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CONNECTIONS BETWEEN $p = x^2 + 3y^2$ AND FRANEL NUMBERS

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Abstract. The Franel numbers are given by $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$ $(n = 0, 1, 2, \ldots)$. Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we show that

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}.$$

We also prove that if $p \equiv 2 \pmod 3$ then

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv -2 \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

In addition, we propose several related conjectures for further research.

1. Introduction

Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. A famous result of Gauss (cf. B.C. Berndt, R.J. Evans and K.S. Williams [BEW, (9.0.1)]) states

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p,$$

which was refined by S. Chowla, B. Dwork and R.J. Evans [CDE] as follows:

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2} \left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

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In 2010 J. B. Cosgrave and K. Dilcher [CD] even determined $\binom{(p-1)/2}{(p-1)/4}$ mod p^3. The author [Su11a, Conjecture 5.5] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\equiv\left(\frac{2}{p}\right)\left(2x-\frac{p}{2x}\right)\ (\mathrm{mod}\ p^2)$$

(where $(\frac{\cdot}{p})$ denotes the Legendre symbol), and this was confirmed by the author's twin brother Z.-H. Sun [S] with the help of Legendre polynomials. Furthermore, the author [Su12] proved that

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{8^k}\equiv 2\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{(-16)^k}\equiv\left(\frac{2}{p}\right)\left(\frac{p}{2x}-x\right)\quad(\mathrm{mod}\ p^2).$$

When $p\equiv 3\pmod 4$ is a prime, the author [Su13b] showed that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv\frac{(-1)^{(p+1)/4}\,2p}{\binom{(p+1)/2}{(p+1)/4}}\quad(\mathrm{mod}\ p^2).$$

For $n\in\mathbb{N}=\{0,1,2,\ldots\}$, we have the combinatorial identities

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}\quad\text{and}\quad\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\binom{2n}{n}\binom{3n}{n}$$

(see, e.g., [G, (3.66) and (6.6)]). Note that $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=0$ for $n=1,3,5,\ldots$. A conjecture of the author [Su11b, Conjecture 5.13] states that if $p>3$ is a prime then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\equiv\begin{cases}\binom{2(p-1)/3}{(p-1)/3}\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1\ (\mathrm{mod}\ 3),\\ p/\binom{2(p+1)/3}{(p+1)/3}\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 2\ (\mathrm{mod}\ 3).\end{cases}$$

It is known that for any prime $p\equiv 1\pmod 3$ we can write $4p=u^2+27v^2$ with $u,v\in\mathbb{Z}$ and $u\equiv 1\pmod 3$, and we have

$$\binom{2(p-1)/3}{(p-1)/3}\equiv\frac{p}{u}-u\quad(\mathrm{mod}\ p^2)$$

(cf. [CD, Theorem 6]).

In [Su13a] the author introduced the polynomials $S_n(x)=\sum_{k=0}^{n}\binom{n}{k}^4x^k$ ($n=0,1,2,\ldots$) and posed 13 related conjectures one of which states that for any prime $p>2$ we have

$$\sum_{n=0}^{p-1}S_n(12)\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1\ (\mathrm{mod}\ 12)\ \&\ p=x^2+y^2\ (3\nmid x),\\ (\frac{xy}{3})4xy\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 5\ (\mathrm{mod}\ 12)\ \&\ p=x^2+y^2\ (x,y\in\mathbb{Z}),\\ 0\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 3\ (\mathrm{mod}\ 4).\end{cases}$$

In view of the above work, it is natural to investigate similar congruences involving the Franel numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n \in \mathbb{N}) \tag{1.1}$$

(cf. [Sl, A000172]). These numbers were first introduced by J. Franel in 1894 who noted the recurrence relation

$$(n+1)^2 f_{n+1} = (7n(n+1)+2)f_n + 8n^2 f_{n-1} \ (n = 1, 2, 3, \dots).$$

For a combinatorial interpretation of the Franel numbers, the reader may consult D. Callan [C].

It is well known that any prime $p \equiv 1 \pmod 3$ can be written uniquely in the form $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ (cf. [Co, p. 7]). In this paper we reveal somewhat surprising connections between the Franel numbers and the representation $p = x^2 + 3y^2$.

Now we state our main result.

Theorem 1.1. *Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we have*

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}. \tag{1.2}$$

If $p \equiv 2 \pmod 3$, then

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv -2 \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}, \tag{1.3}$$

and also

$$\sum_{n=0}^{p-1} \frac{\sum_{k=0}^{n} \binom{n}{k}^3 (m-1)^k}{m^n} \equiv 0 \pmod p \tag{1.4}$$

for any p-adic integer $m \not\equiv 0 \pmod p$.

