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SUPERCONGRUENCES INVOLVING PRODUCTS OF TWO BINOMIAL COEFFICIENTS

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Abstract. In this paper we deduce some new supercongruences modulo powers of a prime $p>3$. Let $d\in\{0,1,\ldots,(p-1)/2\}$. We show that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{2k}{k+d}}{8^k}\equiv 0\ (\mathrm{mod}\ p)\quad\text{if } d\equiv\frac{p+1}{2}\ (\mathrm{mod}\ 2),$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{2k}{k+d}}{16^k}\equiv\left(\frac{-1}{p}\right)+p^2\frac{(-1)^d}{4}E_{p-3}\left(d+\frac12\right)\pmod{p^3},$$

where $E_{p-3}(x)$ denotes the Euler polynomial of degree $p-3$, and $(-)$ stands for the Legendre symbol. The paper also contains some other results such as

$$\sum_{k=0}^{p-1}k^{(1+(\frac{-1}{p}))/2}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}\equiv 0\pmod{p^2}.$$

1. Introduction

Let p be an odd prime and let $(\frac{\cdot}{p})$ be the Legendre symbol. For each $d\in\mathbb{N}=\{0,1,\ldots\}$ and any rational p-adic integer λ, we define

$$a_p^{(d)}(\lambda):=\sum_{x=0}^{p-1}x^d\left(\frac{x(x-1)(x-\lambda)}{p}\right). \tag{1.1}$$

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Note that $a_p^{(0)}(\lambda)$ arises naturally from counting the number of points on the cubic curve $\mathbb{E}_p(\lambda):\ y^2=x(x-1)(x-\bar{\lambda})$ over the finite field $\mathbb{F}_p=\mathbb{Z}/p\mathbb{Z}$, where $\bar{\lambda}$ is the residue class $\lambda \pmod p$.

The following theorem in the case $d=0$ is a known result (cf. S. Ahlgren [A, Theorem 2]).

Theorem 1.1. *Let p be an odd prime and let $d\in\{0,\ldots,(p-1)/2\}$. Then, for any rational p-adic integer λ we have*

$$a_p^{(d)}(\lambda)\equiv(-1)^{(p+1)/2}\frac{\lambda^d}{4^d}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{2(k+d)}{k+d}}{16^k}\lambda^k-\delta_{d,(p-1)/2}\pmod p,\tag{1.2}$$

where the Kronecker symbol $\delta_{s,t}$ takes 1 or 0 according as $s=t$ or not.

Remark 1.1. Let $d\in\{0,\ldots,(p-1)/2\}$ with p an odd prime. Clearly

$$a_p^{(d)}(1)=\sum_{x=0}^{p-1}x^d\left(\frac{x}{p}\right)-1\equiv\sum_{x=1}^{p-1}x^{d+(p-1)/2}-1\equiv-\delta_{d,(p-1)/2}-1\pmod p.$$

Thus (1.2) with $\lambda=1$ gives the congruence

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{2k+2d}{k+d}}{16^k}\equiv 4^d\left(\frac{-1}{p}\right)\pmod p.$$

Soon we will see that this congruence even holds modulo p^2.

Recall that the Euler numbers $E_0,E_1,E_2,\ldots$ are integers defined by

$$E_0=1\quad\text{and}\quad\sum_{\substack{k=0\\2\mid k}}^{n}\binom{n}{k}E_{n-k}=0\quad\text{for } n\in\mathbb{Z}^+=\{1,2,3,\ldots\}.$$

For each $n\in\mathbb{N}$, the Euler polynomial of degree n is given by

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}.$$

Clearly $E_n(1/2)=E_n/2^n$.

Now we state our second theorem.

Theorem 1.2. *Let $p > 3$ be a prime and let $d \in \{0, \dots, (p-1)/2\}$. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{2k}{k+d}}{16^k} \equiv \left(\frac{-1}{p}\right) + p^2 \frac{(-1)^d}{4} E_{p-3}\left(d + \frac{1}{2}\right) \pmod{p^3}. \tag{1.3}$$

Remark 1.2. Let $p > 3$ be a prime. The supercongruence

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}$$

was a conjecture of Rodriguez-Villegas [RV] confirmed by E. Mortenson [Mo1] via an advanced tool involving the p-adic Gamma function and the Gross-Koblitz formula for character sums. (1.3) with $d = 0$ yields the congruence

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) + p^2 E_{p-3} \pmod{p^3}$$

which was first proved in [S4] with the help of the software `Sigma`.

Corollary 1.1. *Let $p > 3$ be a prime. For any $d = 0, \dots, (p-1)/2$, we have*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{2k+2d}{k+d}}{16^k} \equiv 4^d \left(\frac{-1}{p}\right) \pmod{p^2}. \tag{1.4}$$

Let $p \equiv 1 \pmod 4$ be a prime. It is well known that $p = x^2 + y^2$ for some $x, y \in \mathbb{Z}$ with $x \equiv 1 \pmod 4$. A celebrated result of Gauss asserts that $\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p$ (see, e.g., [BEW, (9.0.1)]). This was refined in [CDE] as follows:

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

Recently, J. B. Cosgrave and K. Dilcher [CD] even determined $\binom{(p-1)/2}{(p-1)/4} \bmod p^3$. Recall that $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$. Z.-H. Sun [Su] confirmed the author's following conjecture (cf. [S3, Conjecture 5.5]):

$$\begin{aligned}\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{8^k} &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv \left(\frac{2}{p}\right) \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{32^k} \\ &\equiv \left(\frac{2}{p}\right)\left(2x - \frac{p}{2x}\right) \pmod{p^2}.\end{aligned}$$

In [S5] the author showed that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(k+1)8^k}\equiv-2\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{8^k}\equiv\frac{1}{2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(k+1)(-16)^k}$$
$$\equiv-4\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{(-16)^k}\equiv\left(\frac{2}{p}\right)\left(2x-\frac{p}{x}\right)\pmod{p^2}.$$

Note that those integers $\binom{2k}{k}/(k+1)=\binom{2k}{k}-\binom{2k}{k+1}$ are called Catalan numbers and they occur naturally in many enumeration problems in combinatorics (see, e.g., [St, pp. 219–229]).

Motivated by (1.2) in the cases $\lambda=-1,2$ we obtain the following result.

