining the above, we finally obtain

$$f_{p-1}-1\equiv -3p\left(-q_p(2)+\frac{p}{2}q_p(2)^2\right)+\frac{9}{2}p^2q_p(2)^2 \pmod{p^3}$$

and hence (2.4) holds. $\square$

Lemma 2.6. *Let p be any prime. Then*

$$\binom{p-1}{k}\binom{p+k}{k}\equiv(-1)^k \pmod{p^2}\quad \textit{for } k=0,1,\ldots,p-1,$$

and

$$\binom{2k}{k}\sum_{n=k}^{p-1}(2n+1)\binom{n+k}{2k}\equiv p^2\,\frac{(-1)^k}{k+1} \pmod{p^4}\quad \textit{for } k=0,\ldots,p-2.$$

Proof. Let $k\in\{0,1,\ldots,p-1\}$. Clearly

$$\binom{p-1}{k}\binom{p+k}{k}=\prod_{0<j\leqslant k}\left(\frac{p-j}{j}\cdot\frac{p+j}{j}\right)\equiv(-1)^k \pmod{p^2}.$$

In view of the known identity $\sum_{n=0}^m\binom{n}{l}=\binom{m+1}{l+1}$ $(l,m=0,1,\ldots)$ (see, e.g., (1.52) of [G, p. 7]) which can be easily proved by induction, we have

$$\begin{aligned}\sum_{n=k}^{p-1}\frac{2n+1}{2k+1}\binom{n+k}{2k}=&\sum_{n=k}^{p-1}\left(\frac{2(n+k+1)}{2k+1}-1\right)\binom{n+k}{2k}\\ =&2\sum_{n=k}^{p-1}\binom{n+k+1}{2k+1}-\sum_{n=k}^{p-1}\binom{n+k}{2k}\\ =&2\binom{p+k+1}{2k+2}-\binom{p+k}{2k+1}=\frac{p}{k+1}\binom{p+k}{2k+1}\end{aligned}$$

and hence

$$\binom{2k}{k}\sum_{n=k}^{p-1}(2n+1)\binom{n+k}{2k}=p\frac{2k+1}{k+1}\binom{2k}{k}\binom{p+k}{2k+1}=\frac{p^2}{k+1}\binom{p-1}{k}\binom{p+k}{k}.$$

Thus, if $k<p-1$ then

$$\binom{2k}{k}\sum_{n=k}^{p-1}(2n+1)\binom{n+k}{2k}\equiv\frac{p^2}{k+1}(-1)^k \pmod{p^4}$$

as desired. □

Proof of Theorem 1.1. In view of Lemma 2.1, (2.1) with $x=1$ gives (1.4).

(2.1) with $r=0$ yields the congruence

$$\sum_{k=0}^{p-1}(-1)^k f_k(x)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}x^k \pmod{p^2}.$$

In the case $x=1$, this gives (1.5) since $\sum_{k=0}^{p-1}\binom{2k}{k}\equiv(\frac{p}{3}) \pmod{p^2}$ by [ST11, (1.9)].

By (2.1) with $r=0,1$,

$$\begin{aligned}&\sum_{k=0}^{p-1}(3(k+1)-1)(-1)^k f_k(x)\\ \equiv&\sum_{k=0}^{p-1}\binom{2k}{k}x^k\left(3(k+1)^2-1\right)=\sum_{k=0}^{p-1}P_2(k)\binom{2k}{k}x^k \pmod{p^2}\end{aligned}$$

where $P_2(x)=2(2x+1)(x+1)-x^2=3x^2+6x+2$. Thus, with the help of Lemmas 2.1-2.2, we have

$$\sum_{k=0}^{p-1}(3k+2)(-1)^k f_k\equiv 0 \pmod{p^2} \tag{2.5}$$

and hence (1.6) holds in view of (1.5).

Taking $r=2$ in (2.1) we get

$$2\sum_{k=0}^{p-1}(k^2+3k+2)(-1)^k f_k(x)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}x^k((k+1)(k+2))^2 \pmod{p^2}.$$

In view of (2.5), this yields

$$2\sum_{k=0}^{p-1}(-1)^k k^2 f_k\equiv\sum_{k=0}^{p-1}\binom{2k}{k}(k^2+3k+2)^2 \pmod{p^2}.$$

Note that

$$27(k^2+3k+2)^2 = 9P_4(k)+12P_3(k)+23P_2(k)+20$$

where $P_m(x)$ is given by Lemma 2.2. Therefore, with the help of Lemma 2.3 and [ST11, (1.9)], we have

$$54\sum_{k=0}^{p-1}(-1)^k k^2 f_k \equiv \sum_{k=0}^{p-1}(9P_4(k)+12P_3(k)+23P_2(k)+20)\binom{2k}{k} \equiv 20\left(\frac{p}{3}\right) \pmod{p^2}$$

and hence (1.7) follows.

Putting $r=-1/2$ in (2.1) and noting that $\binom{k-1/2}{k}=\binom{2k}{k}/4^k$, we then obtain

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k(x)}{(-4)^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}x^k \pmod{p^2}. \tag{2.6}$$

In the case $x=1$ this gives (1.8).