Remark 1.1. For any prime $p > 3$, we are also able to show $\sum_{k=1}^{p-1} (-1)^k f_k / k^2 \equiv 0 \pmod p$ and determine $\sum_{k=1}^{p-1} (-1)^k k^r f_k$ modulo p^2 for $r = 0, \pm 1, 2$.

Next we pose five related conjectures for further research.

Conjecture 1.1. *Let $p>2$ be a prime. Then*

$$\sum_{n=0}^{p-1}(-1)^n\sum_{k=0}^{n}\binom{n}{k}^3 4^k \equiv \sum_{n=0}^{p-1}\frac{f_n}{2^n} \pmod{p^2}. \tag{1.5}$$

Provided $p\equiv 1 \pmod 3$ we have

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k}\equiv\sum_{k=0}^{p-1}\frac{f_k}{(-4)^k} \pmod{p^3}. \tag{1.6}$$

If $p=x^2+3y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv 1 \pmod 3$, then

$$x\equiv\frac{1}{4}\sum_{k=0}^{p-1}(3k+4)\frac{f_k}{2^k}\equiv\frac{1}{2}\sum_{k=0}^{p-1}(3k+2)\frac{f_k}{(-4)^k} \pmod{p^2} \tag{1.7}$$

and

$$\sum_{n=0}^{p-1}(-1)^n n\sum_{k=0}^{n}\binom{n}{k}^3 4^k\equiv-\frac{5}{3}x \pmod{p}.$$

It is known that $\sum_{k=0}^{n}\binom{n}{k}f_k$ coincides with $g_n:=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$ (cf. [St]). In view of this, Theorem 1.1 has the following consequence.

Corollary 1.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{g_k}{3^k}\equiv\sum_{k=0}^{p-1}\frac{g_k}{(-3)^k}\equiv\begin{cases}2x \pmod p & \text{if } p=x^2+3y^2\ (x,y\in\mathbb{Z}\ \&\ 3\mid x-1),\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases} \tag{1.8}$$

The following conjecture is a refinement of Corollary 1.1.

Conjecture 1.2. *Let $p>3$ be a prime. When $p\equiv 1 \pmod 3$ and $p=x^2+3y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv 1 \pmod 3$, we have*

$$\sum_{k=0}^{p-1}\frac{g_k}{3^k}\equiv\sum_{k=0}^{p-1}\frac{g_k}{(-3)^k}\equiv 2x-\frac{p}{2x} \pmod{p^2} \tag{1.9}$$

and

$$x\equiv\sum_{k=0}^{p-1}(k+1)\frac{g_k}{3^k}\equiv\sum_{k=0}^{p-1}(k+1)\frac{g_k}{(-3)^k} \pmod{p^2}. \tag{1.10}$$

If $p\equiv 2 \pmod 3$, then

$$2\sum_{k=0}^{p-1}\frac{g_k}{3^k}\equiv-\sum_{k=0}^{p-1}\frac{g_k}{(-3)^k}\equiv\frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}. \tag{1.11}$$

Conjecture 1.3. *For any positive integer n,*

$$\frac{1}{2n^2}\sum_{k=0}^{n-1}(3k+2)(-1)^k f_k \in \mathbb{Z} \quad \text{and} \quad \frac{1}{n^2}\sum_{k=0}^{n-1}(4k+1)g_k 9^{n-1-k} \in \mathbb{Z}. \qquad (1.12)$$

Moreover, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1}(3k+2)(-1)^k f_k \equiv 2p^2(2^p-1)^2 \pmod{p^5},$$

$$\sum_{k=0}^{p-1}(4k+1)\frac{g_k}{9^k} \equiv \frac{p^2}{2}\left(3-\left(\frac{p}{3}\right)\right) - p^2(3^p-3) \pmod{p^4}.$$

Note that the sequences $(f_n)_{n\geqslant 0}$ and $(g_n)_{n\geqslant 0}$ are two of the five sporadic sequences (cf. D. Zagier [Z, Section 4]) which are integral solutions of certain Apéry-like recurrence equations and closely related to the theory of modular forms. Concerning sequences D and E in [Z, p. 354], we have not found interesting congruences similar to those in Theorem 1.1. For the sequence

$$w_n = \sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k 3^{n-3k}\binom{n}{3k}\binom{2k}{k}\binom{3k}{k} \quad (n = 0, 1, 2, \ldots) \qquad (1.13)$$

(which is sequence B in [Z, p. 354]), we have the following conjecture.