Theorem 1.3. (i) *If $p\equiv3\pmod 4$ is a prime, then*

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(k+1)8^k}&\equiv-2\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{8^k}\\&\equiv-\frac{1}{2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(k+1)(-16)^k}\equiv4\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{(-16)^k}\\&\equiv\frac{(-1)^{(p+1)/4}}{2}\binom{(p+1)/2}{(p+1)/4}\pmod p\end{aligned}\tag{1.5}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{8^k}\equiv-\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv\frac{(-1)^{(p+1)/4}\,2p}{\binom{(p+1)/2}{(p+1)/4}}\pmod{p^2}.\tag{1.6}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{2k}{k+d}}{8^k}\equiv0\pmod p\tag{1.7}$$

for all $d\in\{0,\ldots,(p-1)/2\}$ with $d\equiv(p+1)/2\pmod 2$.

Remark 1.3. In 2009 the author conjectured that $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{2k}{k+1}/8^k\equiv0\pmod p$ for any prime $p\equiv1\pmod 4$ and this was confirmed by his student Yong Zhang in her PhD thesis.

Besides (1.4) with $d=0$, Rodriguez-Villegas [RV] also made the following similar conjectures (confirmed in [Mo2]) on supercongruences with p a prime greater than 3:

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2},\tag{1.8}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},\tag{1.9}$$

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}.\tag{1.10}$$

Note that the denominators 27, 64, 432 come from the following observation via the Stirling formula:

$$\binom{3k}{k}\binom{2k}{k}\sim\frac{\sqrt{3}\times 27^k}{2k\pi},\quad \binom{4k}{2k}\binom{2k}{k}\sim\frac{64^k}{\sqrt{2}k\pi},\quad \binom{6k}{3k}\binom{3k}{k}\sim\frac{432^k}{2k\pi}.$$

Up to now no simple proofs of (1.8)-(1.10) have been found.

Motivated by the work in [PS] and [ST], the author [S2] determined $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ modulo p^2 in terms of Lucas sequences, where p is an odd prime and m is any integer not divisible by p. In [S3] and [S4] the author posed many conjectures on sums of terms involving central binomial coefficients.

For a sequence of $(a_n)_{n\in\mathbb{N}}$ of numbers, as in [S1] we introduce its dual sequence $(a_n^*)_{n\in\mathbb{N}}$ by defining

$$a_n^* := \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k \quad (n=0,1,2,\ldots).$$

It is well-known that $(a_n^*)^* = a_n$ for all $n\in\mathbb{N}$ (see, e.g., (5.48) of [GKP, p. 192]). For Bernoulli numbers $B_0, B_1, B_2, \ldots$, the sequence $((-1)^n B_n)_{n\in\mathbb{N}}$ is self-dual.

Theorem 1.4. *Let $p>3$ be a prime and let $(a_n)_{n\in\mathbb{N}}$ be any sequence of p-adic integers. Then we have*

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}a_k\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}a_k^*\pmod{p^2},\tag{1.11}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}a_k\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}a_k^*\pmod{p^2},\tag{1.12}$$

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}a_k\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}a_k^*\pmod{p^2}.\tag{1.13}$$

Remark 1.4. Z.-H. Sun [Su] recently proved that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\left(a_k-\left(\frac{-1}{p}\right)a_k^*\right)\equiv 0 \pmod{p^2}$$

for any odd prime p via Legendre polynomials. We can also show, for any prime $p>3$, the following result similar to (1.3) and (1.4): If $d\in\{0,\ldots,\lfloor p/3\rfloor\}$ then

$$\frac{1}{4^d}\sum_{k=0}^{(p-1)/2}\frac{\binom{3k}{k}\binom{2k+2d}{k+d}}{27^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{3k}{k}\binom{2k}{k+d}}{27^k}\equiv\left(\frac{p}{3}\right)\pmod{p};$$

if $d\in\{0,\ldots,\lfloor p/4\rfloor\}$ then

$$\frac{1}{4^d}\sum_{k=0}^{(p-1)/2}\frac{\binom{4k}{2k}\binom{2k+2d}{k+d}}{64^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{4k}{2k}\binom{2k}{k+d}}{64^k}\equiv\left(\frac{-2}{p}\right)\pmod{p}.$$

Let p be a prime and let $f(x)\in\mathbb{F}_p[x]$ with $\deg(f)<p$. Then $f(x)$ is identically zero if $f(a)=0$ for all $a\in\mathbb{F}_p$. Thus Theorem 1.4 has the following consequence since $(1-x)^k=\sum_{j=0}^k\binom{k}{j}(-1)^jx^j$ for any $k\in\mathbb{N}$.

Corollary 1.2. *Let $p>3$ be a prime and let $\mathbb{Z}_p$ be the ring of p-adic integers. Then, in the ring $\mathbb{Z}_p[x]$ we have*

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\left(x^k-\left(\frac{p}{3}\right)(1-x)^k\right)\equiv 0 \pmod{p^2},\tag{1.14}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\left(x^k-\left(\frac{-2}{p}\right)(1-x)^k\right)\equiv 0 \pmod{p^2},\tag{1.15}$$

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\left(x^k-\left(\frac{-1}{p}\right)(1-x)^k\right)\equiv 0 \pmod{p^2}.\tag{1.16}$$

Also,

$$\sum_{k=1}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{27^k}\left(x^{k-1}+\left(\frac{p}{3}\right)(1-x)^{k-1}\right)\equiv 0 \pmod{p^2},\tag{1.17}$$

$$\sum_{k=1}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{64^k}\left(x^{k-1}+\left(\frac{-2}{p}\right)(1-x)^{k-1}\right)\equiv 0 \pmod{p^2},\tag{1.18}$$

$$\sum_{k=1}^{p-1}\frac{k\binom{6k}{3k}\binom{3k}{k}}{432^k}\left(x^{k-1}+\left(\frac{-1}{p}\right)(1-x)^{k-1}\right)\equiv 0 \pmod{p^2}.\tag{1.19}$$

Remark 1.5. (1.17)-(1.19) can be easily deduced from (1.14)-(1.16) by taking derivatives. Z.-H. Sun [Su, Theorem 2.4] noted that for any prime $p > 3$ we have

$$\sum_{k=0}^{\lfloor p/3\rfloor}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}(x^k-(-1)^{\lfloor p/3\rfloor}(1-x)^k)\equiv 0 \pmod p.$$

Taking $x = 1/2$ in (1.14)-(1.19) we immediately get the following result.