Now we prove (1.9). Observe that

$$\sum_{l=1}^{p-1}\frac{(-1)^l}{l}\sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2k}{k}x^k = \sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}x^k\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k}.$$

If $1 \leqslant k \leqslant (p-1)/2$, then

$$\begin{aligned}\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k} &= \sum_{l=k}^{2k}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k}\\ &= \sum_{j=0}^{k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j}\\ &= (-1)^k\sum_{j=0}^{k}\binom{-k}{j}\binom{k}{k-j} = (-1)^k\binom{0}{k} = 0\end{aligned}$$

by the Chu-Vandermonde identity. If $(p+1)/2 \leqslant k \leqslant p-1$, then

$$\begin{aligned}\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k} &= \sum_{j=0}^{p-1-k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j}\\ &= (-1)^k\sum_{j=0}^{p-1-k}\binom{-k}{j}\binom{k}{k-j}\end{aligned}$$

and hence applying Lemma 2.3 we get

$$\begin{aligned}
&\sum_{l=k}^{p-1}(-1)^l\binom{l-1}{k-1}\binom{k}{l-k}\\
=&(-1)^k\frac{k-(p-1-k)}{k}\binom{-k-1}{p-1-k}\binom{k}{k-(p-1-k)}\\
=&(-1)^{p-1}\left(\frac{p-k}{k}\right)^2\binom{p-1}{k-1}\binom{k}{p-k}\\
\equiv&(-1)^{k-1}\binom{k}{p-k}=\binom{p-2k-1}{p-k}\\
\equiv&\binom{2(p-k)-1}{p-k}=\frac{1}{2}\binom{2(p-k)}{p-k}\ (\mathrm{mod}\ p).
\end{aligned}$$

Note that $\binom{2k}{k}\equiv 0\ (\mathrm{mod}\ p)$ for $k=(p+1)/2,\ldots,p-1$. By the above,

$$\sum_{l=1}^{p-1}\frac{(-1)^l}{l}f_l(x)\equiv\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}}{k}x^k\frac{\binom{2(p-k)}{p-k}}{2}\equiv p\sum_{k=(p+1)/2}^{p-1}\frac{x^k}{k^2}\pmod{p^2}\quad(2.7)$$

with the help of Lemma 2.4. Hence (1.9) follows from (2.7) in the case $x=1$ since

$$2\sum_{k=(p+1)/2}^{p-1}\frac{1}{k^2}\equiv\sum_{k=(p+1)/2}^{p-1}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=H_{p-1}^{(2)}\equiv 0\pmod p.$$

Instead of proving (1.10) we show its extension (1.12). Clearly,

$$\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}=\sum_{k=1}^{(p-1)/2}\left(\frac{(-1)^{kr}}{k^{r-1}}+\frac{(-1)^{(p-k)r}}{(p-k)^{r-1}}\right)\equiv 0\pmod p.$$

Thus

$$\begin{aligned}
\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}f_l^{(r)}\equiv&\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}\sum_{k=1}^{l}\binom{l}{k}^r=\sum_{k=1}^{p-1}\frac{1}{k^{r-1}}\sum_{l=k}^{p-1}(-1)^{lr}\binom{l-1}{k-1}^{r-1}\binom{l}{k}\\
=&\sum_{k=1}^{p-1}\frac{1}{k^{r-1}}\sum_{j=0}^{p-1-k}(-1)^{(k+j)r}\binom{k+j-1}{j}^{r-1}\binom{k+j}{j}\\
=&\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\sum_{j=0}^{p-1-k}\binom{-k}{j}^{r-1}\binom{-k-1}{j}\\
\equiv&\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\sum_{j=0}^{p-k-1}\binom{p-k}{j}^{r-1}\binom{p-k-1}{j}\ (\mathrm{mod}\ p).
\end{aligned}$$

For any positive integer n, we have

$$f_n^{(r)}=\sum_{k=0}^{n}\left(\frac{k}{n}+\frac{n-k}{n}\right)\binom{n}{k}^r=2\sum_{k=0}^{n}\frac{n-k}{n}\binom{n}{k}^r=2\sum_{k=0}^{n-1}\binom{n}{k}^{r-1}\binom{n-1}{k}.$$

Therefore,

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{(-1)^{lr}}{l^{r-1}}f_l^{(r)}\equiv&\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}\cdot\frac{f_{p-k}^{(r)}}{2}=\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{(p-k)r}f_k^{(r)}}{(p-k)^{r-1}}\\ \equiv&-\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{kr}}{k^{r-1}}f_k^{(r)}\pmod p\end{aligned}$$

and hence (1.12) follows.

Finally we show (1.11). By (1.3) and Lemma 2.6,

$$\begin{aligned}\frac{1}{p}\sum_{n=0}^{p-1}(2n+1)A_n=&\frac{1}{p}\sum_{n=0}^{p-1}(2n+1)\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}f_k\\ =&\frac{1}{p}\sum_{k=0}^{p-1}\binom{2k}{k}f_k\sum_{n=k}^{p-1}(2n+1)\binom{n+k}{2k}\\ \equiv&\frac{f_{p-1}}{p}\binom{2p-2}{p-1}(2p-1)+p\sum_{k=0}^{p-2}\frac{(-1)^kf_k}{k+1}\\ =&\binom{2p-1}{p-1}f_{p-1}-p\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_{k-1}\pmod{p^3}.\end{aligned}$$

Combining this with Wolstenholme's congruence $\binom{2p-1}{p-1}\equiv1\pmod{p^3}$ (cf. [W]) and [S12b, (1.6)] we obtain

$$\sum_{k=1}^{p-1}\frac{(-1)^kf_{k-1}}{k}\equiv\frac{f_{p-1}-1}{p}\equiv3q_p(2)+3p\,q_p(2)^2\pmod{p^2}$$

by Lemma 2.5. □

Acknowledgment. The author would like to thank Prof. Qing-Hu Hou at Nankai Univ. for his comments on the author's original version of Lemma 2.2.

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