Conjecture 1.4. *Let $p > 3$ be a prime. If $p \equiv 1 \pmod 3$ and $4p = u^2 + 27v^2$ with $u, v \in \mathbb{Z}$ and $u \equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{p-1}\frac{w_k}{3^k} \equiv \sum_{k=0}^{p-1}\frac{w_k}{9^k} \equiv \frac{p}{u} - u \pmod{p^2}. \qquad (1.14)$$

If $p \equiv 2 \pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{w_k}{9^k} \equiv 0 \pmod{p^2}.$$

Remark 1.2. For any prime $p > 3$, we are able to prove that

$$\sum_{k=0}^{p-1}\frac{w_k}{3^k} \equiv p\sum_{k=0}^{\lfloor (p-1)/3\rfloor}\frac{\binom{2k}{k}\binom{3k}{k}}{(3k+1)27^k} \pmod{p^2} \text{ and } \sum_{k=0}^{p-1}\frac{w_k}{9^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k} \pmod{p}.$$

Motivated by Remark 1.2, we propose one more conjecture.

Conjecture 1.5. *If $p = x^2 + y^2$ is an odd prime with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 4$, then*

$$p\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(4k+1)64^k} \equiv \left(\frac{2}{p}\right)\left(2x - \frac{p}{2x}\right) \pmod{p^2}. \tag{1.15}$$

If $p = u^2 + 27v^2$ is a prime with $u, v \in \mathbb{Z}$ and $u \equiv 1 \pmod 3$, then

$$p\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{(6k+1)432^k} \equiv 2u - \frac{p}{2u} \pmod{p^2}. \tag{1.16}$$

In the next section we shall provide several basic lemmas. Section 3 is devoted to our proofs of Theorem 1.1 and Corollary 1.1.

2. Some basic lemmas

Lemma 2.1. *Let $m \geqslant k \geqslant 0$ be integers. Then we have*

$$\sum_{n=k}^{m}\binom{n}{k} = \binom{m+1}{k+1}. \tag{2.1}$$

Remark 2.1. The identity is well-known (cf. [G, (1.5)]) and it can be easily proved by induction on m.

Lemma 2.2. *For any $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n}\binom{n}{k}^3 z^k = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+k}{3k}\binom{2k}{k}\binom{3k}{k}z^k(1+z)^{n-2k} \tag{2.2}$$

and

$$f_n = \sum_{k=0}^{n}\binom{n+2k}{3k}\binom{2k}{k}\binom{3k}{k}(-4)^{n-k}. \tag{2.3}$$

Remark 2.2. (2.2) is an identity of MacMahon [M, p. 122] (see also [G, (6.7)] and [R, p. 41]). (2.3) can be easily proved by induction since we have the recurrence relation

$$(n+1)^2u_{n+1} = (7n(n+1)+2)u_n + 8n^2u_{n-1} \ (n = 1, 2, 3, \dots)$$

by applying the Zeilberger algorithm (cf. [PWZ, pp. 101–119]) via `Mathematica` 7, where u_n denotes the right-hand side of (2.3).

Recall that for a prime p and an integer $a \not\equiv 0 \pmod p$, the Fermat quotient $(a^{p-1} - 1)/p$ is denoted by $q_p(a)$.

Lemma 2.3 ([Y]). *Let $p \equiv 1 \pmod 3$ be a prime and write $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$. Then we have*

$$\binom{(p-1)/2}{(p-1)/3} \equiv \left(2x - \frac{p}{2x}\right)\left(1 - \frac{2}{3}p\,q_p(2) + \frac{3}{4}p\,q_p(3)\right) \pmod{p^2}. \tag{2.4}$$

Lemma 2.4. *For any positive integer n we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{3k+1} = \prod_{k=1}^{n}\frac{3k}{3k+1}. \tag{2.5}$$

Proof. Recall that

$$B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

for any positive real numbers a and b. With the help of this, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{3k+1} &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\int_0^1 x^{3k}dx = \int_0^1(1-x^3)^n dx\\ &= \int_0^1 (1-y)^n (y^{1/3})'dy = \frac{1}{3}\int_0^1 y^{1/3-1}(1-y)^{(n+1)-1}\\ &= \frac{1}{3}B\left(\frac{1}{3}, n+1\right) = \frac{1}{3}\cdot\frac{\Gamma(n+1)\Gamma(1/3)}{\Gamma(n+1+1/3)}\\ &= \frac{n!}{(1+1/3)\cdots(n+1/3)} = \prod_{k=1}^{n}\frac{3k}{3k+1}.\end{aligned}$$