Corollary 1.3. *Let $p > 3$ be a prime. Then*

$$\begin{aligned}
\sum_{k=0}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{54^k} &\equiv 0 \pmod{p^2} \quad \text{if } p\equiv 1 \pmod 3,\\
\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{54^k} &\equiv 0 \pmod{p^2} \quad \text{if } p\equiv 2 \pmod 3;\\
\sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{128^k} &\equiv 0 \pmod{p^2} \quad \text{if } p\equiv 1,3 \pmod 8,\\
\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{128^k} &\equiv 0 \pmod{p^2} \quad \text{if } p\equiv 5,7 \pmod 8;\\
\sum_{k=0}^{p-1}\frac{k\binom{6k}{3k}\binom{3k}{k}}{864^k} &\equiv 0 \pmod{p^2} \quad \text{if } p\equiv 1 \pmod 4,\\
\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} &\equiv 0 \pmod{p^2} \quad \text{if } p\equiv 3 \pmod 4.
\end{aligned}$$

Remark 1.6. The first and the second congruences mod p were obtained by Z.-H. Sun [Su]. The author [S4] and Z.-H. Sun [Su] conjectured the first and the second congruences respectively. Inputting
`FullSimplify[Sum[k*Binomial[3k,k]Binomial[2k,k]/54^k,{k,0,Infty}]]`,
we obtain from `Mathematica` the exact result

$$\sum_{k=0}^{\infty}\frac{k\binom{3k}{k}\binom{2k}{k}}{54^k}=\frac{\sqrt{\pi}}{9\Gamma(\frac43)\Gamma(\frac76)},$$

which should follow from certain algorithm hidden in `Mathematica`.

(1.14) and (1.17) in the case $x = 9/8$, and (1.15) and (1.18) in the cases $x = 4/3$, $8/9$, $64/63$, yield the following result.

Corollary 1.4. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{24^k}\equiv\left(\frac p3\right)\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{(-216)^k}\pmod{p^2},\tag{1.20}$$
$$\sum_{k=0}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{24^k}\equiv 9\left(\frac p3\right)\sum_{k=0}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{(-216)^k}\pmod{p^2}.\tag{1.21}$$

Also,

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{48^k}&\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-192)^k}\pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{48^k}&\equiv 4\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{(-192)^k}\pmod{p^2};\\ \sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{72^k}&\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{576^k}\pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{72^k}&\equiv -8\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{576^k}\pmod{p^2}.\end{aligned}$$

If $p\neq 7$, then

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{63^k}&\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k}\pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{63^k}&\equiv 64\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k}\pmod{p^2}.\end{aligned}$$

Remark 1.7. Let $p>3$ be a prime. In [S4, Conjecture 5.13] the author conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv\left(\frac p3\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\equiv\begin{cases}\binom{2(p-1)/3}{(p-1)/3}\pmod{p^2}&\text{if }p\equiv 1\pmod 3,\\ p/\binom{2(p+1)/3}{(p+1)/3}\pmod{p^2}&\text{if }p\equiv 2\pmod 3.\end{cases}$$

The author [S4] also made conjectures on $\sum_{k=0}^{p-1}\binom{4k}{2k}\binom{2k}{k}/m^k$ modulo p^2 with $m=48,63,72,128$.

For any prime $p>3$ and integer $m\not\equiv 0 \pmod p$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{(k+1)m^k}\equiv p+\frac{m-27}{6}\sum_{k=0}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{m^k} \pmod{p^2}, \tag{1.22}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(k+1)m^k}\equiv p+\frac{m-64}{12}\sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{m^k} \pmod{p^2}, \tag{1.23}$$

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(k+1)m^k}\equiv p+\frac{m-432}{60}\sum_{k=0}^{p-1}\frac{k\binom{6k}{3k}\binom{3k}{k}}{m^k} \pmod{p^2}, \tag{1.24}$$

due to the identities

$$\sum_{k=0}^{n-1}\left(\frac{6\binom{2k}{k}}{k+1}+(27-m)k\binom{2k}{k}\right)\frac{\binom{3k}{k}}{m^k}=\frac{n}{m^{n-1}}\binom{2n}{n}\binom{3n}{n},$$

$$\sum_{k=0}^{n-1}\left(\frac{12\binom{2k}{k}}{k+1}+(64-m)k\binom{2k}{k}\right)\frac{\binom{4k}{2k}}{m^k}=\frac{n}{m^{n-1}}\binom{4n}{2n}\binom{2n}{n},$$

$$\sum_{k=0}^{n-1}\left(\frac{60}{k+1}+(432-m)k\right)\frac{\binom{6k}{3k}\binom{3k}{k}}{m^k}=\frac{n}{m^{n-1}}\binom{6n}{3n}\binom{3n}{n},$$

which can be easily proved by induction on n. So, the following result follows from Corollary 1.3 and (1.21).

Corollary 1.5. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{(k+1)54^k}\equiv p \pmod{p^2}\quad \text{if } p\equiv 1 \pmod 3, \tag{1.25}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(k+1)128^k}\equiv p \pmod{p^2}\quad \text{if } p\equiv 1,3 \pmod 8, \tag{1.26}$$

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(k+1)864^k}\equiv p \pmod{p^2}\quad \text{if } p\equiv 1 \pmod 4. \tag{1.27}$$

We also have

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{(k+1)24^k}\equiv p+\frac{1}{9}\left(\frac{p}{3}\right)\left(\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{(k+1)(-216)^k}-p\right) \pmod{p^2}. \tag{1.28}$$

Remark 1.8. Similar to the identity in Remark 1.6, `Mathematica` (version 7) also yields

$$\sum_{k=0}^{\infty}\frac{\binom{3k}{k}\binom{2k}{k}}{(k+1)54^k}=\frac{3\sqrt{\pi}}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{6})},\quad \sum_{k=0}^{\infty}\frac{\binom{4k}{2k}\binom{2k}{k}}{(k+1)128^k}=\frac{4\sqrt{\pi}}{\Gamma(\frac{1}{8})\Gamma(\frac{11}{8})},$$

and
$$\sum_{k=0}^{\infty}\frac{\binom{6k}{3k}\binom{3k}{k}}{(k+1)864^k}=\frac{6\sqrt{\pi}}{\Gamma(\frac{1}{12})\Gamma(\frac{17}{12})}.$$

Theorem 1.5. *Let $p>3$ be a prime. Then*
$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{72^k}\equiv\frac{3}{2}\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(k+1)72^k}\\&\equiv\begin{cases}(\frac{6}{p})x \pmod p & \text{if } p=x^2+y^2 \text{ with } x\equiv 1 \pmod 4,\\ \frac{3}{4}(\frac{6}{p})\binom{(p+1)/2}{(p+1)/4} \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}\tag{1.29}$$

Let A and B be integers. The Lucas sequences $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ and $v_n=v_n(A,B)$ $(n\in\mathbb{N})$ are defined as follows:
$$\begin{aligned}&u_0=0,\ u_1=1,\ \text{and } u_{n+1}=Au_n-Bu_{n-1} \text{ for } n\in\mathbb{Z}^+;\\&v_0=2,\ v_1=A,\ \text{and } v_{n+1}=Av_n-Bv_{n-1} \text{ for } n\in\mathbb{Z}^+.\end{aligned}$$
When $\Delta=A^2-4B=0$, by induction we see that $u_n(A,B)=n(A/2)^{n-1}$ and $v_n(A,B)=2(A/2)^n$ for all $n\in\mathbb{Z}^+$. Our following theorem is an analogue of Corollary 1.3 involving Lucas sequences with $\Delta\neq 0$.

