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FIBONACCI NUMBERS MODULO CUBES OF PRIMES

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Abstract. Let p be an odd prime. It is well known that $F_{p-(\frac{p}{5})} \equiv 0 \pmod p$, where $\{F_n\}_{n\geqslant 0}$ is the Fibonacci sequence and $(-)$ is the Jacobi symbol. In this paper we show that if $p \neq 5$ then we may determine $F_{p-(\frac{p}{5})}$ mod p^3 in the following way:

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{p}{5}\right)\left(1+\frac{F_{p-(\frac{p}{5})}}{2}\right) \pmod{p^3}.$$

We also use Lucas quotients to determine $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}/m^k$ modulo p^2 for any integer $m \not\equiv 0 \pmod p$; in particular, we obtain

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p}\right) \pmod{p^2}.$$

In addition, we pose three conjectures for further research.

1. Introduction

The well known Fibonacci sequence $\{F_n\}_{n\geqslant 0}$, defined by

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_{n+1} = F_n + F_{n-1}\ (n = 1, 2, 3, \ldots),$$

plays an important role in many fields of mathematics. This sequence has nice number-theoretic properties; for example, E. Lucas showed that $(F_m, F_n) = F_{(m,n)}$ for any $m, n \in \mathbb{N} = \{0, 1, \ldots\}$, where (m, n) denotes the greatest common divisor of m and n.

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Let $p \neq 2, 5$ be a prime. It is known that $F_{p-(\frac{p}{5})} \equiv 0 \pmod p$, where $(-)$ denotes the Jacobi symbol. In 1992 Z. H. Sun and Z. W. Sun [SS] proved that if $p^2 \nmid F_{p-(\frac{p}{5})}$ then the Fermat equation $x^p + y^p = z^p$ has no integral solutions with $p \nmid xyz$. When $F_{p-(\frac{p}{5})} \equiv 0 \pmod{p^2}$, p is called a Wall-Sun-Sun prime (cf. [CDP] and [CP, p. 32]). It is conjectured that there should be infinitely many (but rare) Wall-Sun-Sun primes though none of them has been found. There are some congruences for the Fibonacci quotient $F_{p-(\frac{p}{5})}/p$ modulo p (cf. [W], [SS] and [ST]); for example, in 1982 H. C. Williams [W] proved that

$$\frac{F_{p-(\frac{p}{5})}}{p} \equiv \frac{2}{5} \sum_{k=1}^{\lfloor \frac{4}{5}p \rfloor} \frac{(-1)^k}{k} \pmod p.$$

Quite recently H. Pan and Z. W. Sun [PS] proved that for any $a \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ we have

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left(\frac{p^a}{5}\right) \left(1 - 2F_{p^a-(\frac{p^a}{5})}\right) \pmod{p^3},$$

which was a conjecture in [ST].

Now we give the first theorem of this paper.

Theorem 1.1. *Let p be an odd prime and let a be a positive integer. If $p \neq 5$, then*

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{p^a}{5}\right) \left(1 + \frac{F_{p^a-(\frac{p^a}{5})}}{2}\right) \pmod{p^3}. \tag{1.1}$$

If $p \neq 3$, then

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-32)^k} \equiv \left(\frac{2}{p^a}\right) \left(1 + \frac{2^{p^a-1}-1}{6} - \frac{(2^{p^a-1}-1)^2}{8}\right) \pmod{p^3}. \tag{1.2}$$

Let p be an odd prime and let $a \in \mathbb{Z}^+$. For $k = 0, 1, \ldots, (p^a-1)/2$, clearly

$$\frac{\binom{(p^a-1)/2}{k}}{\binom{-1/2}{k}} = \prod_{0 \leqslant j < k} \left(1 - \frac{p^a}{2j+1}\right) \equiv 1 \pmod p$$

and hence

$$\binom{(p^a-1)/2}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod p. \tag{1.3}$$

Thus, for any integer $m \not\equiv 0 \pmod p$ we have

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}}{m^k} \equiv \sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m(m-4)}{p^a}\right) \pmod p, \tag{1.4}$$

since

$$\sum_{k=0}^{(p^a-1)/2} \binom{(p^a-1)/2}{k} \left(-\frac{4}{m}\right)^k = \left(1-\frac{4}{m}\right)^{(p^a-1)/2}$$

and

$$\binom{2k}{k} = \binom{p^a + (2k-p^a)}{0p^a + k} \equiv \binom{2k-p^a}{k} = 0 \pmod p$$

for each $k = (p^a+1)/2, \ldots, p^a-1$ by Lucas' theorem (cf. [St, p. 44]). Recently the author [Su10] determined $\sum_{k=0}^{p^a-1} \binom{2k}{k}/m^k \bmod p^2$ in terms of Lucas sequences. See also [SSZ], [GZ] and [Su11a] for related results on p-adic valuations.

Let $A, B \in \mathbb{Z}$. The Lucas sequences $u_n = u_n(A,B)$ $(n \in \mathbb{N})$ and $v_n = v_n(A,B)$ $(n \in \mathbb{N})$ are defined by

$$u_0 = 0,\ u_1 = 1, \text{ and } u_{n+1} = Au_n - Bu_{n-1}\ (n = 1,2,3,\ldots)$$

and

$$v_0 = 2,\ v_1 = A, \text{ and } v_{n+1} = Av_n - Bv_{n-1}\ (n = 1,2,3,\ldots).$$

The sequence $\{v_n\}_{n\geqslant 0}$ is called the companion of $\{u_n\}_{n\geqslant 0}$. (Note that $F_n = u_n(1,-1)$, and those $L_n = v_n(1,-1)$ are called Lucas numbers.) It is known that for any prime p not dividing $2B$ we have

$$u_p \equiv \left(\frac{\Delta}{p}\right) \pmod p \quad \text{and} \quad u_{p-(\frac{\Delta}{p})} \equiv 0 \pmod p$$

where $\Delta = A^2 - 4B$ (see, e.g., [Su10, Lemma 2.3]); the integer $u_{p-(\frac{\Delta}{p})}/p$ is called a *Lucas quotient*. The reader may consult [Su06] for connections between Lucas quotients and quadratic fields.

Our second theorem is as follows.