This proves (2.5). □

Lemma 2.5. *Let $p > 3$ be a prime and let $\varepsilon = (\frac{p}{3})$. Then*

$$\sum_{k=1}^{(p-\varepsilon)/3}\frac{1}{2k-1} \equiv -\frac{3}{4}q_p(3) \pmod p \tag{2.6}$$

and

$$\binom{2(p-\varepsilon)/3}{(p-\varepsilon)/3}2^{-2(p-\varepsilon)/3} \equiv \frac{1}{2-\varepsilon}\binom{(p-\varepsilon)/2}{(p-\varepsilon)/3}\left(1 - \frac{3}{4}p\,q_p(3)\right) \pmod{p^2}. \tag{2.7}$$

Proof. (i) It is known that for any $r \in \mathbb{Z}$ we have

$$\begin{bmatrix}p\\ r\end{bmatrix}_6 := \sum_{k\equiv r \,(\mathrm{mod}\ 6)}\binom{p}{k} = \frac{2^{p-1}-1}{3} + \frac{\delta_r}{2}\left((-1)^{\lfloor (p+1-2r)/6\rfloor}3^{(p-1)/2}+1\right), \tag{2.8}$$

where δ_r takes 1 or 0 according as $3 \nmid p + r$ or not. This essentially follows from [G, (1.54)], and the present form was given in [Su02]. Note that for any $k = 1, \ldots, p-1$ we have

$$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1} \equiv \frac{p}{k}(-1)^{k-1} \pmod{p^2}.$$

Clearly

$$\begin{aligned} q_p(3) =& \frac{(-3)^{(p-1)/2} - (\frac{-3}{p})}{p}\left((-3)^{(p-1)/2} + \left(\frac{-3}{p}\right)\right) \\ \equiv& \frac{2\varepsilon}{p}\left((-3)^{(p-1)/2} - \varepsilon\right) \pmod{p} \end{aligned}$$

and hence by (2.8) with $r = 0$ we have

$$\begin{aligned} \sum_{k=1}^{(p-\varepsilon)/3} \frac{1}{2k-1} =& \sum_{k=1}^{(p-1)/2} \frac{1}{2k-1} - \sum_{k=1}^{\lfloor p/6 \rfloor} \frac{1}{2((p+1)/2 - k) - 1} \\ \equiv& \frac{1}{p}\sum_{k=1}^{(p-1)/2}\binom{p}{2k-1} - \frac{3}{p}\sum_{k=1}^{\lfloor p/6 \rfloor}\binom{p}{6k} \\ =& \frac{1}{p}\Bigg(\sum_{k \equiv 1 \pmod{2}}\binom{p}{k} - 1\Bigg) - \frac{3}{p}\left(\begin{bmatrix} p \\ 0 \end{bmatrix}_6 - 1\right) \\ =& \frac{2^{p-1}-1}{p} - \frac{3}{p}\left(\frac{2^{p-1}-1}{3} + \frac{\varepsilon(-3)^{(p-1)/2}+1}{2} - 1\right) \\ =& -\frac{3}{p}\varepsilon \times \frac{(-3)^{(p-1)/2} - \varepsilon}{2} \equiv -\frac{3}{4}q_p(3) \pmod{p}. \end{aligned}$$

This proves (2.6)

(ii) Now we deduce (2.7). Observe that

$$\begin{aligned} \binom{(p-\varepsilon)/2}{(p-\varepsilon)/3} =& \prod_{k=1}^{(p-\varepsilon)/3} \frac{(p-\varepsilon)/2 + 1 - k}{k} = \prod_{k=1}^{(p-\varepsilon)/3} \frac{2k - 2 + \varepsilon - p}{2k} \\ =& \prod_{k=1}^{(p-\varepsilon)/3}\left(1 - \frac{p}{2k-2+\varepsilon}\right) \times \prod_{k=1}^{(p-\varepsilon)/3} \frac{2k-2+\varepsilon}{2k} \\ \equiv& \left(1 - \sum_{k=1}^{(p-\varepsilon)/3} \frac{p}{2k-2+\varepsilon}\right) \times \frac{\varepsilon}{2}P \pmod{p^2}, \end{aligned}$$

where

$$\begin{aligned}P &:= \prod_{k=2}^{(p-\varepsilon)/3} \frac{(2k-1+\varepsilon)(2k-2+\varepsilon)}{2k(2k-1+\varepsilon)} \\ &= \frac{(2(p-\varepsilon)/3-1+\varepsilon)!/(1+\varepsilon)!}{2^{2((p-\varepsilon)/3-1)}((p-\varepsilon)/3)!((p-\varepsilon)/3+(\varepsilon-1)/2)!}.\end{aligned}$$