Theorem 1.6. *Let $A,B\in\mathbb{Z}$ with $A\neq 0$ and $A^2\neq 4B$, and let $u_k=u_k(A,B)$ and $v_k=v_k(A,B)$ for all $k\in\mathbb{N}$. Let $p>3$ be a prime with $p\nmid A$.*

(i) *If $p\equiv 1 \pmod 3$, then*
$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{(27A)^k}u_k\equiv\sum_{k=1}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{(27A)^k}v_{k-1}\equiv 0 \pmod{p^2}.\tag{1.30}$$
If $p\equiv 2 \pmod 3$, then
$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{(27A)^k}v_k\equiv\sum_{k=1}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{(27A)^k}u_{k-1}\equiv 0 \pmod{p^2}.\tag{1.31}$$

(ii) *If $p\equiv 1,3 \pmod 8$, then*
$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(64A)^k}u_k\equiv\sum_{k=1}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{(64A)^k}v_{k-1}\equiv 0 \pmod{p^2}.\tag{1.32}$$
If $p\equiv 5,7 \pmod 8$, then
$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(64A)^k}v_k\equiv\sum_{k=1}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{(64A)^k}u_{k-1}\equiv 0 \pmod{p^2}.\tag{1.33}$$

(iii) *If* $p\equiv 1\pmod 4$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(432A)^k}u_k\equiv\sum_{k=1}^{p-1}\frac{k\binom{6k}{3k}\binom{3k}{k}}{(432A)^k}v_{k-1}\equiv 0\pmod{p^2}. \tag{1.34}$$

If $p\equiv 3\pmod 4$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{2k}{k}}{(432A)^k}v_k\equiv\sum_{k=1}^{p-1}\frac{k\binom{6k}{3k}\binom{2k}{k}}{(432A)^k}u_{k-1}\equiv 0\pmod{p^2}. \tag{1.35}$$

We will not list corollaries of Theorem 1.6 with respect to some special Lucas sequences like the Fibonacci sequence $F_n=u_n(1,-1)$ $(n\in\mathbb{N})$ and its companion $L_n=v_n(1,-1)$ $(n\in\mathbb{N})$.

In the next section we are going to show Theorems 1.1-1.3 and Corollary 1.1. Sections 3 and 4 are devoted to our proofs of Theorem 1.4 and Theorems 1.5-1.6 respectively.

2. Proofs of Theorems 1.1-1.3 and Corollary 1.1

Proof of Theorem 1.1. Set $n=(p-1)/2$. Then

$$\begin{aligned}
a_p^{(d)}(\lambda)&\equiv\sum_{k=0}^{p-1}x^d\left(x(x-1)(x-\lambda)\right)^n\\
&=\sum_{k=0}^{p-1}x^{n+d}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}x^k\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^l x^{n-l}\\
&=\sum_{k,l=0}^{n}\binom{n}{k}\binom{n}{l}(-1)^{n-k}(-\lambda)^l\sum_{x=1}^{p-1}x^{p-1+d+k-l}\\
&\equiv\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\sum_{\substack{0\leqslant l\leqslant n\\ p-1\mid l-(d+k)}}\binom{n}{l}(-\lambda)^l(p-1)\\
&\equiv-\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{n}{d+k}(-\lambda)^{d+k}-\delta_{d,n}\binom{n}{0}(-\lambda)^0\pmod p.
\end{aligned}$$

Since

$$\binom{(p-1)/2}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod p\quad\text{for all } k=0,\dots,p-1, \tag{2.1}$$

we immediately obtain (1.2) from the above. $\square$

Proof of Theorem 1.2. By induction, we have

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}}{16^k}\left(\binom{2k}{k+m}-\binom{2k}{k+m+1}\right)=\frac{2n+1}{(2m+1)16^n}\binom{2n}{n}\binom{2n+1}{n-m} \tag{2.2}$$

for each $n=m,m+1,\ldots$, where $m\in\mathbb{N}$.

Set $n=(p-1)/2$. If $0\leqslant m<n$, then for the right-hand side R_m of (2.2) we have

$$\begin{aligned}R_m=&\frac{p^2}{(2m+1)((p-1)/2-m)4^{p-1}}\binom{p-1}{n}\binom{p-1}{n-m-1}\\ \equiv&2p^2\frac{(-1)^m}{(2m+1)^2}\pmod{p^3}\end{aligned}$$

since

$$\binom{p-1}{k}=\prod_{0<j\leqslant k}\frac{p-j}{j}\equiv(-1)^k\pmod p\quad\text{for all }k=0,\ldots,p-1. \tag{2.3}$$

As $d\leqslant n$, we have

$$\begin{aligned}&\sum_{k=0}^{n}\frac{\binom{2k}{k}}{16^k}\left(\binom{2k}{k}-\binom{2k}{k+d}\right)\\ =&\sum_{0\leqslant m<d}\sum_{k=0}^{n}\frac{\binom{2k}{k}}{16^k}\left(\binom{2k}{k+m}-\binom{2k}{k+m+1}\right)\\ \equiv&2p^2\sum_{0\leqslant m<d}\frac{(-1)^m}{(2m+1)^2}\equiv\frac{p^2}{2}\sum_{0\leqslant m<d}(-1)^m\left(m+\frac12\right)^{p-3}\\ =&\frac{p^2}{4}\sum_{0\leqslant m<d}(-1)^m\left(E_{p-3}\left(m+\frac12\right)+E_{p-3}\left(m+1+\frac12\right)\right)\\ =&\frac{p^2}{4}\left(E_{p-3}\left(\frac12\right)-(-1)^dE_{p-3}\left(d+\frac12\right)\right)\pmod{p^3}.\end{aligned}$$

