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ON SUMS RELATED TO CENTRAL BINOMIAL AND TRINOMIAL COEFFICIENTS

ZHI-WEI SUN

Abstract. A generalized central trinomial coefficient $T_n(b,c)$ is the coefficient of x^n in the expansion of $(x^2+bx+c)^n$ with $b,c \in \mathbb{Z}$. In this paper we investigate congruences and series for sums of terms related to central binomial coefficients and generalized central trinomial coefficients. The paper contains many conjectures on congruences related to representations of primes by certain binary quadratic forms, and 62 proposed new series for $1/\pi$ motivated by congruences and related dualities.

1. Introduction

Let $\mathbb{N} = \{0,1,2,\ldots\}$. The central binomial coefficients

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \quad (n \in \mathbb{N})$$

play important roles in combinatorics and number theory. In this section we first review some known results on sums involving products of at most three central binomial coefficients.

Let $\mathbb{Z}^+ = \{1,2,3,\ldots\}$. Recall that for given numbers A and B the Lucas sequence $u_n = u_n(A,B)$ $(n \in \mathbb{N})$ and its companion $v_n = v_n(A,B)$ $(n \in \mathbb{N})$ are defined by

$$u_0 = 0,\ u_1 = 1,\ u_{n+1} = Au_n - Bu_{n-1}\ (n \in \mathbb{Z}^+),$$

and

$$v_0 = 2,\ v_1 = A,\ v_{n+1} = Av_n - Bv_{n-1}\ (n \in \mathbb{Z}^+).$$

It is well known that

$$(\alpha-\beta)u_n = \alpha^n - \beta^n \quad \text{and} \quad v_n = \alpha^n + \beta^n \quad \text{for all } n \in \mathbb{N},$$

where $\alpha = (A+\sqrt{\Delta})/2$ and $\beta = (A-\sqrt{\Delta})/2$ with $\Delta = A^2 - 4B$.

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Let p be an odd prime and let m be any integer not divisible by p. The author [Su1] proved that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}\equiv\left(\frac{m^2-4m}{p}\right)+u_{p-(\frac{m^2-4m}{p})}(m-2,1)\pmod{p^2},$$

where $(-)$ denotes the Jacobi symbol.

Let $p\equiv 1\pmod 4$ be a prime. Write $p=x^2+y^2$ with $x\equiv 1\pmod 4$ and $y\equiv 0\pmod 2$. Gauss showed in 1828 the famous congruence

$$\binom{(p-1)/2}{(p-1)/4}\equiv 2x\ (\mathrm{mod}\ p),$$

and this was further refined by S. Chowla, B. Dwork and R. J. Evans [CDE] in 1986 who used Gauss and Jacobi sums to show that

$$\binom{(p-1)/2}{(p-1)/4}\equiv\frac{2^{p-1}+1}{2}\left(2x-\frac{p}{2x}\right)\ (\mathrm{mod}\ p^2).$$

For more such congruences involving products of one or more binomial coefficients, the reader may consult the excellent survey [HW] by R. H. Hudson and K. S. Williams. In 2009 the author (cf. [Su2, Conjecture 5.5]) conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\equiv\left(\frac{2}{p}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2},$$

and this was later confirmed by Z.-H. Sun [S1]. Recently the author [Su5] determined x mod p^2 via the congruence

$$\left(\frac{2}{p}\right)x\equiv\sum_{k=0}^{(p-1)/2}\frac{k+1}{8^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2\pmod{p^2}.$$

Note that $p\mid\binom{2k}{k}$ for all $k=(p+1)/2,\ldots,p-1$.

Let p be an odd prime. By [I, vH],

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x\ \&\ 2\mid y),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

In [Su2, Su4] the author made conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ mod p^2 for $m=1,-8,16,-64,256,-512,4096$; for example, Conjecture 5.3 of [Su2] states that

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1.\end{cases}$$

(Throughout this paper, when we write a multiple of a prime in the form ax^2+by^2, we always assume that x and y are *nonzero* integers.) To attack such conjectures, Z.-H. Sun [S2] deduced the useful combinatorial identity

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2x^k=P_n(\sqrt{1+4x})^2 \tag{1}$$

where $P_n(x)$ is the Legendre polynomial of degree n given by

$$P_n(x):=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k.$$

Actually (1) is just a special case of the well-known Clausen formula for hypergeometric series. We can rewrite (1) in the form

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}(x(x+1))^k=D_n(x)^2 \tag{2}$$

where $D_n(x)$ is the Delannoy polynomial of degree n given by

$$D_n(x):=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k.$$

Note that those $D_n=D_n(1)$ $(n=0,1,2,\ldots)$ are central Delannoy numbers (see, e.g., [CHV], [Su3] and [St, p. 178]. It is well known that $P_n(-x)=(-1)^nP_n(x)$, i.e., $(-1)^nD_n(x)=D_n(-x-1)$ (cf. [Su3, Remark 1.2]). As observed by Z.-H. Sun [S1, Lemma 2.2], if $0\leqslant k\leqslant n=(p-1)/2$ then

$$\binom{n+k}{2k}\equiv\frac{\binom{2k}{k}}{(-16)^k}\pmod{p^2}$$

and hence

$$\binom{n}{k}\binom{n+k}{k}=\binom{n+k}{2k}\binom{2k}{k}\equiv\frac{\binom{2k}{k}^2}{(-16)^k}\pmod{p^2}.$$

This simple trick was also realized by van Hamme [vH, p. 231]. Combining this useful trick with the identity (2), we see that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-16)^k}(x(x+1))^k\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}x^k\right)^2\pmod{p^2}. \tag{3}$$

To study the author's conjectures on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}$$

modulo p^2 (with m suitable integers not divisible by p) given in [Su4, Su7], Z.-H. Sun [S2, S3, S4] managed to prove the following congruences similar to (3):

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-64)^k}(x(x+1))^k\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}x^k\right)^2 \pmod{p^2}, \tag{4}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}(x(x+1))^k\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}x^k\right)^2 \pmod{p^2}\ (p>3), \tag{5}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-432)^k}(x(x+1))^k\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}x^k\right)^2 \pmod{p^2}\ (p>3). \tag{6}$$

In 1859 G. Bauer proved that

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}.$$

In 1914 S. Ramanujan [R] found 16 new series for $1/\pi$ which are quite similar to Bauer's series. The (rational) Ramanujan-type series for $1/\pi$ (cf. B. C. Berndt [Be, pp. 353-354], and also [BB] and [ChCh]) have the following form:

$$\sum_{k=0}^{\infty}(a+dk)\frac{f(k)}{m^k}=\frac{C}{\pi}, \tag{7}$$

where $f(k)$ refers to one of

$$\binom{2k}{k}^3,\ \binom{2k}{k}^2\binom{3k}{k},\ \binom{2k}{k}^2\binom{4k}{2k},\ \binom{2k}{k}\binom{3k}{k}\binom{6k}{k},$$

and a,d,m are integers with $dm\neq 0$, and C^2 is rational. Up to now, 36 such series have been established via the theory of modular forms. The reader may consult [CCL], [CC], and S. Cooper [C] for some other series for $1/\pi$.

Let p be an odd prime. Note that $\Gamma(1/2)^2=\pi$ while $\Gamma_p(1/2)^2=(-1)^{(p+1)/2}$, where $\Gamma_p(x)$ denotes the p-adic Γ-function. In view of this, in 1997 van Hamme [vH] studied p-adic supercongruences for partial sums of some hypergeometric series involving the Gamma function. (If a p-adic congruence happens to be true modulo a higher power of p, then it is called a supercongruence.) For example, Bauer's series led him to conjecture that

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv p\left(\frac{-1}{p}\right)\pmod{p^3},$$

which was later confirmed by E. Mortenson [M2] in 2008. More supercongruences motivated by Ramanujan-type series have been investigated by some followers of van Hamme, see, e.g., L. Long [L] and Zudilin's work stated there. The author [Su6] refined the congruence by van Hamme and Mortenson to the following congruence mod p^4:

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3}\pmod{p^4},$$

where $E_0,E_1,\ldots$ are the Euler numbers defined by

$$E_0=1,\ \text{and}\ \sum_{\substack{k=0\\ 2\mid k}}^{n}\binom{n}{k}E_{n-k}=0\ \ (n\in\mathbb{Z}^+).$$

For more conjectural connections between Ramanujan-type congruences and Euler numbers or Euler polynomials, the reader may consult [Su4].

Gosper announced in 1974 that $\sum_{k=0}^{\infty}(25k-3)/(2^k\binom{3k}{k})=\pi/2$ (see G. Almkvist, C. Krattenthaler and J. Petersson [AKP] for a simple proof). Though this is not a Ramanujan-type series, the author conjectures that for any prime $p>3$ we have

$$2p\sum_{k=0}^{p-1}\frac{25k-3}{2^k\binom{3k}{k}}\equiv 3\left(\frac{-1}{p}\right)+(E_{p-3}-9)p^2\pmod{p^3},$$

$$p\sum_{k=0}^{(p-1)/2}\frac{25k-3}{2^k\binom{3k}{k}}\equiv\left(\frac{-1}{p}\right)-\left(\frac{2}{p}\right)\frac{5p}{2}\pmod{p^2},$$

$$\sum_{k=0}^{p-1}(25k+3)k2^k\binom{3k}{k}\equiv 6\left(\frac{-1}{p}\right)-18p\pmod{p^2}.$$

The author [Su4] found some new series for powers of π motivated by corresponding p-adic congruences. Here is a new example: Immediately after the author discovered the conjectural congruence

$$\sum_{n=0}^{p-1}\frac{28n+5}{576^n}\binom{2n}{n}\sum_{k=0}^{n}\frac{5^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(3+2\left(\frac{2}{p}\right)\right)\pmod{p^2}$$

for any prime $p>3$, he conjectured (on Jan. 14, 2012) that

$$\sum_{n=0}^{\infty}\frac{28n+5}{576^n}\binom{2n}{n}\sum_{k=0}^{n}\frac{5^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}=\frac{9}{\pi}(2+\sqrt{2}).\tag{8}$$

The author [Su4, Su7] found that an identity like (7) usually corresponds to a congruence for $\sum_{k=0}^{p-1}f(k)/m^k$ modulo p^2 in terms of parameters in representations of a prime p or its multiple by certain binary quadratic forms. This was the main starting point of the author's discoveries of many new series for $1/\pi$.