Theorem 1.2. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. Let m be any integer not divisible by p. Then*

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m(m-4)}{p^a}\right) + \left(\frac{-m}{p}\right)\left(\frac{m(m-4)}{p^{a-1}}\right) \bar{m} u_{p-(\frac{4-m}{p})}(4,m) \pmod{p^2}, \tag{1.5}$$

where

$$\bar{m} = \begin{cases} 1 & \text{if } m \equiv 4 \pmod{p}, \\ 2 & \text{if } (\frac{4-m}{p}) = 1, \\ 2/m & \text{if } (\frac{4-m}{p}) = -1. \end{cases}$$

We also have

$$\sum_{k=0}^{(p^a-1)/2} \frac{C_k}{m^k} \equiv \frac{4-m}{2} \sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{m^k} + \frac{m}{2} - 2p\,\delta_{a,1}\left(\frac{-m}{p}\right) \pmod{p^2}, \quad (1.6)$$

where C_k *denotes the Catalan number* $\frac{1}{k+1}\binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1}$, *and the Kronecker symbol* $\delta_{s,t}$ *takes* 1 *or* 0 *according as* $s = t$ *or not.*

Remark 1.1. For any $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$, the sum $\sum_{k=0}^{n} k\binom{2k}{k}/m^k$ is closely related to $\sum_{k=0}^{n} \binom{2k}{k}/m^k$ via the identity

$$\sum_{k=0}^{n} \left(1 - \frac{m-4}{2}k\right) \frac{\binom{2k}{k}}{m^k} = (2n+1)\frac{\binom{2n}{n}}{m^n}$$

which can be easily proved by induction.

Now we present two consequences of Theorem 1.2.

Corollary 1.1. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. Then*

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{8^k} \equiv \left(\frac{2}{p^a}\right) \pmod{p^2} \quad (1.7)$$

and

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a}\right) \pmod{p^2}. \quad (1.8)$$

Corollary 1.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(2k-1)^2 16^k} \equiv \left(\frac{-1}{p}\right) \frac{3(\frac{p}{3})+1}{4} \pmod{p^2}, \quad (1.9)$$

that is,

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(2k-1)^2 16^k} \equiv \begin{cases} 1 \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ -1/2 \pmod{p^2} & \text{if } p \equiv 5 \pmod{12}, \\ -1 \pmod{p^2} & \text{if } p \equiv 7 \pmod{12}, \\ 1/2 \pmod{p^2} & \text{if } p \equiv 11 \pmod{12}. \end{cases} \quad (1.10)$$

We will show Theorems 1.1 and 1.2 in Sections 2 and 3 respectively. Section 4 is devoted to the proofs of Corollaries 1.1–1.2.

To conclude this section we pose three conjectures.

Conjecture 1.1. *For any $n \in \mathbb{N}$ we have*

$$\frac{1}{(2n+1)^2\binom{2n}{n}}\sum_{k=0}^{n}\frac{\binom{2k}{k}}{16^k} \equiv \begin{cases} 1 \pmod 9 & \text{if } 3 \mid n, \\ 4 \pmod 9 & \text{if } 3 \nmid n. \end{cases}$$

Also,

$$\frac{1}{3^{2a}}\sum_{k=0}^{(3^a-1)/2}\frac{\binom{2k}{k}}{16^k} \equiv (-1)^a 10 \pmod{27}$$

for every $a = 1, 2, 3, \ldots$.

Let $p > 3$ be a prime. In 2007 A. Adamchuk [A] conjectured that if $p \equiv 1 \pmod 3$ then

$$\sum_{k=1}^{\lfloor \frac{2}{3}p \rfloor}\binom{2k}{k} \equiv 0 \pmod{p^2}.$$

Motivated by this and Theorems 1.1 and 1.2, we pose the following conjecture based on the author's computation via the software `Mathematica`.

Conjecture 1.2. *Let p be an odd prime and let $a \in \mathbb{Z}^+$.*

(i) *If $p \equiv 1 \pmod 3$ or $a > 1$, then*

$$\sum_{k=0}^{\lfloor \frac{5}{6}p^a \rfloor}\frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a}\right) \pmod{p^2}.$$

(ii) *Suppose $p \neq 5$. If $p^a \equiv 1, 2 \pmod 5$ or $p \equiv 2 \pmod 5$ or $a > 2$, then*

$$\sum_{k=0}^{\lfloor \frac{4}{5}p^a \rfloor}(-1)^k\binom{2k}{k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

If $p^a \equiv 1, 3 \pmod 5$ or $p \equiv 3 \pmod 5$ or $a > 2$, then

$$\sum_{k=0}^{\lfloor \frac{3}{5}p^a \rfloor}(-1)^k\binom{2k}{k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

(iii) *If $p^a \equiv 1, 2 \pmod 5$ or $p \equiv 2 \pmod 5$ or $a > 2$, then*

$$\sum_{k=0}^{\lfloor \frac{7}{10}p^a \rfloor}\frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

If $p^a \equiv 1, 3 \pmod 5$ or $p \equiv 3 \pmod 5$ or $a > 2$, then

$$\sum_{k=0}^{\lfloor \frac{9}{10}p^a \rfloor}\frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

Conjecture 1.3. *Let $p\neq 2,5$ be a prime and set $q:=F_{p-(\frac{p}{5})}/p$. Then*

$$p\sum_{k=1}^{p-1}\frac{F_{2k}}{k^2\binom{2k}{k}}\equiv-\left(\frac{p}{5}\right)\left(\frac{3}{2}q+\frac{5}{4}p\,q^2\right)\pmod{p^2}$$

and

$$p\sum_{k=1}^{p-1}\frac{L_{2k}}{k^2\binom{2k}{k}}\equiv-\frac{5}{2}q-\frac{15}{4}p\,q^2\pmod{p^2}.$$

Remark 1.2. It is interesting to compare Conjecture 1.3 with the two identities

$$\sum_{k=1}^{\infty}\frac{F_{2k}}{k^2\binom{2k}{k}}=\frac{4\pi^2}{25\sqrt{5}}\quad\text{and}\quad\sum_{k=1}^{\infty}\frac{L_{2k}}{k^2\binom{2k}{k}}=\frac{\pi^2}{5}$$

obtained by putting $x=(\sqrt{5}\pm1)/2$ in the known formula

$$2\arcsin^2\frac{x}{2}=\sum_{k=1}^{\infty}\frac{x^{2k}}{k^2\binom{2k}{k}}\quad(|x|<2).$$