Note that

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}\binom{2k}{k+n}}{16^k}=\frac{\binom{2n}{n}}{16^n}=\frac{\binom{p-1}{(p-1)/2}}{4^{p-1}}\equiv\left(\frac{-1}{p}\right)=(-1)^n\pmod{p^3}$$

by Morley's congruence ([M]), and that

$$E_{p-3}\left(n+\frac12\right)=E_{p-3}\left(\frac p2\right)\equiv E_{p-3}(0)=0\pmod p.$$

(It is well known that $E_{2k}(0)=0$ for all $k\in\mathbb{Z}^+$.) Therefore

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{16^k}-(-1)^n\equiv\frac{p^2}{4}E_{p-3}\left(\frac{1}{2}\right)\equiv p^2E_{p-3}\pmod{p^3}$$

and hence (1.3) follows from the above. □

Proof of Corollary 1.1. For $k=0,1,\ldots,$ we have

$$\binom{2k+2d}{k+d}=\sum_{c=-d}^{d}\binom{2k}{k+c}\binom{2d}{d-c}$$

by the Chu-Vandermonde identity (cf. [GKP, p. 169]). (Note that $\binom{2k}{k+c}$ is regarded as zero if $k+c<0$.) In view of this and (1.3),

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{2k+2d}{k+d}}{16^k}&=\sum_{c=-d}^{d}\binom{2d}{d-c}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{2k}{k+|c|}}{16^k}\\&\equiv\sum_{c=-d}^{d}\binom{2d}{d-c}\left(\frac{-1}{p}\right)=2^{2d}\left(\frac{-1}{p}\right)\pmod{p^2}.\end{aligned}$$

So (1.4) is valid and we are done. □

Proof of Theorem 1.3. (i) For $m\in\mathbb{Z}\setminus\{0\}$ and $n\in\mathbb{N}$ we have the combinatorial identity

$$\sum_{k=0}^{n}\left(\frac{16-m}{4}k+\frac{1}{k+1}\right)\frac{\binom{2k}{k}^2}{m^k}=\frac{(2n+1)^2}{(n+1)m^n}\binom{2n}{n}^2\tag{2.4}$$

which can be easily proved by induction on n.

Now let $p=2n+1$ be a prime with $p\equiv 3\pmod 4$. Setting $n=(p-1)/2$ we obtain from (2.4) that

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)m^k}\equiv\frac{m-16}{4}\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2}{m^k}\pmod{p^2}\tag{2.5}$$

for any integer $m\not\equiv 0\pmod p$.

As $n=(p-1)/2$ is odd, by a result of Z.-H. Sun [Su],

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{16^k}(x^k+(1-x)^k)=p^2f(x)$$

for some polynomial $f(x)$ of degree at most $(p-1)/2$ with rational p-adic integer coefficients. In particular,

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{8^k}\equiv-\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k}\pmod{p^2}. \tag{2.6}$$

By integration,

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)16^k}x^{k+1}-\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)16^k}\left((1-x)^{k+1}-1\right)=p^2\int_0^x f(t)dt.$$

Putting $x=-1$ we obtain

$$-\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)(-16)^k}-\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)16^k}\left(2^{k+1}-1\right)\equiv 0\pmod{p^2}.$$

Since

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)16^k}=\frac{(2n+1)^2}{16^n(n+1)}\binom{2n}{n}^2\equiv 0\pmod{p^2},$$

as observed by van Hamme [vH] (see also (2.5) with $m=16$), we have

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)(-16)^k}\equiv-2\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)8^k}\pmod{p^2}. \tag{2.7}$$

With the help of (2.1),

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k}=\sum_{k=0}^{n}(-1)^k\binom{-1/2}{k}^2\equiv\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2=0\pmod{p}.$$

(Note that $(-1)^{n-k}=-(-1)^k$.) Thus

$$\begin{aligned}&\sum_{h=0}^{p-1}\frac{2h+1}{(-16)^h}\sum_{k=0}^{h}\binom{2k}{k}^2\binom{2(h-k)}{h-k}^2\\ \equiv&\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k}\sum_{j=0}^{n}\frac{(2(k+j)+1)\binom{2j}{j}^2}{(-16)^j}\equiv 4\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k}\sum_{j=0}^{n}\frac{j\binom{2j}{j}^2}{(-16)^j}\pmod{p^2}.\end{aligned}$$

By [S5, Lemma 3.2],

$$\sum_{h=0}^{p-1}\frac{2h+1}{(-16)^h}\sum_{k=0}^{h}\binom{2k}{k}^2\binom{2(h-k)}{h-k}^2\equiv p\left(\frac{-1}{p}\right)=-p\pmod{p^2}.$$

Therefore

$$\frac{1}{p}\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k}\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2}{(-16)^k} \equiv -\frac{1}{4} \pmod{p}. \tag{2.8}$$

In view of (2.5)-(2.8), both (1.5) and (1.6) hold if

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)8^k} \equiv \frac{(-1)^{(p+1)/4}}{2}\binom{(p+1)/2}{(p+1)/4} \pmod{p}. \tag{2.9}$$

For $d \in \{0,1\}$, clearly

$$a_p^{(d)}(2) = \sum_{x=1}^{p} x^d\left(\frac{x(x-1)(x-2)}{p}\right) = \sum_{r=0}^{p-1}(r+1)^d\left(\frac{r(r^2-1)}{p}\right)$$

and

$$\begin{aligned}
a_p^{(d)}(-1) =& \sum_{r=0}^{p-1} r^d\left(\frac{r(r^2-1)}{p}\right)\\
\equiv& \sum_{r=0}^{p-1} r^{d+n}(r^2-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\sum_{r=1}^{p-1} r^{n+d+2k}\\
\equiv& -\sum_{\substack{0\leqslant k\leqslant n\\ p-1\mid n+d+2k}}\binom{n}{k}(-1)^{n-k}\\
\equiv& \begin{cases} 0 \pmod{p} & \text{if } d=0,\\ (-1)^{(p-3)/4}\binom{n}{(n-1)/2} - \delta_{p,3} \pmod{p} & \text{if } d=1.\end{cases}
\end{aligned}$$

Thus we have

$$a_p^{(0)}(2) = a_p^{(0)}(-1) \equiv 0 \pmod{p}$$

and

$$a_p^{(1)}(2) = a_p^{(0)}(-1) + a_p^{(1)}(-1) \equiv (-1)^{(p-3)/4}\binom{n}{(n-1)/2} - \delta_{p,3} \pmod{p}.$$

Applying Theorem 1.1 with $\lambda = 2$ and $d = 0,1$, and noting that

$$\frac{1}{2}\binom{2k+2}{k+1} = \binom{2k+1}{k+1} = 2\binom{2k}{k} - \frac{\binom{2k}{k}}{k+1} \quad \text{for all } k \in \mathbb{N},$$

we get

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)8^k} + \delta_{p,3} \equiv 2a_p^{(0)}(2) - a_p^{(1)}(2) \pmod{p}.$$

So (2.9) follows.