Let $n\in\mathbb{N}$. Clearly $\binom{2n}{n}$ is the coefficient of x^n in the expansion of $(x^2+2x+1)^n=(x+1)^{2n}$. The nth central trinomial coefficient

$$T_n=[x^n](x^2+x+1)^n$$

is the coefficient of x^n in the expansion of $(x^2+x+1)^n$. Since T_n is the constant term of $(1+x+x^{-1})^n$, by the multi-nomial theorem we see that

$$T_n=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!}{k!k!(n-2k)!}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k}.$$

Central trinomial coefficients arise naturally in enumerative combinatorics (cf. [Sl]), e.g., T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Given $b,c\in\mathbb{Z}$, we define the *generalized central trinomial coefficients*

$$\begin{aligned}T_n(b,c):=&[x^n](x^2+bx+c)^n=[x^0](b+x+cx^{-1})^n\\=&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}b^{n-2k}c^k.\end{aligned}\tag{9}$$

Clearly $T_n(2,1)=\binom{2n}{n}$ and $T_n(1,1)=T_n$. An efficient way to compute $T_n(b,c)$ is to use the initial values $T_0(b,c)=1$ and $T_1(b,c)=b$, and the recursion

$$(n+1)T_{n+1}(b,c)=(2n+1)bT_n(b,c)-n(b^2-4c)T_{n-1}(b,c)\ \ (n=1,2,\ldots).$$

Note that the recursion is rather simple if $b^2-4c=0$.

Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. It is known that $T_n(b,c)=\sqrt{d}^nP_n(b/\sqrt{d})$ if $d\neq 0$ (see, e.g., [N] and [Su9]). Thus

$$T_n(b,c)=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}\left(\frac{b-\sqrt{d}}{2}\right)^k\sqrt{d}^{n-k}. \tag{10}$$

(In the case $d=0$, (10) holds trivially since $x^2+bx+c=(x+b/2)^2$.) By the Laplace-Heine formula (cf. [Sz, p. 194]), for any complex number $x\notin[-1,1]$ we have

$$P_n(x)\sim\frac{(x+\sqrt{x^2-1})^{n+1/2}}{\sqrt{2n\pi}\sqrt[4]{x^2-1}}\quad\text{ as } n\to+\infty.$$

It follows that if $b>0$ and $c>0$ then

$$T_n(b,c)\sim f_n(b,c):=\frac{(b+2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}\quad\text{ as } n\to+\infty. \tag{11}$$

Note that $T_n(-b,c)=(-1)^nT_n(b,c)$.

The generalized central trinomial coefficients seem to be natural extensions of the central binomial coefficients. To see this, in the next section we study congruences for

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(b,c)}{m^k}\quad\text{and}\quad\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_{2k}(b,c)}{m^k}$$

modulo an odd prime p, where $b,c,m\in\mathbb{Z}$ and $m\not\equiv 0\pmod p$. One may compare them with congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2/m^k$ mod p^2 with $m=8,-16,32$. Since

$$T_k(2,1)=\binom{2k}{k},\ T_{2k}(2,1)=\binom{4k}{2k}\text{ and } T_{3k}(2,1)=\binom{6k}{3k},$$

in Section 3 we are going to investigate general sums

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{m^k}T_k(b,c),\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{m^k}T_k(b,c),\ \sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{m^k}T_k(b,c)$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{m^k}T_{2k}(b,c),\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{m^k}T_{3k}(b,c)$$

modulo p^2, where p is an odd prime, $b,c,m\in\mathbb{Z}$ and $m\not\equiv 0\pmod p$. For this purpose, we need to extend those congruences (3)-(6) in Section 3.

Section 4 contains many conjectural congruences involving generalized central binomial coefficients and they offer backgrounds for those conjectural series for $1/\pi$ in Sect. 5. In the fifth section we first show a theorem on dualities and then propose 61 new conjectural series for $1/\pi$ based on our investigation of congruences.

2. On $\sum_{k=0}^{p-1}\binom{2k}{k}T_k(b,c)/m^k$ and $\sum_{k=0}^{p-1}\binom{2k}{k}T_{2k}(b,c)/m^k$ modulo p

Lemma 2.1. *Let $p=2n+1$ be an odd prime and let $k\in\{0,\ldots,n\}$. Then*

$$\binom{2k}{k}\equiv(-1)^n16^k\binom{2(n-k)}{n-k}\pmod{p}. \tag{12}$$

Given $b,c\in\mathbb{Z}$ with $b^2\not\equiv 4c \pmod p$, we also have

$$T_{2(n-k)}(b,c)\equiv\left(\frac{b^2-4c}{p}\right)\frac{T_{2k}(b,c)}{(b^2-4c)^{2k}}\pmod{p}. \tag{13}$$

Proof. (12) holds because

$$\frac{\binom{2k}{k}}{(-4)^k}=\binom{-1/2}{k}\equiv\binom{n}{k}=\binom{n}{n-k}\equiv\binom{-1/2}{n-k}=\frac{\binom{2(n-k)}{n-k}}{(-4)^{n-k}}\pmod{p}.$$

For $b,c\in\mathbb{Z}$ with $d=b^2-4c\not\equiv 0 \pmod{p}$, we get (13) from the known result $d^jT_{p-1-j}(b,c)\equiv(\frac{d}{p})T_j(b,c)$ for $j=0,\ldots,p-1$ (see [N, (14)] or [Su9, Lemma 2.2]). □

Theorem 2.1. *Let p be an odd prime and let $m,b,c\in\mathbb{Z}$ with $m\not\equiv 0 \pmod p$. If $m\equiv 4b \pmod p$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}T_k(b,c)\equiv\begin{cases}(\frac{m}{p})2xc^{(p-1)/4}\pmod{p} & \text{if } p=x^2+y^2\ (4\mid x-1),\\ 0\pmod{p} & \text{if } p\equiv 3\pmod 4.\end{cases} \tag{14}$$

If $m\not\equiv 4b \pmod p$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}T_k(b,c)\equiv\left(\frac{m(m-4b)}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}c^k}{(m-4b)^{2k}}\pmod{p}. \tag{15}$$

Also, provided that $d=b^2-4c\not\equiv 0 \pmod p$, for any $h\in\mathbb{Z}^+$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^hT_{2k}(b,c)}{m^k}\equiv\left(\frac{(-1)^hdm}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^hT_{2k}(b,c)}{(16^hd^2/m)^k}\pmod{p}. \tag{16}$$

Proof. Set $n=(p-1)/2$. As $\binom{n}{k}\equiv\binom{-1/2}{k}=\binom{2k}{k}/(-4)^k \pmod p$ for all $k=0,\ldots,p-1$, we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}T_k(b,c)\equiv&\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{4}{m}\right)^k[x^0](x+b+cx^{-1})^k\\ =&[x^0]\left(1-\frac{4}{m}\cdot\frac{x^2+bx+c}{x}\right)^n\\ \equiv&\left(\frac{m}{p}\right)[x^n](mx-4(x^2+bx+c))^n\\ \equiv&(-1)^n\left(\frac{m}{p}\right)[x^n]\left(x^2-\frac{m-4b}{4}x+c\right)^n\\ =&\left(\frac{m}{p}\right)T_n\left(\frac{m-4b}{4},c\right)=\left(\frac{m}{p}\right)\frac{T_n(m-4b,16c)}{2^{2n}}\\ \equiv&\left(\frac{m}{p}\right)T_n(m-4b,16c)\pmod p.\end{aligned}$$

Observe that

$$\begin{aligned}T_n(m-4b,16c)=&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}(m-4b)^{n-2k}(16c)^k\\ \equiv&\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{4k}{2k}}{(-4)^{2k}}\binom{2k}{k}(m-4b)^{n-2k}(16c)^k\\ =&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{4k}{2k}\binom{2k}{k}(m-4b)^{n-2k}c^k \pmod p.\end{aligned}$$

Thus (15) holds when $m\not\equiv 4b \pmod p$. If $m\equiv 4b\pmod p$, then

$$T_n(m-4b,16c)\equiv\begin{cases}\binom{2n}{n}\binom{n}{n/2}c^{n/2}\pmod p & \text{if } 2\mid n,\\ 0\pmod p & \text{if } 2\nmid n.\end{cases}$$

Clearly $\binom{2n}{n}=\binom{p-1}{n}\equiv(-1)^n\pmod p$. If $p=2n+1\equiv 1\pmod 4$ and $p=x^2+y^2$ with $x\equiv 1\pmod 4$, then $\binom{n}{n/2}\equiv 2x\pmod p$ as observed by Gauss. Thus, (14) holds when $m\equiv 4b\pmod p$.