2. Proof of Theorem 1.1

Lemma 2.1. *Let p be an odd prime and let $k\in\{0,\ldots,(p^a-1)/2\}$ with $a\in\mathbb{Z}^+$. Then*

$$\begin{aligned}&\binom{(p^a-1)/2+k}{2k}-\frac{\binom{2k}{k}}{(-16)^k}\\ \equiv&(-1)^{k-1}\left(\frac{-1}{p^a}\right)\binom{p^a-1-2k}{(p^a-1)/2-k}\sum_{0<j\leqslant k}\frac{p^{2a}}{(2j-1)^2}\pmod{p^3}.\end{aligned}\tag{2.1}$$

Proof. Clearly (2.1) holds for $k=0$. Below we assume $1\leqslant k\leqslant(p^a-1)/2$. Note that

$$\begin{aligned}\binom{(p^a-1)/2+k}{2k}=&\frac{\prod_{j=1}^k(p^{2a}-(2j-1)^2)}{4^k(2k)!}\\ =&\frac{\prod_{j=1}^k(-(2j-1)^2)}{4^k(2k)!}\prod_{j=1}^k\left(1-\frac{p^{2a}}{(2j-1)^2}\right)\\ \equiv&\frac{\binom{2k}{k}}{(-16)^k}\left(1-\sum_{j=1}^k\frac{p^{2a}}{(2j-1)^2}\right)\pmod{p^4}\end{aligned}$$

(which was observed by Z. H. Sun [S11, Lemma 2.2] in the case $a = 1$). Thus, in view of (1.3) we have

$$\begin{aligned}\frac{\binom{2k}{k}}{(-16)^k} \equiv&\binom{(p^a-1)/2}{k}4^{-k} = \binom{(p^a-1)/2}{(p^a-1)/2-k}4^{-k}\\ \equiv&\binom{p^a-1-2k}{(p^a-1)/2-k}(-4)^{-((p^a-1)/2-k)}4^{-k}\\ \equiv&\left(\frac{-1}{p^a}\right)(-1)^k\binom{p^a-1-2k}{(p^a-1)/2-k} \pmod p.\end{aligned}$$

So (2.1) holds. $\square$

Lemma 2.2. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \neq 3$, then*

$$1+\frac{2^{p^a-1}-1}{6}+\frac{(2^{p^a-1}-1)^2}{24} \equiv \left(\frac{2}{p^a}\right)\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}} \pmod{p^3}. \quad (2.2)$$

When $p \neq 5$, we have

$$\frac{L_{p^a}-1}{5}-\left(\frac{p^a}{5}\right)F_{p^a}+1 \equiv -\frac{1}{2}F^2_{p^a-(\frac{p^a}{5})} \pmod{p^4}. \quad (2.3)$$

Proof. Note that

$$2^{(p^a-1)/2} = \left(2^{\frac{p-1}{2}}\right)^{\sum_{k=0}^{a-1}p^k} \equiv \left(\frac{2}{p}\right)^a = \left(\frac{2}{p^a}\right) \pmod p$$

and

$$\begin{aligned}\frac{2^{p^a-1}-1}{p} =&\frac{2^{(p^a-1)/2}-(\frac{2}{p^a})}{p}\left(2^{(p^a-1)/2}+\left(\frac{2}{p^a}\right)\right)\\ \equiv&2\left(\frac{2}{p^a}\right)\frac{2^{(p^a-1)/2}-(\frac{2}{p^a})}{p} \pmod p.\end{aligned}$$

Thus

$$\begin{aligned}(2^{p^a-1}-1)^2 \equiv&4\left(2^{(p^a-1)/2}-\left(\frac{2}{p^a}\right)\right)^2\\ =&4(2^{p^a-1}-1)+8-8\left(\frac{2}{p^a}\right)2^{(p^a-1)/2} \pmod{p^3}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{2^{(p^a-1)/2}}{2}\left(2^{p^a-1}-1\right)+\frac{1}{8}\left(\frac{2}{p^a}\right)\left(2^{p^a-1}-1\right)^2\\ \equiv&\frac{2^{(p^a-1)/2}}{8}\left(\left(2^{p^a-1}-1\right)^2-8+8\left(\frac{2}{p^a}\right)2^{(p^a-1)/2}\right)\\ &+\frac{1}{8}\left(\frac{2}{p^a}\right)\left(2^{p^a-1}-1\right)^2\\ \equiv&\frac{1}{4}\left(\frac{2}{p^a}\right)\left(2^{p^a-1}-1\right)^2-2^{(p^a-1)/2}+\left(\frac{2}{p^a}\right)2^{p^a-1}\\ \equiv&\left(\frac{2}{p^a}\right)\left(2^{p^a}+1\right)-3\times 2^{(p^a-1)/2}\pmod{p^3},\end{aligned}$$

which is equivalent to (2.2) times $3\times 2^{(p^a-1)/2}$. So (2.2) is valid if $p>3$.

Now assume that $p\neq 5$. As $L_{2n}=5F_n^2+2(-1)^n=L_n^2-2(-1)^n$ for all $n\in\mathbb{N}$, by [SS, Corollary 1] we have $L_{p-(\frac{5}{p})}\equiv 2(\frac{p}{5})\pmod{p^2}$. Thus, in view of [Su10, Lemma 2.3],

$$L_{p^a-(\frac{p^a}{5})}\equiv(-1)^{((\frac{5}{p})-(\frac{5}{p^a}))/2}L_{p-(\frac{5}{p})}\equiv 2\left(\frac{p^a}{5}\right)\pmod{p^2}.$$

Write $L_{p^a-(\frac{p^a}{5})}=2(\frac{p^a}{5})+p^2Q$ with $Q\in\mathbb{Z}$. Then

$$\begin{aligned}5F^2_{p^a-(\frac{p^a}{5})}=&L^2_{p^a-(\frac{p^a}{5})}-4(-1)^{p^a-(\frac{p^a}{5})}\\ =&-4+\left(2\left(\frac{p^a}{5}\right)+p^2Q\right)^2\equiv 4p^2\left(\frac{p^a}{5}\right)Q\pmod{p^4}.\end{aligned}$$