(ii) Let $p=2n+1$ be an odd prime. Now we prove (1.7) for all $d\in\{0,\dots,n\}$ with $d\equiv n+1 \pmod 2$. (1.7) is valid for $d=n-1$ since

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}\binom{2k}{k+n-1}}{8^k}=\frac{\binom{2(n-1)}{n-1}}{8^{n-1}}+\frac{2n\binom{2n}{n}}{8^n}=\frac{2n+1}{2\times 8^{n-1}}\binom{2n-2}{n-1}\equiv 0 \pmod p.$$

Define

$$f(d):=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k+d}(-2)^k \quad \text{for } d=0,1,\dots.$$

Since

$$\binom{n+k}{2k}\equiv\frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2} \quad \text{for } k=0,\dots,n \tag{2.10}$$

(see, e.g., [Su, Lemma 2.2]), we have

$$f(d)\equiv\sum_{k=0}^{n}\frac{\binom{2k}{k}\binom{2k}{k+d}}{8^k} \pmod{p^2}$$

for all $d=0\dots,n$. By applying the Zeilberger algorithm (cf. [PWZ, pp. 101–119]) via `Mathematica` (version 7), we find the recurrence relation

$$\begin{aligned}&(n-d-1)(n+d+2)(2d+1)f(d+2)\\=&(2n+1)^2(d+1)f(d+1)-(n-d)(n+d+1)(2d+3)f(d).\end{aligned}$$

Note that $2n+1=p$. So, if $0\leqslant d\leqslant n-2$, then

$$f(d)\equiv-\frac{(n-d-1)(n+d+2)(2d+1)}{(n-d)(n+d+1)(2d+3)}f(d+2) \pmod{p^2}$$

and hence

$$f(d+2)\equiv 0 \pmod p \implies f(d)\equiv 0 \pmod p.$$

Now it is clear that (1.7) holds for all $d\in\{0,\dots,n\}$ with $d\equiv n+1 \pmod 2$.

3. Proof of Theorem 1.4

Proof of (1.11). Observe that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}a_k^*=&\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\sum_{m=0}^{k}\binom{k}{m}(-1)^m a_m\\=&\sum_{m=0}^{p-1}(-1)^m a_m\sum_{k=m}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\binom{k}{m}.\end{aligned}$$

So it suffices to show that

$$\sum_{k=m}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\binom{k}{m}\equiv\left(\frac{p}{3}\right)\frac{\binom{3m}{m}\binom{2m}{m}}{(-27)^m}\pmod{p^2}$$

for all $m=0,1,\ldots,p-1$.

For $0\leqslant m<n$ define

$$f_n(m)=\sum_{k=m}^{n-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\binom{k}{m}.$$

By Zeilberger's algorithm via `Mathematica` (version 7), we find that

$$\begin{aligned}&9(m+1)^2f_n(m+1)+(3m+1)(3m+2)f_n(m)\\=&\frac{(3n-1)(3n-2)}{27^{n-1}}\binom{n-1}{m}\binom{2n-2}{n-1}\binom{3n-3}{n-1}.\end{aligned}$$

Applying this with $n=p>m+1\geqslant 1$ and noting that

$$\binom{2p-2}{p-1}=\frac{p}{2p-1}\prod_{k=1}^{p-1}\frac{p+k}{k}\equiv -p\pmod{p^2}\tag{3.1}$$

and

$$\binom{3p-3}{p-1}=\frac{p}{3p-2}\prod_{k=1}^{2p-2}\frac{p+k}{k}\equiv -p\pmod{p^2},\tag{3.2}$$

we get

$$\begin{aligned}&9(m+1)^2f_p(m+1)+(3m+1)(3m+2)f_p(m)\\\equiv&\frac{(3p-1)(3p-2)}{27^{p-1}}\binom{p-1}{m}p^2\equiv(-1)^m2p^2\pmod{p^3}\end{aligned}$$

and hence

$$\begin{aligned}&f_p(m+1)-\left(\frac{p}{3}\right)\frac{\binom{3m+3}{m+1}\binom{2m+2}{m+1}}{(-27)^{m+1}}\\&+\frac{(3m+1)(3m+2)}{9(m+1)^2}\left(f_p(m)-\left(\frac{p}{3}\right)\frac{\binom{3m}{m}\binom{2m}{m}}{(-27)^m}\right)\\=&f_p(m+1)+\frac{(3m+1)(3m+2)}{9(m+1)^2}f_p(m)\equiv p^2\frac{2(-1)^m}{9(m+1)^2}\pmod{p^3}.\end{aligned}$$

Thus, for every $m=0,\ldots,p-2$, we have

$$\begin{aligned}&f_p(m)\equiv\left(\frac p3\right)\frac{\binom{3m}{m}\binom{2m}{m}}{(-27)^m}\pmod{p^2}\\ \Longrightarrow\ &f_p(m+1)\equiv\left(\frac p3\right)\frac{\binom{3(m+1)}{m+1}\binom{2(m+1)}{m+1}}{(-27)^{m+1}}\pmod{p^2}.\end{aligned}\tag{3.3}$$

Since

$$f_p(0)=\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\equiv\left(\frac p3\right)\frac{\binom{3\times0}{0}\binom{2\times0}{0}}{(-27)^0}\pmod{p^2}$$

by (1.8), with the help of (3.3) we obtain that

$$f_p(m)\equiv\left(\frac p3\right)\frac{\binom{3m}{m}\binom{2m}{m}}{(-27)^m}\pmod{p^2}\quad\text{for all } m=0,1,\ldots,p-1.$$