Now suppose that $d=b^2-4c\not\equiv 0 \pmod p$ and $h\in\mathbb{Z}^+$. In view of Lemma 2.1, we have

$$\begin{aligned}\sum_{k=0}^{n}\frac{\binom{2k}{k}^hT_{2k}(b,c)}{m^k}\equiv&\sum_{k=0}^{n}\frac{((-1)^n16^k\binom{2(n-k)}{n-k})^h}{m^k}\left(\frac{d}{p}\right)d^{2k}T_{2(n-k)}(b,c)\\ =&(-1)^{hn}\left(\frac{d}{p}\right)\sum_{j=0}^{n}\left(\frac{16^hd^2}{m}\right)^{n-j}\binom{2j}{j}^hT_{2j}(b,c)\\ \equiv&\left(\frac{(-1)^hdm}{p}\right)\sum_{k=0}^{n}\frac{\binom{2k}{k}^hT_{2k}(b,c)}{(16^hd^2/m)^k}\pmod p.\end{aligned}$$

Recall that $p\mid\binom{2k}{k}$ for each $k=n+1,\ldots,p-1$. So (16) follows. $\square$

Corollary 2.1. *Let p be an odd prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}T_k(1,2)\\ \equiv&\begin{cases}(-1)^{(x-1)/2+y/4}2x \pmod{p} & \text{if } 8\mid p-1\ \&\ p=x^2+y^2\ (2\nmid x),\\ (-1)^{(y-2)/4}2y \pmod{p} & \text{if } 8\mid p-5\ \&\ p=x^2+y^2\ (2\mid y),\\ 0 \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

Proof. If $p\equiv 3 \pmod 4$, then $\sum_{k=0}^{p-1}\binom{2k}{k}T_k(1,2)/4^k\equiv 0 \pmod p$ by (14) with $m=4$, $b=1$ and $c=2$.

Now assume that $p\equiv 1 \pmod 4$ and write $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$. Applying (14) with $m=4$, $b=1$ and $c=2$, we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}T_k(1,2)\equiv 2x\times 2^{(p-1)/4} \pmod p.$$

By Exercise 27 of [IR, p. 64] (an observation of Dirichlet),

$$2^{(p-1)/4}\equiv\left(\frac{y}{x}\right)^{xy/2} \pmod p.$$

Note that

$$\left(\frac{y}{x}\right)^2=\frac{y^2}{x^2}\equiv -1 \pmod p \text{ and hence } \left(\frac{y}{x}\right)^4\equiv 1 \pmod 4.$$

So we have

$$2^{(p-1)/4}\equiv\left(\frac{y}{x}\right)^{y/2} \pmod p.$$

If $p \equiv 1 \pmod 8$, then $4 \mid y$ and hence $2^{(p-1)/4} \equiv (-1)^{y/4} \pmod p$. If $p \equiv 5 \pmod 8$, then $y \equiv 2 \pmod 4$ and hence

$$2^{(p-1)/4} \equiv \left(\frac{y}{x}\right)^{2(y-2)/4} \frac{y}{x} \equiv (-1)^{(y-2)/4} \frac{y}{x} \pmod p.$$

Combining the above, we obtain the desired result. □

Corollary 2.2. *For any prime $p > 3$ we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k}{(-4)^k} \equiv \left(\frac{-1}{p}\right) \pmod p \quad \textit{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k}{12^k} \equiv \left(\frac{p}{3}\right) \pmod p.$$

Proof. Applying (15) with $b = c = 1$ and $m \in \{-4, 12\}$ we obtain

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k}{(-4)^k} \equiv \left(\frac{(-4)(-8)}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{64^k} \pmod p$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k}{12^k} \equiv \left(\frac{12 \times 8}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{64^k} \pmod p.$$

It is known that

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

which was conjectured in [RV] and proved in [M1]. So the two congruences in Corollary 2.2 are valid. □

Theorem 2.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_{2k}}{4^k} \equiv \left(\frac{-2}{p}\right) \pmod p.$$

Proof. Set $n=(p-1)/2$. Then

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}T_{2k}\equiv&\sum_{k=0}^{n}\binom{n}{k}(-1)^k[x^0](1+x+x^{-1})^{2k}\\ =&[x^0]\left(1-(1+x+x^{-1})^2\right)^n\\ =&[x^0](-1)^n\left(\frac{x^2+1}{x}\cdot\frac{(x+1)^2}{x}\right)^n\\ =&(-1)^n[x^{2n}](x^2+1)^n(x+1)^{2n}=(-1)^n\sum_{k=0}^{n}\binom{n}{k}\binom{2n}{2k}\\ \equiv&\sum_{k=0}^{n}\binom{n}{k}(-1)^n=(-2)^n\equiv\left(\frac{-2}{p}\right)\pmod p.\end{aligned}$$

This concludes the proof. □

Remark 2.1. For any prime $p>3$ we observe the following congruences:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}T_{2k}(5,4)\equiv1\pmod p,\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}T_{2k}(3,1)\equiv\left(\frac{2}{p}\right)\pmod p,$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{16^k}T_{2k}(4,9)\equiv\left(\frac{p}{3}\right)\pmod p,\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{16^k}T_{2k}(8,25)\equiv\left(\frac{-5}{p}\right)\pmod p.$$

Conjecture 2.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{12^k}T_k\equiv\left(\frac{p}{3}\right)\frac{3^{p-1}+3}{4}\pmod{p^2},$$
$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(4,1)\equiv1\pmod{p^2},$$
$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{(k+1)16^k}T_{2k}(4,1)\equiv\frac{4}{3}\left(\left(\frac{3}{p}\right)-p\left(\frac{-1}{p}\right)\right)\pmod{p^2},$$
$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k}T_{2k}(3,4)\equiv\left(\frac{-1}{p}\right)\frac{7-3^p}{4}\pmod{p^2},$$
$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(8,9)\equiv\left(\frac{3}{p}\right)\pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{432^k}T_{3k}(6,1)\equiv 1 \pmod p.$$

Conjecture 2.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(2,3)\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(4,-3)$$
$$\equiv\begin{cases}(\frac{-1}{p})(\frac{x}{3})(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p=x^2+3y^2,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Also,

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k}T_{2k}(1,-3)\equiv\begin{cases}(-1)^{xy/2}(\frac{x}{3})2x \pmod p & \text{if } p=x^2+3y^2,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3;\end{cases}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(4,3)$$
$$\equiv\begin{cases}(-1)^{\lfloor x/6\rfloor+y/2}2x \pmod p & \text{if } 12\mid p-1\ \&\ p=x^2+y^2\ (2\nmid x),\\ (-1)^{(x+y+1)/2}(\frac{xy}{3})2y \pmod p & \text{if } 12\mid p-5\ \&\ p=x^2+y^2\ (2\nmid x),\\ 0 \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 2.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}T_{2k}(12,-7)$$
$$\equiv\begin{cases}2x(\frac{x}{7}) \pmod p & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2,\\ 0 \pmod p & \text{if } (\frac{p}{7})=-1,\ \text{i.e.},\ p\equiv 3,5,6 \pmod 7.\end{cases}$$

3. Extensions of (2)–(6) with applications to sums involving generalized central trinomial coefficients

Our following theorem is a natural generalization of (2).

Theorem 3.1. *For any $n\in\mathbb{N}$ we have*

$$D_n(x)D_n(y)=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j},\tag{17}$$

Proof. Let a_n denote the left hand side or the right-hand side of (17). It is easy to see that

$$a_0 = 1,\ a_1 = (2x+1)(2y+1),\ a_2 = (6x^2+6x+1)(6y^2+6y+1)$$

and

$$a_3 = (20x^3+30x^2+12x+1)(20y^3+30y^2+12y+1).$$

Applying the Zeilberger algorithm (cf. [PWZ, pp. 101-119]) via `Mathematica` we find the recursion for $n \geqslant 3$:

$$\begin{aligned}&(n+1)^2(2n-3)a_{n+1} - (2n-3)(2n+1)^2(2x+1)(2y+1)a_n\\&+ (2n-1)A(n,x,y)a_{n-1} - (2n-3)^2(2n+1)(2x+1)(2y+1)a_{n-2}\\&+ (n-2)^2(2n+1)a_{n-3}\\&= 0,\end{aligned}$$

where

$$A(n,x,y) := 6n^2 - 6n - 5 + (16n^2 - 16n - 12)(x + y - x^2 - y^2).$$

Thus (17) holds by induction. □

Now we give our extensions of (3) – (6).

Theorem 3.2. *Let p be a prime and let $a \in \mathbb{Z}^+$. Let h be a p-adic integer and set $w_k(h) = \binom{h}{k}\binom{h+k}{k}$ for $k \in \mathbb{N}$. Then*

$$\begin{aligned}&\left(\sum_{k=0}^{p^a-1} w_k(h)x^k\right)\left(\sum_{k=0}^{p-1} w_k(h)y^k\right)\\&\equiv \sum_{k=0}^{p^a-1} w_k(h)\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j} \pmod{p^2}.\end{aligned}\tag{18}$$

In particular, if $p \neq 2$ then

$$\begin{aligned}&\left(\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}^2}{(-16)^k}x^k\right)\left(\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}^2}{(-16)^k}y^k\right)\\&\equiv \sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}^2}{(-16)^k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j} \pmod{p^2}\end{aligned}\tag{19}$$

and

$$\begin{aligned}&\left(\sum_{k=0}^{p^a-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}x^k\right)\left(\sum_{k=0}^{p^a-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}y^k\right)\\&\equiv \sum_{k=0}^{p^a-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j} \pmod{p^2};\end{aligned}\tag{20}$$

provided $p>3$ we have

$$\begin{aligned}&\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}x^k\bigg)\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}y^k\bigg)\\ \equiv&\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j}\pmod{p^2}\end{aligned}\tag{21}$$

and

$$\begin{aligned}&\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}x^k\bigg)\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}y^k\bigg)\\ \equiv&\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j}\pmod{p^2}.\end{aligned}\tag{22}$$

Remark 3.1. Note that

$$\begin{aligned}w_k\left(-\frac{1}{2}\right)&=\frac{\binom{2k}{k}^2}{(-16)^k},\quad w_k\left(-\frac{1}{4}\right)=\frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k},\\ w_k\left(-\frac{1}{3}\right)&=\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k},\quad w_k\left(-\frac{1}{6}\right)=\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}.\end{aligned}$$

Also, (19)-(22) in the case $x=y$ and $a=1$ yield (3)-(6) respectively.