Note that

$$L_{p^a}=F_{p^a}+2F_{p^a-1}=2F_{p^a+1}-F_{p^a}=2F_{p^a-(\frac{p^a}{5})}+\left(\frac{p^a}{5}\right)F_{p^a}$$

and

$$2L_{p^a}=5F_{p^a-1}+L_{p^a-1}=5F_{p^a+1}-L_{p^a+1}=5F_{p^a-(\frac{p^a}{5})}+\left(\frac{p^a}{5}\right)L_{p^a-(\frac{p^a}{5})}.$$

Therefore

$$\begin{aligned}&\frac{L_{p^a}-1}{5}-\left(\frac{p^a}{5}\right)F_{p^a}+1\\ =&2F_{p^a-(\frac{p^a}{5})}-\frac{4}{5}(L_{p^a}-1)\\ =&2F_{p^a-(\frac{p^a}{5})}-\frac{4}{5}\left(\frac{5}{2}F_{p^a-(\frac{p^a}{5})}+\frac{1}{2}\left(\frac{p^a}{5}\right)L_{p^a-(\frac{p^a}{5})}-1\right)\\ =&\frac{4}{5}-\frac{2}{5}\left(\frac{p^a}{5}\right)\left(2\left(\frac{p^a}{5}\right)+p^2Q\right)\equiv-\frac{1}{2}F^2_{p^a-(\frac{p^a}{5})}\pmod{p^4}.\end{aligned}$$

This proves (2.3). □

The following Lemma was posed as [Su12, Conjecture 1.1].

Lemma 2.3. *Let p be an odd prime and let $H_k^{(2)} = \sum_{0<j\leqslant k} 1/j^2$ for $k = 0,1,2,\ldots$. If $p \neq 5$, then*

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}H_k^{(2)} \equiv \left(\frac{p}{5}\right)\frac{5}{2}q^2 - \delta_{p,3} \pmod{p}, \tag{2.4}$$

where q denotes the Fibonacci quotient $F_{p-(\frac{p}{5})}/p$. If $p > 3$, then

$$\sum_{k=0}^{p-1}(-2)^k\binom{2k}{k}H_k^{(2)} \equiv \frac{2}{3}q_p(2)^2 \pmod{p}, \tag{2.5}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Proof. It is easy to verify (2.4) for $p = 3$. Below we assume $p > 3$.

The desired congruences essentially follow from [MT, (37)]. Here we provide the details. Putting $t = -1, -1/2$ in [MT, (37)] we get

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}H_k^{(2)} \equiv -2\sum_{k=1}^{p-1}\frac{u_k(3,1)}{k^2} \pmod{p} \tag{2.6}$$

and

$$\sum_{k=0}^{p-1}(-2)^k\binom{2k}{k}H_k^{(2)} \equiv -2\sum_{k=1}^{p-1}\frac{u_k(5/2,1)}{k^2} \pmod{p}. \tag{2.7}$$

Note that $u_k(3,1) = F_{2k}$ and $u_k(5/2,1) = 2(2^k - 2^{-k})/3$ for all $k = 0,1,2,\ldots$.

Since

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2} \equiv -q_p(2)^2 \pmod{p} \quad\text{and}\quad \sum_{k=1}^{p-1}\frac{1}{k^2 2^k} \equiv -\frac{q_p(2)^2}{2} \pmod{p}$$

by [Gr] and [S08, Theorem 4.1(iv)] respectively, (2.7) implies that

$$\sum_{k=0}^{p-1}(-2)^k\binom{2k}{k}H_k^{(2)} \equiv -\frac{4}{3}\sum_{k=1}^{p-1}\left(\frac{2^k}{k^2} - \frac{1}{k^2 2^k}\right) \equiv \frac{2}{3}q_p(2)^2 \pmod{p}.$$

Now we work with $p > 5$. Recall that for any $n \in \mathbb{N}$ we have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad\text{and}\quad L_n = \alpha^n + \beta^n,$$

where α and β are the two roots of the equation $x^2-x-1=0$. By [PS, (3.2) and (3.7)],

$$\begin{aligned}2\beta^{2p}\sum_{k=1}^{p-1}\frac{\alpha^{2k}}{k^2}\equiv&-2\sum_{k=1}^{p-1}\frac{\alpha^k}{k^2}-\left(\frac{L_p-1}{p}\right)^2\\ \equiv&\left(\frac{2\alpha^p-1}{5}-1\right)\left(\frac{L_p-1}{p}\right)^2 \pmod p.\end{aligned}$$

Since $\alpha\beta=-1$ and $\alpha^{2p}=(\alpha+1)^p\equiv\alpha^p+1 \pmod p$, we have

$$\sum_{k=1}^{p-1}\frac{\alpha^{2k}}{k^2}\equiv\frac{(\alpha^p+1)(\alpha^p-3)}{5}\left(\frac{L_p-1}{p}\right)^2\equiv-\frac{\alpha^p+2}{5}\left(\frac{L_p-1}{p}\right)^2 \pmod p.$$

Hence

$$\begin{aligned}5\sum_{k=1}^{p-1}\frac{F_{2k}}{k^2}=&(\alpha-\beta)^2\sum_{k=1}^{p-1}\frac{F_{2k}}{k^2}=(\alpha-\beta)\sum_{k=1}^{p-1}\frac{\alpha^{2k}-\beta^{2k}}{k^2}\\ \equiv&(\alpha-\beta)\frac{\beta^p-\alpha^p}{5}\left(\frac{L_p-1}{p}\right)^2\equiv-\frac{(\alpha-\beta)^{p+1}}{5}\left(\frac{L_p-1}{p}\right)^2\\ =&-5^{(p-1)/2}\left(\frac{L_p-1}{p}\right)^2\equiv-\left(\frac{5}{p}\right)\frac{25}{4}q^2 \pmod p\end{aligned}$$

since $2(L_p-1)\equiv 5F_{p-(\frac{p}{5})} \pmod{p^2}$ by [ST, p. 139]. Combining this with (2.6) we obtain

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}H_k^{(2)}\equiv 2\times\left(\frac{5}{p}\right)\frac{5}{4}q^2=\left(\frac{p}{5}\right)\frac{5}{2}q^2 \pmod p.$$