If $\varepsilon = 1$, then

$$\sum_{k=1}^{(p-1)/3} \frac{1}{2k-1} \equiv -\frac{3}{4}q_p(3) \pmod{p}$$

by (2.6), and

$$\frac{\varepsilon}{2}P = 2^{-2(p-1)/3}\binom{2(p-1)/3}{(p-1)/3}.$$

If $\varepsilon = -1$, then

$$\begin{aligned}\sum_{k=1}^{(p+1)/3} \frac{1}{2k-3} &= \sum_{k=1}^{(p-2)/3} \frac{1}{2k-1} - 1 \\ &\equiv -\frac{3}{4}q_p(3) - \frac{1}{2(p+1)/3-1} - 1 \equiv 2 - \frac{3}{4}q_p(3) \pmod{p}\end{aligned}$$

by (2.6), and

$$\frac{\varepsilon}{2}P = -2^{-2(p+1)/3}\binom{2(p+1)/3}{(p+1)/3}\Big/\frac{2p-1}{3}.$$

Therefore

$$\begin{aligned}\binom{(p-\varepsilon)/2}{(p-\varepsilon)/3} &\equiv \left(1 + p\left(\frac{3}{4}q_p(3) + \varepsilon - 1\right)\right) 2^{-2(p-\varepsilon)/3}\binom{2(p-\varepsilon)/3}{(p-\varepsilon)/3}\frac{2-\varepsilon}{1+p(\varepsilon-1)} \\ &\equiv (2-\varepsilon)\left(1 + \frac{3}{4}p\,q_p(3)\right) 2^{-2(p-\varepsilon)/3}\binom{2(p-\varepsilon)/3}{(p-\varepsilon)/3} \pmod{p^2}\end{aligned}$$

and hence (2.7) follows.

The proof of Lemma 2.5 is now complete. □

Lemma 2.6. *Let $p \equiv 1 \pmod 3$ be a prime. Then*

$$\binom{p + 2(p-1)/3}{(p-1)/3} \equiv \binom{2(p-1)/3}{(p-1)/3} \pmod{p^2} \tag{2.9}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{1}{3k-1} \equiv -\frac{2}{3}q_p(2) \pmod{p}. \tag{2.10}$$

Proof. Clearly

$$\frac{\binom{p+2(p-1)/3}{(p-1)/3}}{\binom{2(p-1)/3}{(p-1)/3}}=\prod_{k=1}^{(p-1)/3}\frac{p+k+(p-1)/3}{k+(p-1)/3}=\prod_{k=1}^{(p-1)/3}\left(1+\frac{3p}{p-1+3k}\right)$$
$$\equiv 1+3p\sum_{k=1}^{(p-1)/3}\frac{1}{3k-1}=1+3p\sum_{\substack{k=1\\ k\equiv 2\,(\mathrm{mod}\ 3)}}^{p-1}\frac{1}{k}\pmod{p^2}.$$

It is trivial that

$$2\sum_{\substack{k=1\\ k\equiv 2\,(\mathrm{mod}\ 3)}}^{p-1}\frac{1}{k}=\sum_{\substack{k=1\\ k\equiv 2\,(\mathrm{mod}\ 3)}}^{p-1}\left(\frac{1}{k}+\frac{1}{p-k}\right)\equiv 0 \pmod p.$$

So (2.9) holds.

By (2.8),

$$\begin{bmatrix}p\\2\end{bmatrix}_6=\frac{2^{p-1}-1}{3}=\frac{p}{3}q_p(2).$$

Note that

$$\sum_{k=1}^{(p-1)/2}\frac{1}{3k-1}=\sum_{k=1}^{(p-1)/3}\frac{1}{3k-1}+\sum_{k=1}^{(p-1)/6}\frac{1}{3((p-1)/3+k)-1}$$
$$\equiv\sum_{\substack{k=1\\ k\equiv 2\,(\mathrm{mod}\ 3)}}^{p-1}\frac{1}{k}+\sum_{k=1}^{(p-1)/6}\frac{2}{6k-4}$$
$$\equiv -\frac{2}{p}\sum_{\substack{k=1\\ k\equiv 2\,(\mathrm{mod}\ 6)}}^{p-1}\binom{p}{k}=-\frac{2}{p}\begin{bmatrix}p\\2\end{bmatrix}_6=-\frac{2}{3}q_p(2) \pmod p.$$

This proves (2.10). □

3. Proofs of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1. For convenience we write $p=2l+1$ and divide the proof into three parts.