The reader may wonder how we found Theorem 3.2. In fact, (17) is our main clue to the congruence (19). By refining our proof of (19)-(22) we found (18).

To prove Theorem 3.2 we need two lemmas.

Lemma 3.1. *For $m,n\in\mathbb{N}$ we have*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k+m}{n}w_{k+m}(h)=\frac{w_m(h)w_n(h)}{\binom{m+n}{n}},\tag{23}$$

where $w_k(h)=\binom{h}{k}\binom{h+k}{k}$ as defined in Theorem 3.2.

Proof. Let u_n denote the left-hand side of (23). By applying the Zeilberger algorithm via `Mathematica`, we find the recursion:

$$(n+1)(m+n+1)u_{n+1}=(h-n)(h+n+1)u_n\ (n=0,1,2,\ldots).$$

Thus (23) can be easily proved by induction on n. □

Lemma 3.2. *For $k,m,n\in\mathbb{N}$ we have the combinatorial identity*

$$\begin{aligned}\sum_{j=0}^{m}(-1)^{m-j}\binom{m+j}{2j}\binom{2j}{j}\binom{j+k+m}{k}\binom{j}{n}\\=\binom{k+m+n}{m}\binom{k+m}{m}\binom{m}{n}.\end{aligned}\tag{24}$$

Proof. If $m<n$ then both sides of (24) vanish. (24) in the case $m=n$ can be directly verified. Let s_m denote the left-hand side of (24). By the Zeilberger algorithm we find the recursion

$$(m+1)(m-n+1)s_{m+1}=(k+m+1)(k+m+n+1)s_m\ \ (m=n,n+1,\ldots).$$

So we can show (24) by induction. □

Proof of Theorem 3.2. In view of Remark 3.1, it suffices to prove (18). Note that both sides of (18) are polynomials in x and y and the degrees with respect to x or y are all smaller than p^a.

Fix $m,n\in\{0,\ldots,p^a-1\}$ and let $c(m,n)$ denote the coefficient of x^ny^m in the right-hand side of (18). Define $\binom{z}{-k}=0$ for $k=1,2,3,\ldots$. Then $c(m,n)$ coincides with

$$\begin{aligned}&[x^n]\sum_{0\leqslant j\leqslant k<p^a}w_k(h)\binom{k+j}{2j}\binom{2j}{j}(x+1)^j\binom{k-j}{m-j}(-1)^{m-j}x^{k-m}\\&=\sum_{k=m}^{p^a-1}w_k(h)\sum_{j=0}^{m}(-1)^{m-j}\binom{k+j}{2j}\binom{2j}{j}\binom{k-j}{m-j}\binom{j}{m+n-k}\\&=\sum_{k=0}^{p^a-1-m}w_{k+m}(h)\sum_{j=0}^{m}(-1)^{m-j}\binom{k+m+j}{2j}\binom{2j}{j}\binom{k+m-j}{k}\binom{j}{n-k}\\&=\sum_{k=0}^{p^a-1-m}w_{k+m}(h)\sum_{j=0}^{m}(-1)^{m-j}\binom{m+j}{2j}\binom{2j}{j}\binom{k+m+j}{k}\binom{j}{n-k}.\end{aligned}$$

Applying Lemma 3.2 we get

$$\begin{aligned}c(m,n)=&\binom{m+n}{m}\sum_{k=0}^{p^a-1-m}w_{k+m}(h)\binom{k+m}{m}\binom{m}{n-k}\\=&\binom{m+n}{m}\sum_{k=0}^{p^a-1-m}w_{k+m}(h)\binom{k+m}{n}\binom{n}{k}.\end{aligned}$$

By Lemma 3.1,

$$\begin{aligned}&\sum_{k=0}^{p^a-1} w_{k+m}(h)\binom{k+m}{n}\binom{n}{k}\\&=\sum_{k=0}^{n} w_{k+m}(h)\binom{k+m}{n}\binom{n}{k}=\frac{w_m(h)w_n(h)}{\binom{m+n}{m}}.\end{aligned}$$

So, it remains to show

$$\binom{m+n}{m}\sum_{k=p^a-m}^{p^a-1} w_{k+m}(h)\binom{k+m}{n}\binom{n}{k}\equiv 0 \pmod{p^2}. \tag{25}$$

To prove (25) we only need to show

$$\binom{m+n}{m}\equiv\binom{k+m}{n}\equiv 0 \pmod{p}$$

under the supposition $n\geqslant k\geqslant p^a-m$. Note that $m+n\geqslant k+m\geqslant p^a$ and $0<p^a-n\leqslant k+m-n\leqslant m<p^a$. As the addition of m and n in base p has at least one carry, we have $p\mid\binom{m+n}{m}$ by Kummer's theorem (cf. [Ri, p. 24]). Similarly, $p\mid\binom{k+m}{n}$.

So far we have completed the proof of Theorem 3.2. □

Theorem 3.2 implies the following useful result on congruences for sums of central binomial coefficients and generalized central trinomial coefficients.

Theorem 3.3. *Let p be an odd prime and let x be a p-adic integer. Let $a\in\mathbb{Z}^+$, $b,c\in\mathbb{Z}$ and $d=b^2-4c$. Set $D:=1+2bx+dx^2$. Then we have*

$$\begin{aligned}&\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{(-16)^k}T_k(b,c)x^k\\&\equiv\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{32^k}(1-\sqrt{D}+\sqrt{d}\,x)^k\bigg)\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{32^k}(1-\sqrt{D}-\sqrt{d}\,x)^k\bigg)\\&\equiv P_{(p^a-1)/2}(\sqrt{D}+\sqrt{d}\,x)P_{(p^a-1)/2}(\sqrt{D}-\sqrt{d}\,x) \pmod{p^2}\end{aligned} \tag{26}$$

and

$$\sum_{k=0}^{p^a-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}T_k(b,c)x^k\equiv\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{128^k}(1-\sqrt{D}+\sqrt{d}\,x)^k\bigg)\times\sum_{k=0}^{p^a-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{128^k}(1-\sqrt{D}-\sqrt{d}\,x)^k\pmod{p^2}.\tag{27}$$

If $p>3$, *then*

$$\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}T_k(b,c)x^k\equiv\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{54^k}(1-\sqrt{D}+\sqrt{d}\,x)^k\bigg)\times\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{54^k}(1-\sqrt{D}-\sqrt{d}\,x)^k\pmod{p^2}\tag{28}$$

and

$$\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}T_k(b,c)x^k\equiv\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}(1-\sqrt{D}+\sqrt{d}\,x)^k\bigg)\times\sum_{k=0}^{p^a-1}\frac{\binom{6k}{k}\binom{3k}{k}}{864^k}(1-\sqrt{D}-\sqrt{d}\,x)^k\pmod{p^2}.\tag{29}$$

Remark 3.2. Note that $\sqrt{d}$ and $\sqrt{D}$ in Theorem 3.3 are viewed as algebraic p-adic integers.

Proof of Theorem 3.3. Let $n=(p^a-1)/2$. For $k=0,\ldots,n$ we have

$$\binom{n+k}{2k}=\frac{\binom{2k}{k}}{(-16)^k}\prod_{0<j\leqslant k}\left(1-\frac{p^{2a}}{(2j-1)^2}\right)\equiv\frac{\binom{2k}{k}}{(-16)^k}\pmod{p^2}$$

and hence

$$\binom{n}{k}\binom{n+k}{k}=\binom{n+k}{2k}\binom{2k}{k}\equiv\frac{\binom{2k}{k}^2}{(-16)^k}\pmod{p^2}.$$

Note also that $p \mid \binom{2k}{k}$ for $k = n+1, \ldots, p^a - 1$ by Kummer's theorem. Thus

$$\begin{aligned} P_n(t) =& \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{t-1}{2}\right)^k \\ \equiv& \sum_{k=0}^{n} \frac{\binom{2k}{k}^2}{(-16)^k}\left(\frac{t-1}{2}\right)^k \equiv \sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}^2}{32^k}(1-t)^k \pmod{p^2}, \end{aligned}$$

and hence the second congruence in (26) follows.

Set

$$u = \frac{\sqrt{D} + \sqrt{d}\,x - 1}{2} \quad \text{and} \quad v = \frac{\sqrt{D} - \sqrt{d}\,x - 1}{2}.$$

Then

$$uv + v = \frac{D - (\sqrt{d}\,x + 1)^2}{4} = \frac{b - \sqrt{d}}{2}x \quad \text{and} \quad u - v = \sqrt{d}\,x.$$

In view of (10), for any $k \in \mathbb{N}$ we have

$$\begin{aligned} &\sum_{j=0}^{k} \binom{k+j}{2j}\binom{2j}{j}(uv+v)^j(u-v)^{k-j} \\ =& x^k \sum_{j=0}^{k} \binom{k+j}{2j}\binom{2j}{j}\left(\frac{b-\sqrt{d}}{2}\right)^j \sqrt{d}^{\,k-j} = x^k T_k(b,c). \end{aligned}$$

So the first congruence in (26) follows from (19). Similarly, (27)-(29) are consequences of (20)-(22) respectively. $\square$

For $d \in \{2, 3, 4, 7\}$, it is well known that an odd prime p can be written in the form $x^2 + dy^2$ with $x, y \in \mathbb{Z}$ if and only if $(\frac{-d}{p}) = 1$ (see, e.g., [BEW] and [Co]).