The proof of Lemma 2.3 is now complete. □

Proof of Theorem 1.1. Let us first recall the following two identities:

$$F_{n+1}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\quad\text{and}\quad\sum_{k=0}^{n}\frac{\binom{n+k}{2k}}{2^k}=\frac{2^{2n+1}+1}{3\times 2^n}.$$

Thus we have

$$F_{p^a}=\sum_{k=0}^{(p^a-1)/2}\binom{p^a-1-k}{p^a-1-2k}=\sum_{j=0}^{(p^a-1)/2}\binom{(p^a-1)/2+j}{2j}$$

and

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{(p^a-1)/2+k}{2k}}{2^k}=\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}}.$$

Therefore, with the help of (2.1),

$$\begin{aligned}&\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-16)^k}-F_{p^a}\\ \equiv&\sum_{k=0}^{(p^a-1)/2}\binom{p^a-1-2k}{(p^a-1)/2-k}(-1)^{(p^a-1)/2-k}\sum_{0<j\leqslant k}\frac{p^{2a}}{(2j-1)^2}\\ =&\sum_{k=0}^{(p^a-1)/2}\binom{2k}{k}(-1)^k\sum_{0<j\leqslant (p^a-1)/2-k}\frac{p^{2a}}{(2j-1)^2}\pmod{p^3}\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-32)^k}-\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}}\\ \equiv&\sum_{k=0}^{(p^a-1)/2}\binom{2k}{k}\frac{(-1)^k}{2^{(p^a-1)/2-k}}\sum_{0<j\leqslant (p^a-1)/2-k}\frac{p^{2a}}{(2j-1)^2}\pmod{p^3}.\end{aligned}$$

For $k=0,\ldots,(p^a-1)/2$, clearly

$$\begin{aligned}\sum_{j=1}^{(p^a-1)/2-k}\frac{p^{2a}}{(2j-1)^2}=&\sum_{j=k+1}^{(p^a-1)/2}\frac{p^{2a}}{(2((p^a-1)/2-j+1)-1)^2}\\ \equiv&\sum_{j=1}^{(p^a-1)/2}\frac{p^{2a}}{4j^2}-\sum_{0<j\leqslant k}\frac{p^{2a}}{4j^2}\\ \equiv&\sum_{i=1}^{(p-1)/2}\frac{p^{2a}}{4(p^{a-1}i)^2}-\sum_{0<i\leqslant\lfloor k/p^{a-1}\rfloor}\frac{p^{2a}}{4(p^{a-1}i)^2}\pmod{p^3}.\end{aligned}$$

Since

$$2\sum_{i=1}^{(p-1)/2}\frac{1}{i^2}\equiv\sum_{i=1}^{(p-1)/2}\left(\frac{1}{i^2}+\frac{1}{(p-i)^2}\right)=\sum_{i=1}^{p-1}\frac{1}{i^2}\equiv 2\delta_{p,3}\pmod p$$

with the help of Wolstenholme's congruence (cf. [Wo] and [Z]), by the above we have

$$\begin{aligned}&\frac{-4}{p^2}\left(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-16)^k}-F_{p^a}\right)\\ \equiv&\sum_{k=0}^{p^a-1}\binom{2k}{k}(-1)^k\left(H^{(2)}_{\lfloor k/p^{a-1}\rfloor}-\delta_{p,3}\right)\pmod p\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}&\frac{-4}{p^2}\bigg(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-32)^k}-\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}}\bigg)\\&\equiv\left(\frac{2}{p^a}\right)\sum_{k=0}^{p^a-1}\binom{2k}{k}(-2)^kH^{(2)}_{\lfloor k/p^{a-1}\rfloor}\pmod p.\end{aligned}\tag{2.9}$$

Our next task is to simplify the right-hand sides of congruences (2.8) and (2.9). Let $u\in\{1,2\}$. Then

$$\begin{aligned}&\sum_{k=0}^{p^a-1}\binom{2k}{k}(-u)^kH^{(2)}_{\lfloor k/p^{a-1}\rfloor}\\=&\sum_{k=0}^{p-1}\sum_{r=0}^{p^{a-1}-1}\binom{2p^{a-1}k+2r}{p^{a-1}k+r}(-u)^{p^{a-1}k+r}H_k^{(2)}\\\equiv&\sum_{k=0}^{p-1}(-u)^kH_k^{(2)}\sum_{r=0}^{p^{a-1}-1}\binom{2p^{a-1}k+2r}{p^{a-1}k+r}(-u)^r\pmod p.\end{aligned}$$

For $k\in\{0,\ldots,p-1\}$ and $r\in\{0,\ldots,p^{a-1}-1\}$, by the Chu-Vandermonde identity (cf. [GKP, p. 169]) we have

$$\binom{2p^{a-1}k+2r}{p^{a-1}k+r}=\sum_{j=0}^{p^{a-1}k+r}\binom{2p^{a-1}k}{j}\binom{2r}{p^{a-1}k+r-j}.$$

If $p^{a-1}\nmid j$, then

$$\binom{2p^{a-1}k}{j}=\frac{2p^{a-1}k}{j}\binom{2p^{a-1}k-1}{j-1}\equiv 0\pmod p.$$

Thus

$$\begin{aligned}\binom{2p^{a-1}k+2r}{p^{a-1}k+r}\equiv&\sum_{j=0}^{k}\binom{2p^{a-1}k}{p^{a-1}j}\binom{2r}{p^{a-1}(k-j)+r}=\binom{2p^{a-1}k}{p^{a-1}k}\binom{2r}{r}\\\equiv&\binom{2k}{k}\binom{2r}{r}\pmod p\quad\text{(by Lucas' theorem)}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{k=0}^{p^a-1}\binom{2k}{k}(-u)^kH^{(2)}_{\lfloor k/p^{a-1}\rfloor}\\\equiv&\sum_{k=0}^{p-1}(-u)^kH_k^{(2)}\binom{2k}{k}\sum_{r=0}^{p^{a-1}-1}\binom{2r}{r}(-u)^r\pmod p.\end{aligned}\tag{2.10}$$