(I) Let m be any p-adic integer with $m\not\equiv 0 \pmod p$. By (2.2) we have

$$\sum_{n=0}^{p-1}\frac{\sum_{k=0}^{n}\binom{n}{k}^3(m-1)^k}{m^n}=\sum_{n=0}^{p-1}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+k}{3k}\binom{2k}{k}\binom{3k}{k}\frac{(m-1)^k}{m^{2k}}$$
$$=\sum_{k=0}^{l}\binom{2k}{k}\binom{3k}{k}\left(\frac{m-1}{m^2}\right)^k\sum_{n=2k}^{p-1}\binom{n+k}{3k}$$
$$=\sum_{k=0}^{l}\binom{2k}{k}\binom{3k}{k}\left(\frac{m-1}{m^2}\right)^k\binom{p+k}{3k+1}\quad\text{(by Lemma 2.1).}$$

For each $k = 0, \dots, l$, clearly

$$\begin{aligned}\binom{2k}{k}\binom{3k}{k}\binom{p+k}{3k+1} =& \frac{p\prod_{0<j\leqslant k}(p-k-j)}{(3k+1)\times(k!)^3}\prod_{0<j\leqslant k}(p^2-j^2) \\ \equiv& \frac{p(-1)^k}{3k+1}\binom{p-1-k}{k} \pmod{p^2},\end{aligned}$$

hence when $3k+1 \neq p$ we have

$$\begin{aligned}\binom{2k}{k}\binom{3k}{k}\binom{p+k}{3k+1} \equiv& \frac{p(-1)^k}{3k+1}\binom{-1-k}{k} = \frac{p\binom{2k}{k}}{3k+1} = \frac{p(-4)^k}{3k+1}\binom{-1/2}{k} \\ \equiv& \frac{p(-4)^k}{3k+1}\binom{l}{k} \pmod{p^2}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{\sum_{k=0}^{n}\binom{n}{k}^3(m-1)^k}{m^n} \\ \equiv& p\sum_{\substack{k=0\\3k+1\neq p}}^{l}\left(\frac{4(m-1)}{m^2}\right)^k\binom{l}{k}\frac{(-1)^k}{3k+1} \\ &+\begin{cases}\left(\frac{m-1}{m^2}\right)^{(p-1)/3}\binom{p-1-(p-1)/3}{(p-1)/3} \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}\end{aligned} \tag{3.1}$$

Clearly this implies (1.4) in the case $p \equiv 2 \pmod 3$.

When $p \equiv 2 \pmod 3$, (3.1) with $m = 2$ gives

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{f_n}{2^n} \equiv& p\sum_{k=0}^{l}\binom{l}{k}\frac{(-1)^k}{3k+1} = p\prod_{k=1}^{l}\frac{3k}{3k+1} \quad \text{(by Lemma 2.4)} \\ \equiv& p\prod_{k=1}^{l}\frac{k}{k+(p+1)/3} = \frac{p}{\binom{l+(p+1)/3}{(p+1)/3}} = (-1)^{(p+1)/3}\frac{p}{\binom{-l-1}{(p+1)/3}} \\ \equiv& \frac{p}{\binom{l}{(p+1)/3}} = \frac{p}{\binom{(p+1)/2-1}{(p+1)/6-1}} = \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.\end{aligned}$$

(II) In view of (2.3) and Lemma 2.1,

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{f_n}{(-4)^n} =& \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-4)^k}\sum_{n=k}^{p-1}\binom{n+2k}{3k} = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-4)^k}\binom{p+2k}{3k+1} \\ =& \sum_{k=0}^{p-1}\frac{p(p+k+1)\cdots(p+2k)\prod_{0<j\leqslant k}(p^2-j^2)}{(3k+1)(-4)^k(k!)^3} \\ \equiv& \sum_{k=0}^{p-1}\frac{p\binom{p+2k}{k}}{(3k+1)4^k} \pmod{p^2}.\end{aligned}$$