This concludes the proof. □

Proof of (1.12). Similar to the proof of (1.11), we only need to show that

$$\sum_{k=m}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\binom km\equiv\left(\frac{-2}{p}\right)\frac{\binom{4m}{2m}\binom{2m}{m}}{(-64)^m}\pmod{p^2}$$

for all $m=0,1,\ldots,p-1$. Since this congruence holds for $m=0$ by (1.9), it suffices to prove that for any fixed $0\leqslant m<p-1$ we have

$$\begin{aligned}&g_p(m)\equiv\left(\frac{-2}{p}\right)\frac{\binom{4m}{2m}\binom{2m}{m}}{(-64)^m}\pmod{p^2}\\ \Longrightarrow\ &g_p(m+1)\equiv\left(\frac{-2}{p}\right)\frac{\binom{4(m+1)}{2(m+1)}\binom{2(m+1)}{m+1}}{(-64)^{m+1}}\pmod{p^2},\end{aligned}\tag{3.4}$$

where

$$g_n(m):=\sum_{k=m}^{n-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\binom km$$

with $n>m$. By the Zeilberger algorithm, we find that

$$\begin{aligned}&16(m+1)^2g_n(m+1)+(4m+1)(4m+3)g_n(m)\\ =&\frac{(4n-1)(4n-3)}{64^{n-1}}\binom{n-1}{m}\binom{2n-2}{n-1}\binom{4n-4}{2n-2}.\end{aligned}\tag{3.5}$$

Clearly

$$\binom{4p-2}{2p-2}=\prod_{k=1}^{2p-2}\frac{2p+k}{k}=\frac{3p}{p}\prod_{\substack{k=1\\ k\neq p}}^{2p-2}\left(1+\frac{2p}{k}\right)\equiv3\pmod p$$

and hence

$$\binom{4p-4}{2p-2} = \frac{2p(2p-1)}{(4p-2)(4p-3)}\binom{4p-2}{2p-2} \equiv -p \pmod{p^2}.$$

In view of this and (3.1), from (3.5) with $n = p$ we get

$$16(m+1)^2 g_p(m+1) + (4m+1)(4m+3)g_p(m) \equiv 3(-1)^m p^2 \pmod{p^3}.$$

This implies (3.4) since

$$-\frac{(4m+1)(4m+3)}{16(m+1)^2}\cdot\frac{\binom{4m}{2m}\binom{2m}{m}}{(-64)^m} = \frac{\binom{4(m+1)}{2(m+1)}\binom{2(m+1)}{m+1}}{(-64)^{m+1}}.$$

We are done. $\square$

Proof of (1.13). For $0 \leqslant m < n$ define

$$h_n(m) := \sum_{k=m}^{n-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\binom{k}{m}.$$

By the Zeilberger algorithm, for $m, n \in \mathbb{N}$ with $m < n-1$, we have

$$\begin{aligned}&36(m+1)^2 h_n(m+1) + (6m+1)(6m+5)h_n(m)\\ =&\frac{(6n-1)(6n-5)}{432^{n-1}}\binom{n-1}{m}\binom{3n-3}{n-1}\binom{6n-6}{3n-3}.\end{aligned}$$

Recall the congruence (3.2) and note that if $p > 5$ then

$$\begin{aligned}\binom{6p-6}{3p-3} =&\frac{3p(3p-1)(3p-2)}{(6p-3)(6p-4)(6p-5)}\binom{6p-3}{3p-3}\\ \equiv& -\frac{p}{10}\prod_{k=1}^{3p-3}\frac{3p+k}{k} \equiv -\frac{p}{10}\cdot\frac{3p+p}{p}\cdot\frac{3p+2p}{2p} = -p \pmod{p^2}.\end{aligned}$$

So, no matter $p = 5$ or not, for every $m = 0, \ldots, p-2$ we have

$$36(m+1)^2 h_p(m+1) + (6m+1)(6m+5)h_p(m) \equiv 0 \pmod{p^2}. \tag{3.6}$$

For $0 \leqslant m < p-1$, since

$$-\frac{(6m+1)(6m+5)}{36(m+1)^2}\cdot\frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} = \frac{\binom{6(m+1)}{3(m+1)}\binom{3(m+1)}{m+1}}{(-432)^{m+1}},$$

by (3.6) we have

$$\begin{aligned} h_p(m) &\equiv \left(\frac{-1}{p}\right)\frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} \pmod{p^2} \\ \Longrightarrow h_p(m+1) &\equiv \left(\frac{-1}{p}\right)\frac{\binom{6(m+1)}{3(m+1)}\binom{3(m+1)}{m+1}}{(-432)^{m+1}} \pmod{p^2}. \end{aligned} \tag{3.7}$$

This, together with (1.10), yields that

$$h_p(m) \equiv \left(\frac{-1}{p}\right)\frac{\binom{6m}{3m}\binom{3m}{m}}{(-432)^m} \pmod{p^2}$$

for all $m = 0, \ldots, p-1$. It follows that

$$\begin{aligned} &\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\sum_{m=0}^{k}\binom{k}{m}(-1)^m a_m \\ =&\sum_{m=0}^{p-1}(-1)^m a_m h_p(m) \equiv \left(\frac{-1}{p}\right)\sum_{m=0}^{p-1} a_m\frac{\binom{6m}{3m}\binom{3m}{m}}{432^m} \pmod{p^2}. \end{aligned}$$

This proves (1.13). $\square$

4. Proofs of Theorems 1.5–1.6

Proof of Theorem 1.5. By (1.23),

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(k+1)72^k} \equiv p + \frac{72-64}{12}\sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{72^k} \pmod{p^2}$$

and hence

$$\sum_{k=0}^{p-1}\frac{k\binom{4k}{2k}\binom{2k}{k}}{72^k} \equiv \frac{3}{2}\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(k+1)72^k} \pmod{p}.$$

So it suffices to determine $\sum_{k=0}^{n} k\binom{4k}{2k}\binom{2k}{k}/72^k \bmod p$, where $n = (p-1)/2$. (Note that $p \mid \binom{2k}{k}$ for $k = n+1, \ldots, p-1$.)