In view of (1.4),

$$\sum_{r=0}^{p^{a-1}-1}\binom{2r}{r}(-1)^r\equiv\left(\frac{5}{p^{a-1}}\right)\pmod p,$$

and also

$$\sum_{r=0}^{p^{a-1}-1}\binom{2r}{r}(-2)^r\equiv\left(\frac{(p-1)/2\times((p-1)/2-4)}{p^{a-1}}\right)=1\pmod p$$

provided $p\neq 3$. Combining this with (2.8) and (2.10), we obtain

$$\begin{aligned}&\frac{-4}{p^2}\bigg(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-16)^k}-F_{p^a}\bigg)\\ \equiv&\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^kH_k^{(2)}\sum_{r=0}^{p^{a-1}-1}\binom{2r}{r}(-1)^r-\delta_{p,3}\sum_{k=0}^{p^a-1}\binom{2k}{k}(-1)^k\\ \equiv&\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^kH_k^{(2)}\left(\frac{5}{p^{a-1}}\right)-\delta_{p,3}\left(\frac{5}{p^a}\right)\pmod p,\end{aligned}$$

and hence

$$\begin{aligned}&\frac{-4}{p^2}\bigg(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-16)^k}-F_{p^a}\bigg)\\ \equiv&\left(\frac{5}{p^{a-1}}\right)\bigg(\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^kH_k^{(2)}+\delta_{p,3}\bigg)\pmod p.\end{aligned}\tag{2.11}$$

Similarly, when $p\neq 3$ we have

$$\begin{aligned}&\frac{-4}{p^2}\bigg(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-32)^k}-\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}}\bigg)\\ \equiv&\left(\frac{2}{p^a}\right)\sum_{k=0}^{p-1}\binom{2k}{k}(-2)^kH_k^{(2)}\pmod p.\end{aligned}\tag{2.12}$$

Now assume that $p\neq 3$. By (2.5) and (2.12),

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-32)^k}-\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}}\equiv-\left(\frac{2}{p^a}\right)\frac{p^2}{6}q_p(2)^2\pmod{p^3}.$$

Since $p^a \equiv p \pmod{\varphi(p^2)}$, we have $2^{p^a} \equiv 2^p \pmod{p^2}$ and hence

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-32)^k} - \frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}} \equiv -\left(\frac{2}{p^a}\right)\frac{(2^{p^a-1}-1)^2}{6} \pmod{p^3}.$$

Combining this with (2.2) we immediately obtain (1.2).

Below we suppose that $p \neq 5$. By (2.4) and (2.11),

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} - F_{p^a} \equiv -\frac{5}{8}\left(\frac{p^a}{5}\right) F^2_{p-(\frac{p}{5})} \pmod{p^3}.$$

In view of [Su10, Lemma 2.3],

$$\frac{F_{p^a-(\frac{p^a}{5})}}{p} \equiv (-1)^{((\frac{5}{p})-(\frac{5}{p^a}))/2}\left(\frac{5}{p^{a-1}}\right)\frac{F_{p-(\frac{p}{5})}}{p} = \frac{F_{p-(\frac{p}{5})}}{p} \pmod{p}$$

and thus

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} - F_{p^a} \equiv -\frac{5}{8}\left(\frac{p^a}{5}\right) F^2_{p^a-(\frac{p^a}{5})} \pmod{p^3}.$$

Combining this with (2.3) we get

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} - F_{p^a} \equiv \frac{L_{p^a}-1}{4}\left(\frac{p^a}{5}\right) - \frac{5}{4}F_{p^a} + \frac{5}{4}\left(\frac{p^a}{5}\right) \pmod{p^3}.$$

Therefore

$$\begin{aligned}\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} &\equiv \left(\frac{p^a}{5}\right)\frac{L_{p^a}}{4} - \frac{F_{p^a}}{4} + \left(\frac{p^a}{5}\right)\\ &= \left(\frac{p^a}{5}\right)\left(1 + \frac{1}{4}\left(L_{p^a} - \left(\frac{p^a}{5}\right)F_{p^a}\right)\right)\\ &= \left(\frac{p^a}{5}\right)\left(1 + \frac{1}{2}F_{p^a-(\frac{p^a}{5})}\right) \pmod{p^3}.\end{aligned}$$

This proves (1.1).

So far we have completed the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

We need some preliminary results about Lucas sequences.

Let $A, B \in \mathbb{Z}$ and $\Delta = A^2 - 4B$. The equation $x^2 - Ax + B = 0$ has two roots

$$\alpha = \frac{A + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{\Delta}}{2}$$

which are algebraic integers. It is well known that for any $n \in \mathbb{N}$ we have

$$u_n(A, B) = \sum_{0 \leqslant k < n} \alpha^k \beta^{n-1-k} \quad \text{and} \quad v_n(A, B) = \alpha^n + \beta^n.$$

If p is a prime then

$$v_p(A, B) = \alpha^p + \beta^p \equiv (\alpha + \beta)^p = A^p \equiv A \pmod{p}.$$

Lemma 3.1. *Let $A, B \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then*

$$u_{n+1}(A, B) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} A^{n-2k}(-B)^k. \tag{3.1}$$

Remark 3.1. (3.1) is a well-known formula due to Lagrange, see, e.g., H. Gould [G, (1.60)].

Lemma 3.2. *Let $A, B \in \mathbb{Z}$ and let p be an odd prime not dividing $B\Delta$ where $\Delta = A^2 - 4B$. Then*

$$u_p(A, B) \equiv \frac{A}{2} B^{((\frac{\Delta}{p})-1)/2} u_{p-(\frac{\Delta}{p})}(A, B) + \left(\frac{\Delta}{p}\right) \frac{B^{p-1} + 1}{2} \pmod{p^2}. \tag{3.2}$$

Proof. For convenience we let $u_n = u_n(A, B)$ and $v_n = v_n(A, B)$ for all $n \in \mathbb{N}$.