If $l < k \leqslant p-1$, then $p+k+1 \leqslant 2p \leqslant p+2k$ and hence $p \mid \binom{p+2k}{k}$. For $k = 0, \ldots, l$ we have

$$\binom{p+2k}{k} = \prod_{0<j\leqslant k} \frac{p+k+j}{j} \equiv \prod_{0<j\leqslant k} \frac{k+j}{j} \equiv \binom{l}{k}(-4)^k \pmod p.$$

Thus

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv & p \sum_{\substack{k=0\\3k+1\neq p}}^{l} \binom{l}{k} \frac{(-1)^k}{3k+1} \\ & + \begin{cases} \binom{p+2(p-1)/3}{(p-1)/3}/4^{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \\ p\binom{p+2(2p-1)/3}{(2p-1)/3}/(2p \times 4^{(2p-1)/3}) \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}\end{aligned} \tag{3.2}$$

Combining this with (3.1) in the case $m = 2$, we find that

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} - \sum_{k=0}^{p-1} \frac{f_k}{2^k} \\ \equiv & \begin{cases} (\binom{p+2(p-1)/3}{(p-1)/3} - \binom{2(p-1)/3}{(p-1)/3})/2^{2(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \\ \binom{p+2(2p-1)/3}{(2p-1)/3}/2^{2(2p-1)/3+1} \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}\end{aligned} \tag{3.3}$$

By Lemma 2.6, if $p \equiv 1 \pmod 3$ then (3.3) yields

$$\sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{2^k} \pmod{p^2}.$$

In the case $p \equiv 2 \pmod 3$,

$$\begin{aligned} & \binom{p+2(2p-1)/3}{(2p-1)/3} = \binom{2p+(p-2)/3}{(2p-1)/3} \\ = & \frac{2p}{(p+1)/3} \prod_{k=1}^{(p-2)/3} \frac{2p+(p+1)/3-k}{k} \times \prod_{k=(p+4)/3}^{(2p-1)/3} \frac{2p+(p+1)/3-k}{k} \\ \equiv & 6p(-1)^{(2p-1)/3-(p+1)/3} \prod_{k=(p+4)/3}^{(2p-1)/3} \frac{k-(p+1)/3}{k} \\ = & -6p \times \frac{((p-2)/3)!}{\prod_{k=(p+4)/3}^{(2p-1)/3} k} = -\frac{12p}{\binom{2(p+1)/3}{(p+1)/3}} \pmod{p^2}, \end{aligned}$$

hence

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{f_k}{(-4)^k}-\sum_{k=0}^{p-1}\frac{f_k}{2^k}&\equiv-\frac{12p}{\binom{2(p+1)/3}{(p+1)/3}2^{(4p+1)/3}}\\&\equiv-\frac{12p}{\frac13\binom{(p+1)/2}{(p+1)/3}2^{2(p+1)/3+(4p+1)/3}}\quad\text{(by (2.7))}\\&=-\frac{36p}{\binom{(p+1)/2}{(p+1)/3}2^{2(p-1)+3}}\equiv-\frac{9p}{2\binom{(p+1)/2}{(p+1)/6}}\pmod{p^2}\end{aligned}$$

and thus

$$\sum_{k=0}^{p-1}\frac{f_k}{(-4)^k}\equiv\sum_{k=0}^{p-1}\frac{f_k}{2^k}-\frac{9p}{2\binom{(p+1)/2}{(p+1)/6}}\equiv-\frac{3p}{2\binom{(p+1)/2}{(p+1)/6}}\pmod{p^2}.$$

(III) Below we assume $p\equiv 1\pmod 3$ and write $p=x^2+3y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv 1\pmod 3$. We want to show that

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k}\equiv 2x-\frac{p}{2x}\pmod{p^2}.\tag{3.4}$$

By (3.1) with $m=2$, we have

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k}\equiv p\sum_{k=0}^{l}\binom{l}{k}\frac{(-1)^k}{3k+1}-\binom{l}{(p-1)/3}(-1)^{(p-1)/3}+\frac{\binom{2(p-1)/3}{(p-1)/3}}{2^{2(p-1)/3}}\pmod{p^2}.\tag{3.5}$$