Applying (26) we get the following new results.

Theorem 3.4. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 T_k(1,-2)}{32^k} \equiv \begin{cases} (\frac{2}{p})(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + 4y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \tag{30}$$

Also,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 T_k(2,-1)}{8^k} \equiv \begin{cases} (\frac{-1}{p})4x^2 \pmod{p} & \text{if } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}; \end{cases} \tag{31}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 T_k(4,1)}{(-4)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}; \end{cases} \tag{32}$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k(16,1)\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}\\ &\equiv\begin{cases}4x^2\ (\mathrm{mod}\ p) & \text{if } (\frac{p}{7})=1 \text{ and } p=x^2+7y^2,\\ 0\ (\mathrm{mod}\ p^2) & \text{if } (\frac{p}{7})=-1,\ i.e.,\ p\equiv 3,5,6\ (\mathrm{mod}\ 7).\end{cases}\end{aligned}\tag{33}$$

Remark 3.3. Let p be an odd prime. We guess that $4x^2\ (\mathrm{mod}\ p)$ in (31)-(33) can be replaced by $4x^2-2p\ (\mathrm{mod}\ p^2)$.

To prove Theorem 3.4 we need a lemma.

Lemma 3.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}x^k\equiv\left(\frac{2}{p}\right)x^nP_n\left(1-\frac{4}{x}\right)\ (\mathrm{mod}\ p)\tag{34}$$

and

$$P_n(x)\equiv(2x+2)^nP_n\left(\frac{3-x}{1+x}\right)\ (\mathrm{mod}\ p).\tag{35}$$

Proof. With the help of Lemma 2.1, we get

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}x^k&\equiv\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}x^k\equiv\sum_{k=0}^{n}\frac{256^k\binom{2(n-k)}{n-k}^2}{32^k}x^k=\sum_{k=0}^{n}\binom{2k}{k}^2(8x)^{n-k}\\ &\equiv\left(\frac{8}{p}\right)x^n\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k}\left(-\frac{2}{x}\right)^k\\ &\equiv\left(\frac{2}{p}\right)x^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{(1-4/x)-1}{2}\right)^k\\ &=\left(\frac{2}{p}\right)x^nP_n\left(1-\frac{4}{x}\right)\ (\mathrm{mod}\ p).\end{aligned}$$

This proves (34).

(35) follows from [S1, Theorem 2.6] and its proof. □

Proof of Theorem 3.4. For convenience we set $n=(p-1)/2$.

(i) Applying (26) with $b=1$, $c=-2$ and $x=-1/2$, we obtain that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}T_k(1,-2)\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\right)\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\right)\ (\mathrm{mod}\ p^2).$$

The author [Su2, Conjecture 5.5] conjectured that $\sum_{k=0}^{p-1}\binom{2k}{k}^2/32^k \equiv 0 \pmod{p^2}$ if $p\equiv 3 \pmod 4$, and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv 2x-\frac{p}{2x} \pmod{p^2}$$

if $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$. This was confirmed by Z.-H. Sun [S1]. So the desired (30) follows.

(ii) Applying (26) with $b=2$, $c=-1$ and $x=-2$ we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(2,-1)}{8^k}\equiv\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\alpha^k\times\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\beta^k \pmod{p^2}.$$

where $\alpha=-4(1+\sqrt{2})$ and $\beta=-4(1-\sqrt{2})$. Clearly $\alpha\beta=-16$. By Lemma 3.3,

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\alpha^k\equiv\left(\frac{2}{p}\right)\alpha^nP_n(\sqrt{2}) \pmod p$$

and

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\beta^k\equiv\left(\frac{2}{p}\right)\beta^nP_n(-\sqrt{2})=\left(\frac{-2}{p}\right)\beta^nP_n(\sqrt{2}) \pmod p.$$

By [S2, Theorem 2.7], $P_n(\sqrt{2})\equiv 0 \pmod p$ if $(\frac{-2}{p})=-1$, and $P_n(\sqrt{2})^2\equiv(\frac{-1}{p})4x^2 \pmod p$ if $(\frac{-2}{p})=1$ and $p=x^2+2y^2$ $(x,y\in\mathbb{Z})$. So (31) holds.

(iii) (26) with $b=x=4$ and $c=1$ yields that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-4)^k}T_k(4,1)\equiv P_n(15+8\sqrt{3})P_n(15-8\sqrt{3}) \pmod{p^2}.$$

By Lemma 3.3,

$$\begin{aligned}&(\pm 1)^nP_n\left(\frac{\sqrt{3}}{2}\right)=P_n\left(\pm\frac{\sqrt{3}}{2}\right)\\ \equiv&(2\pm\sqrt{3})^nP_n\left(\frac{3\mp\sqrt{3}/2}{1\pm\sqrt{3}/2}\right)=(2\pm\sqrt{3})^nP_n(15\mp 8\sqrt{3}) \pmod p.\end{aligned}$$

By [S2, Theorem 2.8], $P_n(\sqrt{3}/2)\equiv 0 \pmod p$ if $p\equiv 2 \pmod 3$, and $P_n(\sqrt{3}/2)^2\equiv(-1)^n4x^2 \pmod p$ if $p\equiv 1 \pmod 3$ and $p=x^2+3y^2$ $(x,y\in\mathbb{Z})$. Therefore (32) is valid.

(iv) Applying (26) with $b=1$, $c=16$ and $x=1/16$ we obtain that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}\equiv\left(\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\alpha^k\right)\times\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\beta^k \pmod{p^2},$$

where $\alpha=(1+3\sqrt{-7})/16$ and $\beta=(1-3\sqrt{-7})/16$. Note that $\alpha\beta=1/4$. By Lemma 3.3,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\alpha^k\equiv\left(\frac{2}{p}\right)\alpha^nP_n(\sqrt{-63}) \pmod{p}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\beta^k\equiv\left(\frac{2}{p}\right)\beta^nP_n(-\sqrt{-63})=\left(\frac{-2}{p}\right)\beta^nP_n(\sqrt{-63}) \pmod{p}.$$

(26) with $b=16$, $c=1$ and $x=16$ yields that

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k(16,1)\equiv P_n(255+96\sqrt{7})P_n(255-96\sqrt{7}) \pmod{p^2}.$$

By Lemma 3.3,

$$(\pm1)^nP_n\left(\frac{3\sqrt{7}}{8}\right)\equiv(8\pm3\sqrt{7})^nP_n(255\mp96\sqrt{7}) \pmod{p}.$$

Therefore

$$(8^2-9\times7)^n\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k(16,1)\equiv(-1)^nP_n\left(\frac{3\sqrt{7}}{8}\right)^2 \pmod{p}.$$

By [S2, Theorem 2.5], $P_n(\sqrt{-63})\equiv P_n(3\sqrt{7}/8)\equiv0 \pmod{p}$ if $(\frac{p}{7})=-1$, and

$$P_n(\sqrt{-63})^2\equiv(-1)^nP_n\left(\frac{3\sqrt{7}}{8}\right)^2\equiv4x^2 \pmod{p}$$

if $(\frac{p}{7})=1$ and $p=x^2+7y^2$ $(x,y\in\mathbb{Z})$. Therefore (33) holds. $\square$

Motivated by Theorem 3.4 and the congruence

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3\equiv8p+16p^4B_{p-3} \pmod{p^5}$$

proved in [Su4] (where $B_0, B_1, \ldots$ are Bernoulli numbers), we conjecture that

$$\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^2 T_k(1,-2)}{32^k} \equiv \left(\frac{-2}{p}\right)\frac{2p}{3-(\frac{-1}{p})} \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(5k+2)\frac{\binom{2k}{k}^2 T_k(2,-1)}{8^k} \equiv p+p\left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(5k+2)\frac{\binom{2k}{k}^2 T_k(4,1)}{(-4)^k} \equiv \frac{2}{3}p\left(2\left(\frac{-1}{p}\right)+1\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(255k+112)(-1)^k\binom{2k}{k}^2 T_k(16,1) \equiv 16p\left(3+4\left(\frac{-1}{p}\right)\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2 T_k(1,16)}{(-256)^k} \equiv 7p\left(\frac{-1}{p}\right) \pmod{p^2}.$$

The last congruence led the author to find the conjectural identity

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2 T_k(1,16) = \frac{24}{\pi}$$

in Jan. 2011 which was the starting point of the discovery of many series for $1/\pi$ of new types given in Section 5.