Let α and β be the two roots of the equation $x^2 - Ax + B = 0$. Then

$$v_n^2 - \Delta u_n^2 = (\alpha^n + \beta^n)^2 - (\alpha^n - \beta^n)^2 = 4(\alpha\beta)^n = 4B^n$$

for any $n \in \mathbb{N}$. As $p \mid u_{p-(\frac{\Delta}{p})}$ (see, e.g., [Su10, Lemma 2.3]), p^2 divides

$$\begin{aligned} &v_{p-(\frac{\Delta}{p})}^2 - 4B^{p-(\frac{\Delta}{p})} \\ =& \left(v_{p-(\frac{\Delta}{p})} - 2\left(\frac{B}{p}\right) B^{(p-(\frac{\Delta}{p}))/2}\right)\left(v_{p-(\frac{\Delta}{p})} + 2\left(\frac{B}{p}\right) B^{(p-(\frac{\Delta}{p}))/2}\right). \end{aligned}$$

On the other hand, by [Su10, Lemma 2.3] we have

$$v_{p-(\frac{\Delta}{p})} \equiv 2B^{(1-(\frac{\Delta}{p}))/2} \equiv 2\left(\frac{B}{p}\right)B^{(p-(\frac{\Delta}{p}))/2} \pmod p.$$

Therefore

$$v_{p-(\frac{\Delta}{p})} \equiv 2\left(\frac{B}{p}\right)B^{(p-(\frac{\Delta}{p}))/2} \pmod{p^2}.$$

By induction, for $\varepsilon = \pm 1$ we have

$$Au_n + \varepsilon v_n = 2B^{(1-\varepsilon)/2}u_{n+\varepsilon}$$

for all $n \in \mathbb{Z}^+$. Thus

$$\begin{aligned}2B^{(1-(\frac{\Delta}{p}))/2}u_p =& Au_{p-(\frac{\Delta}{p})} + \left(\frac{\Delta}{p}\right)v_{p-(\frac{\Delta}{p})}\\ \equiv& Au_{p-(\frac{\Delta}{p})} + \left(\frac{\Delta}{p}\right)2\left(\frac{B}{p}\right)B^{(p-(\frac{\Delta}{p}))/2} \pmod{p^2}\end{aligned}$$

and hence

$$\begin{aligned}&2u_p - AB^{((\frac{\Delta}{p})-1)/2}u_{p-(\frac{\Delta}{p})}\\ \equiv&\left(\frac{\Delta}{p}\right)\left(2\left(\frac{B}{p}\right)\left(B^{(p-1)/2}-\left(\frac{B}{p}\right)\right)+2\right)\\ \equiv&\left(\frac{\Delta}{p}\right)\left(B^{p-1}-1+2\right) \pmod{p^2}.\end{aligned}$$

So (3.2) is valid. $\square$

Lemma 3.3. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. Let m be an integer not divisible by p. Then*

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k+1}}{m^k} \equiv \frac{m-2}{2}\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{m^k} - \frac{m}{2} + 2p\delta_{a,1}\left(\frac{-m}{p}\right) \pmod{p^2}. \tag{3.3}$$

Proof. Observe that

$$\begin{aligned}&\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}+\binom{2k}{k+1}}{m^k}\\ =&\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k+1}{k+1}}{m^k} = \frac{\binom{p^a}{(p^a+1)/2}}{m^{(p^a-1)/2}} + \frac{1}{2}\sum_{k=0}^{(p^a-3)/2}\frac{\binom{2k+2}{k+1}}{m^k}\\ =&\frac{p^a/m^{(p^a-1)/2}}{(p^a+1)/2}\binom{p^a-1}{(p^a-1)/2} + \frac{m}{2}\sum_{k=1}^{(p^a-1)/2}\frac{\binom{2k}{k}}{m^k}\\ \equiv&2p\delta_{a,1}\left(\frac{-m}{p}\right) + \frac{m}{2}\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{m^k} - \frac{m}{2} \pmod{p^2}\end{aligned}$$

and hence (3.3) follows. □

Proof of Theorem 1.2. Set $n = (p^a - 1)/2$. By Lemma 3.3,

$$\sum_{k=0}^{n} \frac{C_k}{m^k} \equiv \left(1 - \frac{m-2}{2}\right) \sum_{k=0}^{n} \frac{\binom{2k}{k}}{m^k} + \frac{m}{2} - 2p\delta_{a,1} \left(\frac{-m}{p}\right) \pmod{p^2}.$$

This proves (1.6). It remains to show (1.5).

By Lemmas 3.1 and 2.1,

$$\begin{aligned} u_{p^a}(4, m) =& \sum_{k=0}^{n} \binom{2n-k}{k} 4^{2n-2k}(-m)^k \\ =& \sum_{k=0}^{n} \binom{2n-k}{2(n-k)} 16^{n-k}(-m)^k = \sum_{k=0}^{n} \binom{n+k}{2k} 16^k(-m)^{n-k} \\ \equiv& \sum_{k=0}^{n} \binom{2k}{k} (-1)^k(-m)^{n-k} = (-m)^n \sum_{k=0}^{n} \frac{\binom{2k}{k}}{m^k} \pmod{p^2}. \end{aligned}$$

Note that

$$(-m)^n = \left((-m)^{(p-1)/2}\right)^{\sum_{s=0}^{a-1} p^s} \equiv \left(\frac{-m}{p}\right)^{\sum_{s=0}^{a-1} p^s} = \left(\frac{-m}{p^a}\right) \pmod{p}$$

and hence

$$\begin{aligned} (-m)^n - \left(\frac{-m}{p^a}\right) \equiv& \left((-m)^n - \left(\frac{-m}{p^a}\right)\right) \frac{(-m)^n + \left(\frac{-m}{p^a}\right)}{2(\frac{-m}{p^a})} \\ \equiv& \frac{(-m)^{p^a-1} - 1}{2} \left(\frac{-m}{p^a}\right) \pmod{p^2}. \end{aligned}$$