Applying Lemma 2.4 and noting that

$$\binom{l+(p-1)/3}{(p-1)/3}=\binom{-l-1}{(p-1)/3}=\binom{l-p}{(p-1)/3},$$

we get

$$\begin{aligned}&p\sum_{k=0}^{l}\binom{l}{k}\frac{(-1)^k}{3k+1}=p\prod_{k=1}^{l}\frac{3k}{3k+1}\\=&p\prod_{k=1}^{l}\frac{(3k)^2(3k-1)}{(3k+1)3k(3k-1)}=\frac{3^{p-1}(l!)^2p\prod_{k=1}^{l}(3k-1)}{(p-1)!p(p+1)\cdots(p+l)}\\=&\frac{l!3^{3l}}{\prod_{k=1}^{l}(p^2-k^2)}\prod_{k=1}^{l}\left(k-\frac13\right)\\\equiv&(-3)^{3l}\binom{l-1/3}{l}=(-3)^{3l}\binom{l+(p-1)/3}{l}\Big/\prod_{k=1}^{l}\frac{k+(p-1)/3}{k-1/3}\\=&(-3)^{3l}\binom{l-p}{(p-1)/3}\Big/\prod_{k=1}^{l}\left(1+\frac{p}{3k-1}\right)\pmod{p^2}.\end{aligned}$$

Clearly

$$\begin{aligned}(-3)^{3l}-1 =&((-3)^l-1)((-3)^{2l}+(-3)^l+1)\\ \equiv&\frac{3}{2}((-3)^l-1)((-3)^l+1)=\frac{3}{2}p\,q_p(3)\pmod{p^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\binom{l-p}{(p-1)/3}}{\binom{l}{(p-1)/3}}=&\prod_{k=1}^{(p-1)/3}\frac{l+1-p-k}{l+1-k}=\prod_{k=1}^{(p-1)/3}\left(1-\frac{p}{l+1-k}\right)\\ \equiv&1-p\sum_{k=1}^{(p-1)/3}\frac{1}{(p+1)/2-k}\equiv 1+2p\sum_{k=1}^{(p-1)/3}\frac{1}{2k-1}\\ \equiv&1-\frac{3}{2}p\,q_p(3)\pmod{p^2}\quad\text{(by Lemma 2.5)}.\end{aligned}$$

Therefore

$$\begin{aligned}&p\sum_{k=0}^{l}\binom{l}{k}\frac{(-1)^k}{3k+1}\\ \equiv&\left(1+\frac{3}{2}p\,q_p(3)\right)\left(1-\frac{3}{2}p\,q_p(3)\right)\binom{l}{(p-1)/3}\prod_{k=1}^{l}\left(1-\frac{p}{3k-1}\right)\\ \equiv&\binom{l}{(p-1)/3}\left(1-\sum_{k=1}^{l}\frac{p}{3k-1}\right)\equiv\binom{l}{(p-1)/3}\left(1+\frac{2}{3}p\,q_p(2)\right)\pmod{p^2}\end{aligned}$$

with the help of (2.10). Combining this with (3.5) and (2.7) we get

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k}\equiv\binom{l}{(p-1)/3}\left(1+\frac{2}{3}p\,q_p(2)-\frac{3}{4}p\,q_p(3)\right)\pmod{p^2}.$$

This, together with Lemma 2.3, implies the desired (3.4).

So far we have completed the proof of Theorem 1.1. □

Proof of Corollary 1.1. Let m be 3 or -3. Then $m-1\in\{2,-4\}$. Observe that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{g_n}{m^n}=&\sum_{n=0}^{p-1}\frac{1}{m^n}\sum_{k=0}^{n}\binom{n}{k}f_k=\sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{n=k}^{p-1}\binom{n}{k}\frac{1}{m^{n-k}}\\ =&\sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{j=0}^{p-1-k}\binom{k+j}{j}\frac{1}{m^j}=\sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{j=0}^{p-1-k}\binom{-k-1}{j}\frac{1}{(-m)^j}\\ \equiv&\sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{j=0}^{p-1-k}\binom{p-1-k}{j}\left(-\frac{1}{m}\right)^j=\sum_{k=0}^{p-1}\frac{f_k}{m^k}\left(1-\frac{1}{m}\right)^{p-1-k}\\ \equiv&\sum_{k=0}^{p-1}\frac{f_k}{m^k}\left(\frac{m}{m-1}\right)^k=\sum_{k=0}^{p-1}\frac{f_k}{(m-1)^k}\pmod{p}.\end{aligned}$$

By Theorem 1.1,

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1}\frac{f_k}{(-4)^k} \equiv \begin{cases} 2x \pmod p & \text{if } p = x^2+3y^2\ (x,y\in\mathbb{Z}\ \&\ 3\mid x-1),\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

So the desired (1.8) follows. □

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