4. Conjectural congruences related to representations of primes by binary quadratic forms

In view of (26), our following conjecture implies that for any prime $p = x^2 + 7y^2$ with $x, y \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2 T_k(1,16)}{(-256)^k} \equiv \left(\frac{-1}{p}\right)(4x^2-2p) \pmod{p^2}.$$

Conjecture 4.1. *Let p be an odd prime with $(\frac{p}{7}) = 1$. Write $p = x^2 + 7y^2$ with $x, y \in \mathbb{Z}$ such that $x \equiv 1 \pmod{4}$ if $p \equiv 1 \pmod{4}$, and $y \equiv 1 \pmod{4}$ if $p \equiv 3 \pmod{4}$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{256^k}u_k(1,16) \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{1}{3}(\frac{2}{p})(\frac{p}{7y}-4y) \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}; \end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{256^k}v_k(1,16) \equiv \begin{cases} 2(\frac{2}{p})(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

When $p\equiv 1 \pmod{4}$, we have

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{16^k}u_k(1,16)\equiv\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{256^k}u_k(1,16)\equiv\frac{(\frac{2}{p})}{42}\left(x-\frac{p}{2x}\right)\pmod{p^2}$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}(4k+3)\frac{\binom{2k}{k}^2}{16^k}v_k(1,16)\\ \equiv&3\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^2}{256^k}v_k(1,16)\equiv 6\left(\frac{2}{p}\right)x\pmod{p^2}.\end{aligned}$$

When $p\equiv 3 \pmod{4}$, we can determine y mod p^2 in the following way:

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{16^k}u_k(1,16)\equiv\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{16^k}v_k(1,16)\equiv-\left(\frac{2}{p}\right)\frac{y}{2}\pmod{p^2}$$

and

$$3\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{256^k}u_k(1,16)\equiv\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{256^k}v_k(1,16)\equiv\left(\frac{2}{p}\right)\frac{y}{2}\pmod{p^2}.$$

Just like $\mathbb{Q}(\sqrt{-7})$, the imaginary quadratic field $\mathbb{Q}(\sqrt{-11})$ also has class number one. Let p be an odd prime. Whenever $(\frac{p}{11})=1$ we can write p in the form $(x^2+11y^2)/4$ with $x,y\in\mathbb{Z}$. We guess that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(46,1)}{512^k}\equiv\begin{cases}x^2-2p\pmod{p^2}&\text{if }4p=x^2+11y^2,\\0\pmod{p^2}&\text{if }(\frac{p}{11})=-1.\end{cases}$$

To attack this we note that (28) with $b=46$, $c=1$ and $x=-27/512$ yields

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{512^k}T_k(46,1)\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}\alpha^k\right)\times\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}\beta^k\pmod{p^2},$$

where $\alpha=(1+\sqrt{33})/2$ and $\beta=(1-\sqrt{33})/2$. Observe that $2\alpha^k=v_k(1,-8)+(\alpha-\beta)u_k(1,-8)$ and $2\beta^k=v_k(1,-8)-(\alpha-\beta)u_k(1,-8)$. So we have

$$\begin{aligned}&4\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(46,1)}{512^k}\\ \equiv&\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\right)^2-33\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\right)^2\pmod{p^2}.\end{aligned}$$

This, together with the author's conjecture on $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/64^k$ mod p^2 (cf. [Su2, Conjecture 5.4]) leads us to raise the following conjecture.

Conjecture 4.2. *Let $p>3$ be a prime. If $(\frac{p}{11})=-1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\equiv 0 \pmod p.$$

When $(\frac{p}{11})=1$, $p\equiv 1 \pmod 3$, and $4p=x^2+11y^2$ with $x\equiv 1 \pmod 3$, we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)&\equiv 0 \pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)&\equiv\frac{114}{11}\left(\frac{2p}{x}-x\right) \pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{216^k}u_k(8,27)&\equiv\frac{4}{99}\left(\frac{2p}{x}-x\right) \pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)&\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{216^k}v_k(8,27)\equiv 2\left(\frac{p}{x}-x\right) \pmod{p^2},\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{p-1}(k+60)\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)&\equiv -60x \pmod{p^2},\\ \sum_{k=0}^{p-1}(9k+2)\frac{\binom{2k}{k}\binom{3k}{k}}{216^k}v_k(8,27)&\equiv -2x \pmod{p^2}.\end{aligned}$$

When $(\frac{p}{11})=1$, $p\equiv 2 \pmod 3$, and $4p=x^2+11y^2$ with $y\equiv 1 \pmod 3$, we have

$$\begin{aligned}&11\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)\\ \equiv&-3\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\equiv\frac{3}{2}\left(\frac{p}{y}-11y\right) \pmod{p^2},\end{aligned}$$

$$\sum_{k=0}^{p-1}(2k-155)\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)\equiv\frac{759}{2}y \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(2k-243)\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\equiv-\frac{4359}{2}y \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{216^k}u_k(8,27)\equiv y-\frac{p}{11y} \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{216^k}u_k(8,27)\equiv\frac{1}{8}\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{216^k}v_k(8,27)\equiv-\frac{y}{9} \pmod{p^2}.$$

Motivated by the author's investigation of $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}T_k(3,1)/27^k$ mod p^2 (with $p>3$ a prime) and the congruence (28), we pose the following conjecture which involves the well-known Fibonacci numbers $F_k=u_k(1,-1)$ ($k\in\mathbb{N}$) and Lucas numbers $L_k=v_k(1,-1)$ ($k\in\mathbb{N}$). Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-15})$ has class number 2.

Conjecture 4.3. *Let $p>5$ be a prime. If $p\equiv 1,4 \pmod{15}$ and $p=x^2+15y^2$ ($x,y\in\mathbb{Z}$) with $x\equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}F_k\equiv\frac{2}{15}\left(\frac{p}{x}-2x\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}L_k\equiv 4x-\frac{p}{x} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(3k+2)\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}L_k\equiv 4x \pmod{p^2}.$$

If $p\equiv 2,8 \pmod{15}$ and $p=3x^2+5y^2$ ($x,y\in\mathbb{Z}$) with $y\equiv 1 \pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}F_k\equiv\frac{p}{5y}-4y \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}F_k\equiv\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}L_k\equiv\frac{4}{3}y \pmod{p^2}.$$

Remark 4.1. By [Su8, Theorem 1.6], for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}F_k\equiv 0 \pmod{p^2}\ \text{ if }\ p\equiv 1 \pmod 3,$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}L_k\equiv 0 \pmod{p^2}\ \text{ if }\ p\equiv 2 \pmod 3.$$

In fact, we have many other conjectures similar to Conjectures 4.1-4.3; for the sake of brevity we don't include them in this paper.

Conjecture 4.4. *Let $p>3$ be a prime.*

(i) *If $p\equiv 1,4 \pmod{15}$ and $p=x^2+15y^2$ with $x,y\in\mathbb{Z}$, then*

$$P_{(p-1)/2}(7\sqrt{-15}\pm 16\sqrt{-3})\equiv\left(\frac{-\sqrt{-15}}{p}\right)\left(\frac{x}{15}\right)\left(2x-\frac{p}{2x}\right) \pmod{p^2}.$$

(ii) *Suppose that $(\frac{p}{5})=(\frac{p}{7})=1$ and write $4p=x^2+35y^2$ with $x,y\in\mathbb{Z}$. If $p\equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{3456^k}(64+27\sqrt5\pm\sqrt{-35})^k\equiv\left(\frac{x}{3}\right)\left(\frac{p}{x}-x\right) \pmod{p^2}.$$

If $p\equiv 2 \pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{3456^k}(64+27\sqrt5\pm\sqrt{-35})^k\equiv\pm\sqrt{-35}\left(\frac{y}{3}\right)\left(y-\frac{p}{35y}\right) \pmod{p^2}.$$

(iii) *If $(\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1$ and $p=x^2+30y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{2916^k}(54-35\sqrt2\pm\sqrt5)^k\equiv\left(\frac{x}{3}\right)\left(2x-\frac{p}{2x}\right) \pmod{p^2}.$$

(iv) *If $(\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{7})=1$, and $p=x^2+42y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{13500^k}(250-99\sqrt6\pm 2\sqrt{14})^k\equiv\left(\frac{x}{3}\right)\left(2x-\frac{p}{2x}\right) \pmod{p^2}.$$

(v) *If $(\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1$ and $p=x^2+78y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{530604^k}(9826-6930\sqrt2\pm 5\sqrt{26})^k\equiv\left(\frac{x}{3}\right)\left(2x-\frac{p}{2x}\right) \pmod{p^2}.$$

(vi) *If* $(\frac{2}{p})=(\frac{p}{3})=(\frac{p}{17})=1$ *and* $p=x^2+102y^2$ *with* $x,y\in\mathbb{Z}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{3881196^k}(71874-17420\sqrt{17}\pm35\sqrt{2})^k\equiv\left(\frac{x}{3}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

(vii) *If* $(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{11})=1$ *and* $p=x^2+33y^2$ *with* $x,y\in\mathbb{Z}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2^{12}3)^k}(96-5\sqrt{11}\pm65\sqrt{3})^k\equiv\left(\frac{x}{3}\right)\left(\frac{p}{2x}-2x\right)\pmod{p^2}.$$

Remark 4.2. Let $p\equiv1,4\pmod{15}$ be a prime with $p=x^2+15y^2$ $(x,y\in\mathbb{Z})$. Applying (26) we see that

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k$$
$$\equiv P_{(p-1)/2}(7\sqrt{-15}+16\sqrt{-3})P_{(p-1)/2}(7\sqrt{-15}-16\sqrt{-3})\pmod{p^2}.$$

Thus part (i) of Conjecture 4.4 implies that

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k\equiv\left(2x-\frac{p}{2x}\right)^2\equiv4x^2-2p\pmod{p^2}.$$

We omit here similar comments on parts (ii)-(vii) of Conjecture 4.4. We also have many other conjectures similar to Conjecture 4.4.