Thus

$$(-m)^n \equiv \left(\frac{-m}{p^a}\right) \left(1 + \frac{m^{p^a-1} - 1}{2}\right) \equiv \frac{(\frac{-m}{p^a})}{1 - (m^{p^a-1} - 1)/2} \pmod{p^2}$$

and hence

$$\begin{aligned} \sum_{k=0}^{n} \frac{\binom{2k}{k}}{m^k} \equiv& u_{p^a}(4, m) \left(\frac{-m}{p^a}\right) \left(1 - \frac{m^{p^a-1} - 1}{2}\right) \\ \equiv& u_{p^a}(4, m) \left(\frac{-m}{p^a}\right) \left(1 - \frac{m^{p-1} - 1}{2}\right) \pmod{p^2} \end{aligned}$$

since $m^{p^a-1}\equiv m^{p-1}\pmod{p^2}$ by Euler's theorem. By [Su10, Lemma 2.3],

$$\begin{aligned}u_{p^a}(4,m)&\equiv\left(\frac{4^2-4m}{p^{a-1}}\right)u_p(4,m)\pmod{p^2}\\&\equiv\left(\frac{4^2-4m}{p^a}\right)u_1(4,m)=\left(\frac{4-m}{p^a}\right)\pmod{p}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=0}^{n}\frac{\binom{2k}{k}}{m^k}\equiv&u_{p^a}(4,m)\left(\frac{-m}{p^a}\right)-u_{p^a}(4,m)\left(\frac{-m}{p^a}\right)\frac{m^{p-1}-1}{2}\\\equiv&\left(\frac{4-m}{p^{a-1}}\right)\left(\frac{-m}{p^a}\right)u_p(4,m)-\left(\frac{-m(4-m)}{p^a}\right)\frac{m^{p-1}-1}{2}\\=&\left(\frac{-m}{p}\right)\left(\frac{m(m-4)}{p^{a-1}}\right)u_p(4,m)-\left(\frac{m(m-4)}{p^a}\right)\frac{m^{p-1}-1}{2}\pmod{p^2}.\end{aligned}$$

In view of Lemma 3.2,

$$u_p(4,m)-\left(\frac{4-m}{p}\right)\frac{m^{p-1}-1}{2}\equiv\bar{m}u_{p-(\frac{4-m}{p})}(4,m)+\left(\frac{4-m}{p}\right)\pmod{p^2}.$$

So, by the above, $\sum_{k=0}^{n}\binom{2k}{k}/m^k$ is congruent to

$$\begin{aligned}&\left(\frac{m(m-4)}{p^{a-1}}\right)\left(\frac{-m}{p}\right)\left(\bar{m}u_{p-(\frac{4-m}{p})}(4,m)+\left(\frac{4-m}{p}\right)\right)\\=&\left(\frac{m(m-4)}{p^a}\right)+\left(\frac{-m}{p}\right)\left(\frac{m(m-4)}{p^{a-1}}\right)\bar{m}u_{p-(\frac{4-m}{p})}(4,m)\end{aligned}$$

modulo p^2. This proves (1.5). We are done. □

4. Proofs of Corollaries 1.1–1.2

Proof of Corollary 1.1. Note that $n=p-(\frac{4-8}{p})\equiv 0\pmod 4$. The equation $x^2-4x+8=0$ has two roots $2\pm 2i$ where $i=\sqrt{-1}$. Thus

$$u_n(4,8)=\frac{(2+2i)^n-(2-2i)^n}{4i}=\frac{(i(2-2i))^n-(2-2i)^n}{4i}=0$$

and hence by Theorem 1.2 we have

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{8^k}\equiv\left(\frac{8(8-4)}{p^a}\right)=\left(\frac{2}{p^a}\right)\pmod{p^2}.$$

Clearly $q = p - (\frac{4-16}{p}) = p - (\frac{p}{3})$ is divisible by 3 and the two roots of the equation $x^2 - 4x + 16 = 0$ are

$$2 + 2\sqrt{-3} = -4\omega^2 \text{ and } 2 - 2\sqrt{-3} = -4\omega,$$

where $\omega = (-1 + \sqrt{-3})/2$ is a primitive cubic root of unity. Thus

$$u_q(4, 16) = \frac{(-4\omega^2)^q - (-4\omega)^q}{4\sqrt{-3}} = 0$$

since $3 \mid q$. Applying (1.5) with $m = 16$ we get

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{16(16-4)}{p^a}\right) = \left(\frac{3}{p^a}\right) \pmod{p^2}.$$

The proof of Corollary 1.1 is now complete. □

Proof of Corollary 1.2. Set $n = (p-1)/2$. Then

$$\begin{aligned}&\sum_{k=1}^{n} \frac{\binom{2k}{k}}{16^k} \left(\frac{1}{2k-1} + \frac{1}{(2k-1)^2}\right)\\ =&\sum_{k=1}^{n} \frac{2\binom{2k-1}{k}}{16^k} \cdot \frac{2k}{(2k-1)^2} = \frac{1}{4} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv 0 \pmod{p^2}\end{aligned}$$

with the help of [Su11b, (1.4)].

Observe that

$$\begin{aligned}\sum_{k=1}^{n} \frac{\binom{2k}{k}}{(2k-1)16^k} =&\sum_{k=1}^{n} \frac{2\binom{2k-1}{k}}{(2k-1)16^k} = 2\sum_{k=1}^{n} \frac{\binom{2k-2}{k-1}}{k16^k}\\ =&2\sum_{j=0}^{n-1} \frac{C_j}{16^{j+1}} = \frac{1}{8} \sum_{k=0}^{n} \frac{C_k}{16^k} - \frac{C_n}{8 \times 4^{2n}}.\end{aligned}$$

Also,

$$\frac{C_n}{4^{2n}} = \frac{\binom{p-1}{(p-1)/2}}{4^{p-1}(p+1)/2} \equiv (-1)^{(p-1)/2} 2(1-p) \pmod{p^2}$$

in view of Morley's congruence ([Mo])

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}.$$

By (1.6) and (1.8),

$$\sum_{k=0}^{n}\frac{C_k}{16^k}\equiv -6\left(\frac{3}{p}\right)+8-2p\left(\frac{-1}{p}\right)\pmod{p^2}.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{(2k-1)16^k}\equiv&\frac{8-6(\frac{3}{p})-2p(\frac{-1}{p})}{8}-\frac{2(1-p)(\frac{-1}{p})}{8}\\ =&1-\frac{(\frac{-1}{p})+3(\frac{3}{p})}{4}=1-\left(\frac{-1}{p}\right)\frac{3(\frac{p}{3})+1}{4}\pmod{p^2}\end{aligned}$$

and hence

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}}{(2k-1)^2 16^k}\equiv-\sum_{k=1}^{n}\frac{\binom{2k}{k}}{(2k-1)16^k}\equiv -1+\left(\frac{-1}{p}\right)\frac{3(\frac{p}{3})+1}{4}\pmod{p^2},$$

which yields (1.9) and its equivalent form (1.10). We are done. □

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