The Legendre polynomial of degree n is given by

$$P_n(x) := \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\left(\frac{x-1}{2}\right)^k.$$

It is known (see, e.g., [N]) that

$$\sum_{k=0}^{n}\binom{n}{2k}\binom{2k}{k}x^k = (\sqrt{1-4x})^n P_n\left(\frac{1}{\sqrt{1-4x}}\right).$$

Taking derivatives of both sides of this identity, we get

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{2k}\binom{2k}{k}kx^{k-1}\\ &=-2n(1-4x)^{n/2-1}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\left(\frac{(1-4x)^{-1/2}-1}{2}\right)^{k}\\ &\quad+(1-4x)^{(n-3)/2}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}k\left(\frac{(1-4x)^{-1/2}-1}{2}\right)^{k-1}.\end{aligned}$$

In view of (2.1) and (2.10), by putting $x=2/9$ in the last equality we obtain

$$\frac{1}{2}\sum_{k=0}^{n}\frac{k\binom{4k}{2k}\binom{2k}{k}}{72^k}\equiv\frac{1}{3^n}\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k}+\frac{3}{3^n}\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2}{(-16)^k}\pmod p$$

and hence

$$\left(\frac{3}{p}\right)\sum_{k=0}^{n}\frac{k\binom{4k}{2k}\binom{2k}{k}}{72^k}\equiv 2\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k}+6\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2}{(-16)^k}\pmod p.$$

Since $\sum_{k=0}^{n}\binom{2k}{k}^2/(-16)^k$ and $\sum_{k=0}^{n}k\binom{2k}{k}^2/(-16)^k$ modulo p have been determined (cf. Theorem 1.3(i) and the paragraph after (1.4)), we finally obtain the desired result for $\sum_{k=0}^{n}k\binom{4k}{2k}\binom{2k}{k}/72^k$ modulo p.

The proof of Theorem 1.5 is now complete. □

Proof of Theorem 1.6. We just show the first part in detail. Parts (ii) and (iii) can be proved similarly.

By (1.14) and (1.17), there are p-adic integers $a_0,\ldots,a_{p-1},b_0,\ldots,b_{p-2}$ such that

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{27^k}\left(x^k-\left(\frac{p}{3}\right)(1-x)^k\right)=p^2\sum_{k=0}^{p-1}a_kx^k \tag{4.1}$$

and

$$\sum_{k=1}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{27^k}\left(x^{k-1}+\left(\frac{p}{3}\right)(1-x)^{k-1}\right)=p^2\sum_{k=0}^{p-2}b_kx^k. \tag{4.2}$$

Let α and β be the two distinct roots of the equation $x^2-Ax+B=0$. It is well known that

$$u_k=\frac{\alpha^k-\beta^k}{\alpha-\beta}\quad\text{and}\quad v_k=\alpha^k+\beta^k\qquad\text{for all }k\in\mathbb{N}.$$

As $\alpha/A+\beta/A=1$, by (4.1) and (4.2) we have

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{(27A)^k}\left(u_k+\left(\frac{p}{3}\right)u_k\right)=p^2\sum_{k=0}^{p-1}\frac{a_k}{A^k}u_k,$$
$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{2k}{k}}{(27A)^k}\left(v_k-\left(\frac{p}{3}\right)v_k\right)=p^2\sum_{k=0}^{p-1}\frac{a_k}{A^k}v_k,$$
$$\sum_{k=1}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{27^kA^{k-1}}\left(u_{k-1}-\left(\frac{p}{3}\right)u_{k-1}\right)=p^2\sum_{k=0}^{p-2}\frac{b_k}{A^k}u_k,$$
$$\sum_{k=1}^{p-1}\frac{k\binom{3k}{k}\binom{2k}{k}}{27^kA^{k-1}}\left(v_{k-1}+\left(\frac{p}{3}\right)v_{k-1}\right)=p^2\sum_{k=0}^{p-2}\frac{b_k}{A^k}v_k.$$

Thus (1.30) holds when $p\equiv 1\pmod 3$, and (1.31) holds when $p\equiv 2\pmod 3$. We are done. $\square$

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References

[A] S. Ahlgren, *Gaussian hypergeometric series and combinatorial congruences*, in: Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics (Gainesville, FI, 1999), pp. 1-12, Dev. Math., Vol. 4, Kluwer, Dordrecht, 2001.

[BEW] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi Sums*, John Wiley & Sons, 1998.

[CDE] S. Chowla, B. Dwork and R. J. Evans, *On the mod p^2 determination of $\binom{(p-1)/2}{(p-1)/4}$*, J. Number Theory **24** (1986), 188–196.

[CD] J. B. Cosgrave and K. Dilcher, *Mod p^3 analogues of theorems of Gauss and Jacobi on binomial coefficients*, Acta Arith. **142** (2010), 103–118.

[GKP] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, New York, 1994.

[M] F. Morley, *Note on the congruence $2^{4n}\equiv(-1)^n(2n)!/(n!)^2$, where $2n+1$ is a prime*, Ann. Math. **9** (1895), 168–170.

[Mo1] E. Mortenson, *A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function*, J. Number Theory **99** (2003), 139–147.

[Mo2] E. Mortenson, *Supercongruences between truncated ${}_2F_1$ by geometric functions and their Gaussian analogs*, Trans. Amer. Math. Soc. **355** (2003), 987–1007.

[N] T. D. Noe, *On the divisibility of generalized central trinomial coefficients*, J. Integer Seq. **9** (2006), Article 06.2.7, 12pp.

[PS] H. Pan and Z.-W. Sun, *A combinatorial identity with application to Catalan numbers*, Discrete Math. **306** (2006), 1921–1940.

[PWZ] M. Petkovšek, H. S. Wilf and D. Zeilberger, *$A=B$*, A K Peters, Wellesley, 1996.

[RV] F. Rodriguez-Villegas, *Hypergeometric families of Calabi-Yau manifolds*, in: Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), pp. 223-231, Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, RI, 2003.

[St] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Univ. Press, Cambridge, 1999.

[Su] Z.-H. Sun, *Congruences concerning Legendre polynomials*, Proc. Amer. Math. Soc. **139** (2011), 1915–1929.

[S1] Z.-W. Sun, *Combinatorial identities in dual sequences*, European J. Combin. **24** (2003), 709–718.

[S2] Z.-W. Sun, *Binomial coefficients, Catalan numbers and Lucas quotients*, Sci. China Math. **53** (2010), 2473–2488.

[S3] Z.-W. Sun, *On congruences related to central binomial coefficients*, J. Number Theory **131** (2011), 2219–2238.

[S4] Z.-W. Sun, *Super congruences and Euler numbers*, Sci. China Math. **54** (2011), 2509–2535.

[S5] Z.-W. Sun, *On sums involving products of three binomial coefficients*, Acta Arith. **156** (2012), 123–141.

[ST] Z.-W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math. **45** (2010), 125–148.

[vH] L. van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, in: p-adic Functional Analysis (Nijmegen, 1996), pp. 223–236, Lecture Notes in Pure and Appl. Math., Vol. 192, Dekker, 1997.