Conjecture 4.5. *Let* $p>5$ *be a prime. Then*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(62,1)}{(-128^2)^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(62,1)}{(-480^2)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv1,9\pmod{20}\ \&\ p=x^2+5y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } p\equiv3,7\pmod{20}\ \&\ 2p=x^2+5y^2,\\ 0\pmod{p^2} & \text{if } p\equiv11,13,17,19\pmod{20}.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(340k+111)\frac{\binom{2k}{k}^2T_{2k}(62,1)}{(-128^2)^k}\equiv3p\left(\frac{-1}{p}\right)\left(22+15\left(\frac{p}{15}\right)\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(340k+59)\frac{\binom{2k}{k}^2T_{2k}(62,1)}{(-480^2)^k}\equiv p\left(\frac{-1}{p}\right)\left(51+8\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

Conjecture 4.6. *Let $p>3$ be a prime. Then*

$$\begin{aligned}
&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2}{4^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(4,1)}{16^k}\\
&\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(10,1)}{(-64)^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(6,1)}{256^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(6,1)}{1024^k}\\
&\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24},\ p=x^2+6y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24},\ p=2x^2+3y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1;\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(6,1)}{192^k}\\
&\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24},\ p=x^2+6y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24},\ p=2x^2+3y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1.\end{cases}
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^2T_k(10,1)}{(-64)^k}&\equiv\frac{p}{4}\left(3\left(\frac{p}{3}\right)+1\right)\pmod{p^2},\\
\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}T_k^2(6,1)}{192^k}&\equiv p\left(\frac{-6}{p}\right)\left(4-3\left(\frac{2}{p}\right)\right)\pmod{p^2}.
\end{aligned}$$

Conjecture 4.7. *Let $p>5$ be a prime.*

(i) *We have*

$$\begin{aligned}
&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(3,1)}{36^k}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(34,1)}{(-64)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(18,1)}{4096^k}\\
&\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,9,11,19\pmod{40},\ p=x^2+10y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } p\equiv 7,13,23,37\pmod{40},\ p=2x^2+5y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-10}{p})=-1.\end{cases}
\end{aligned}$$

Also,

$$\sum_{k=0}^{p-1}(16k+5)\frac{\binom{2k}{k}T_k(3,1)^2}{36^k}\equiv 5p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(60k+23)\frac{\binom{2k}{k}^2T_k(34,1)}{(-64)^k}\equiv p\left(8\left(\frac{2}{p}\right)+15\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

Conjecture 4.8. *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(18,1)}{512^k}\equiv\left(\frac{10}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(6,1)}{(-512)^k}$$
$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=1\ \&\ 4p=x^2+35y^2,\\ 2p-5x^2 \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ 4p=5x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{35})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(35k+9)\frac{\binom{2k}{k}\binom{3k}{k}T_k(18,1)}{512^k}\equiv\frac{9p}{2}\left(7-5\left(\frac{p}{5}\right)\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(35k+9)\frac{\binom{2k}{k}^2T_{3k}(6,1)}{(-512)^k}\equiv\frac{9p}{32}\left(\frac{2}{p}\right)\left(25+7\left(\frac{p}{7}\right)\right) \pmod{p^2}.$$

Conjecture 4.9. *Let $p\neq 2,29$ be a prime. When $p\neq 5,7$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(19602,1)}{78400^{2k}}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{29}{p})=1\ \&\ p=x^2+58y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{29}{p})=-1\ \&\ p=2x^2+29y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-58}{p})=-1.\end{cases}$$

Provided $p\neq 13$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(19602,1)}{78416^{2k}}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{29}{p})=1\ \&\ p=x^2+58y^2,\\ 2p-8x^2 \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{29}{p})=-1\ \&\ p=2x^2+29y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-58}{p})=-1.\end{cases}$$

Conjecture 4.10. *Let $p > 5$ be a prime. Then*

$$\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(26,1)}{(-24)^{3k}} \equiv \left(\frac{15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}$$

$$\equiv \begin{cases} x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=1\ \&\ 4p=x^2+91y^2, \\ 2p-7x^2 \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=-1\ \&\ 4p=7x^2+13y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{91})=-1. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(819k+239)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(26,1)}{(-24)^{3k}} \equiv \frac{p}{32}\left(\frac{-6}{p}\right)\left(949+6699\left(\frac{p}{7}\right)\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(1638k+277)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}} \equiv \frac{p}{40}\left(\frac{-105}{p}\right)\left(8701+2379\left(\frac{p}{7}\right)\right) \pmod{p^2}.$$

Remark 4.3. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number two for $d = 5, 6, 10, 15, 35, 58, 91$.

Conjecture 4.11. *Let $p > 3$ be a prime. We have*

$$\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}T_k(110,1)}{(-96^2)^k}$$

$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=1,\ p=x^2+21y^2, \\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{7})=-1, (\frac{p}{3})=1,\ p=3x^2+7y^2, \\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1, (\frac{p}{7})=1,\ 2p=x^2+21y^2, \\ 6x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=1, (\frac{p}{3})=(\frac{p}{7})=-1,\ 2p=3x^2+7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-21}{p})=-1, \end{cases}$$

and

$$\sum_{k=0}^{p-1}(28k+5)\frac{\binom{4k}{2k}\binom{2k}{k}T_k(110,1)}{(-96^2)^k} \equiv \frac{p}{8}\left(\frac{-6}{p}\right)\left(33+7\left(\frac{p}{7}\right)\right) \pmod{p^2}.$$

Conjecture 4.12. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(18,1)}{256^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p3)=(\frac p5)=1,\ p=x^2+30y^2,\\12x^2-2p\pmod{p^2}&\text{if }(\frac p3)=1,(\frac 2p)=(\frac p5)=-1,\ p=3x^2+10y^2,\\2p-8x^2\pmod{p^2}&\text{if }(\frac 2p)=1,(\frac p3)=(\frac p5)=-1,\ p=2x^2+15y^2,\\2p-6x^2\pmod{p^2}&\text{if }(\frac p5)=1,(\frac 2p)=(\frac p3)=-1,\ 2p=3x^2+10y^2,\\0\pmod{p^2}&\text{if }(\frac{-30}p)=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(30,1)}{256^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-2}p)=(\frac p3)=(\frac p7)=1,\ p=x^2+42y^2,\\12x^2-2p\pmod{p^2}&\text{if }(\frac{-2}p)=1,(\frac p3)=(\frac p7)=-1,\ p=3x^2+14y^2,\\2p-8x^2\pmod{p^2}&\text{if }(\frac p7)=1,(\frac{-2}p)=(\frac p3)=-1,\ p=2x^2+21y^2,\\2p-6x^2\pmod{p^2}&\text{if }(\frac p3)=1,(\frac{-2}p)=(\frac p7)=-1,\ 2p=3x^2+14y^2,\\0\pmod{p^2}&\text{if }(\frac{-42}p)=-1.\end{cases}$$

Conjecture 4.13. *Let $p>3$ be a prime. When $p\neq 13,17$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(102,1)}{102^{3k}}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p3)=(\frac p{13})=1,\ p=x^2+78y^2,\\2p-8x^2\pmod{p^2}&\text{if }(\frac 2p)=1,(\frac p3)=(\frac p{13})=-1,\ p=2x^2+39y^2,\\12x^2-2p\pmod{p^2}&\text{if }(\frac p{13})=1,(\frac 2p)=(\frac p3)=-1,\ p=3x^2+26y^2,\\2p-24x^2\pmod{p^2}&\text{if }(\frac p3)=1,(\frac 2p)=(\frac p{13})=-1,\ p=6x^2+13y^2,\\0\pmod{p^2}&\text{if }(\frac{-78}p)=-1.\end{cases}$$

Provided $p \neq 11, 17$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_k(198,1)}{198^{3k}}$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{3}) = (\frac{p}{17}) = 1,\ p = x^2+102y^2, \\ 2p-8x^2 \pmod{p^2} & \text{if } (\frac{p}{17}) = 1, (\frac{2}{p}) = (\frac{p}{3}) = -1,\ p = 2x^2+51y^2, \\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3}) = 1, (\frac{2}{p}) = (\frac{p}{17}) = -1,\ p = 3x^2+34y^2, \\ 2p-24x^2 \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, (\frac{p}{3}) = (\frac{p}{17}) = -1,\ p = 6x^2+17y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-102}{p}) = -1. \end{cases}$$

Conjecture 4.14. *Let p be an odd prime and let m belong to the set $\{2, 3, 6, 10, 18, 30, 102, 198\}$. If $p \nmid m$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_k(m,1)}{m^{3k}} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 T_{2k}(m,1)}{256^k} \pmod{p^2}. \tag{36}$$

If $m^2 \not\equiv -12 \pmod{p}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 T_{2k}(m,1)}{256^k} \equiv \left(\frac{m^2+12}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}T_k(m^2-2,1)}{(m^2+12)^{2k}} \pmod{p^2}. \tag{37}$$

Remark 4.4. We note that (36) holds mod p for any integer $m \not\equiv 0 \pmod{p}$, and (37) holds mod p for any $m \in \mathbb{Z}$ with $m^2 \not\equiv -12 \pmod{p}$.

Conjecture 4.15. *Let $p \neq 2, 5, 19$ be a prime. We have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 T_{2k}(5778,1)}{1216^{2k}}$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{5}) = (\frac{p}{19}) = 1,\ p = x^2+190y^2, \\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, (\frac{p}{5}) = (\frac{p}{19}) = -1,\ p = 2x^2+95y^2, \\ 2p-20x^2 \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{5}) = -1, (\frac{p}{19}) = 1,\ p = 5x^2+38y^2, \\ 2p-40x^2 \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{19}) = -1, (\frac{p}{5}) = 1,\ p = 10x^2+19y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-190}{p}) = -1, \end{cases}$$

and

$$\sum_{k=0}^{p-1} (57720k+24893) \frac{\binom{2k}{k}^2 T_{2k}(5778,1)}{1216^{2k}}$$
$$\equiv p\left(11548 + 13345\left(\frac{p}{95}\right)\right) \pmod{p^2}.$$

Provided $p\neq 17$ *we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(5778,1)}{439280^{2k}}\equiv\left(\frac p5\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(5778,1)}{1216^{2k}}\pmod{p^2},$$

$$\begin{aligned}&\sum_{k=0}^{p-1}(57720k+3967)\frac{\binom{2k}{k}^2T_{2k}(5778,1)}{439280^{2k}}\\&\quad\equiv p\left(\frac p{19}\right)\left(3983-16\left(\frac p{95}\right)\right)\pmod{p^2}.\end{aligned}$$

Conjecture 4.16. *Let* $p>5$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(198,1)}{224^{2k}}\equiv\left(\frac p7\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(322,1)}{48^{4k}}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p5)=(\frac p7)=1,\ p=x^2+70y^2,\\8x^2-2p\pmod{p^2}&\text{if }(\frac p7)=1,(\frac 2p)=(\frac p5)=-1,\ p=2x^2+35y^2,\\2p-20x^2\pmod{p^2}&\text{if }(\frac p5)=1,(\frac 2p)=(\frac p7)=-1,\ p=5x^2+14y^2,\\28x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=1,(\frac p5)=(\frac p7)=-1,\ p=7x^2+10y^2,\\0\pmod{p^2}&\text{if }(\frac{-70}p)=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(322,1)}{(-2^{10}3^4)^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}p)=(\frac p5)=(\frac p{17})=1,\ p=x^2+85y^2,\\2p-2x^2\pmod{p^2}&\text{if }(\frac p{17})=1,(\frac{-1}p)=(\frac p5)=-1,\ 2p=x^2+85y^2,\\2p-20x^2\pmod{p^2}&\text{if }(\frac{-1}p)=1,(\frac p5)=(\frac p{17})=-1,\ p=5x^2+17y^2,\\10x^2-2p\pmod{p^2}&\text{if }(\frac p5)=1,(\frac{-1}p)=(\frac p{17})=-1,\ 2p=5x^2+17y^2,\\0\pmod{p^2}&\text{if }(\frac{-85}p)=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(1298,1)}{24^{4k}}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-2}p)=(\frac p5)=(\frac p{13})=1,\ p=x^2+130y^2,\\8x^2-2p\pmod{p^2}&\text{if }(\frac{-2}p)=1,(\frac p5)=(\frac p{13})=-1,\ p=2x^2+65y^2,\\2p-20x^2\pmod{p^2}&\text{if }(\frac p5)=1,(\frac{-2}p)=(\frac p{13})=-1,\ p=5x^2+26y^2,\\2p-40x^2\pmod{p^2}&\text{if }(\frac p{13})=1,(\frac{-2}p)=(\frac p5)=-1,\ p=10x^2+13y^2,\\0\pmod{p^2}&\text{if }(\frac{-130}p)=-1.\end{cases}$$

Remark 4.5. . The imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number four for $d = 21, 30, 42, 70, 78, 85, 102, 130, 190$.

Conjecture 4.17. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_{3k}}{(-27)^k} \equiv \begin{cases} (\frac{p}{3})(4x^2-2p) \pmod{p^2} & \text{if } p = x^2+7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1; \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(26,81)}{24^{3k}} \equiv \begin{cases} (\frac{6}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p = x^2+11y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1; \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(10,1)}{24^{3k}} \equiv \begin{cases} (\frac{6}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p = x^2+19y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{19}) = -1. \end{cases}$$

If $p \neq 13$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(106,1)}{312^{3k}} \equiv \begin{cases} (\frac{78}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p = x^2+43y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{43}) = -1. \end{cases}$$

If $p \neq 73$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(586,1)}{1752^{3k}}$$
$$\equiv \begin{cases} (\frac{438}{p})(x^2-2p) \pmod{p^2} & \text{if } (\frac{p}{67}) = 1 \ \&\ 4p = x^2+67y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{67}) = -1. \end{cases}$$

If $p \neq 8893$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(71146,1)}{213432^{3k}}$$
$$\equiv \begin{cases} (\frac{53358}{p})(x^2-2p) \pmod{p^2} & \text{if } (\frac{p}{163}) = 1 \ \&\ 4p = x^2+163y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{163}) = -1. \end{cases}$$

Also,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(2,-1)}{(-3456)^k} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(2,9)}{24^{3k}}$$
$$\equiv \begin{cases} (\frac{2}{p})(x^2-2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \ \&\ 4p = x^2+27y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(15k+2)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(2,-1)}{(-3456)^k}$$
$$\equiv\begin{cases}2p(\frac{2}{p})\ (\mathrm{mod}\ p^2) & \text{if } 3\mid p-1 \text{ and } 2 \text{ is a cubic residue mod } p,\\ 0\ (\mathrm{mod}\ p) & \text{otherwise.}\end{cases}$$

Remark 4.6. The imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number one for $d=7,11,19,43,67,163$. We observe that if $p>3$ is a prime and m is an integer with $m(3m+8)\not\equiv 0\ (\mathrm{mod}\ \ p)$ then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(m+2,1)}{(3m)^{3k}}$$
$$\equiv\left(\frac{-m(3m+8)}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-9m-24)^{3k}}\pmod{p}.$$

Conjecture 4.18. *Let p be an odd prime.*

(i) *When $p>5$ we have*

$$\sum_{k=0}^{p-1}\left(\frac{T_k(38,21^2)}{(-16)^k}\right)^3\equiv\left(\frac{-5}{p}\right)\sum_{k=0}^{p-1}\left(\frac{T_k(38,21^2)}{20^k}\right)^3$$
$$\equiv\begin{cases}4x^2-2p\,(\mathrm{mod}\ \ p^2) & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1,\ p=x^2+30y^2,\\ 12x^2-2p\,(\mathrm{mod}\ \ p^2) & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1,\ p=3x^2+10y^2,\\ 2p-8x^2\,(\mathrm{mod}\ \ p^2) & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ p=2x^2+15y^2,\\ 20x^2-2p\,(\mathrm{mod}\ \ p^2) & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1,\ p=5x^2+6y^2,\\ p\delta_{p,7}\,(\mathrm{mod}\ \ p^2) & \text{if } (\frac{-30}{p})=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(28k+15)\frac{T_k^3(38,21^2)}{(-16)^{3k}}\equiv\frac{p}{7}\left(124-19\left(\frac{p}{3}\right)\right)\pmod{p^2}.$$

If $p\neq 7$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(4,9)}{28^{2k}}\equiv\sum_{k=0}^{p-1}\left(\frac{T_k(38,21^2)}{(-16)^k}\right)^3\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k^2(4,9)\equiv p\left(\frac{-6}{p}\right)\left(4+\left(\frac{2}{p}\right)\right)\pmod{p^2}.$$

(ii) *When $p\neq 7$ we have*

$$\sum_{k=0}^{p-1}\left(\frac{T_k(110,57^2)}{32^k}\right)^3\equiv\left(\frac{-14}{p}\right)\sum_{k=0}^{p-1}\left(\frac{T_k(110,57^2)}{(-28)^k}\right)^3$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{7})=1,\ p=x^2+42y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=1,(\frac{-2}{p})=(\frac{p}{3})=-1,\ p=2x^2+21y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=1,(\frac{p}{3})=(\frac{p}{7})=-1,\ p=3x^2+14y^2,\\ 24x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,(\frac{-2}{p})=(\frac{p}{7})=-1,\ p=6x^2+7y^2,\\ p\delta_{p,19}\pmod{p^2} & \text{if } (\frac{-42}{p})=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(684k+329)\frac{T_k^3(110,57^2)}{2^{15k}}\equiv\frac{p}{19}\left(5160\left(\frac{-2}{p}\right)+1091\right)\pmod{p^2}.$$

If $p>3$ and $p\neq 11,19$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(5,1)}{22^{2k}}\equiv\sum_{k=0}^{p-1}\left(\frac{T_k(110,57^2)}{32^k}\right)^3\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{560k+71}{22^{2k}}\binom{2k}{k}T_k^2(5,1)\equiv\frac{p}{3}\left(280\left(\frac{p}{7}\right)-67\right)\pmod{p^2}.$$

Conjecture 4.19. *Let $p>3$ be a prime. When $p\neq 5$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(9,12)}{900^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{11})=1,\ p=x^2+33y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=1,(\frac{p}{3})=(\frac{p}{11})=-1,\ 2p=x^2+33y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{11})=1,(\frac{-1}{p})=(\frac{p}{3})=-1,\ p=3x^2+11y^2,\\ 6x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,(\frac{-1}{p})=(\frac{p}{11})=-1,\ 2p=3x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-33}{p})=-1.\end{cases}$$

Provided $p\ne 29$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(171,-171)}{(-5177196)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{7})=(\frac{p}{19})=1,\ p=x^2+133y^2,\\ 2x^2-2p\pmod{p^2}&\text{if }(\frac{p}{7})=1,(\frac{-1}{p})=(\frac{p}{19})=-1,\ 2p=x^2+133y^2,\\ 2p-28x^2\pmod{p^2}&\text{if }(\frac{p}{19})=1,(\frac{-1}{p})=(\frac{p}{7})=-1,\ p=7x^2+19y^2,\\ 2p-14x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=1,(\frac{p}{7})=(\frac{p}{19})=-1,\ 2p=7x^2+19y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-133}{p})=-1.\end{cases}$$

Conjecture 4.20. *Let p be an odd prime. When $p\ne 23$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(7,1)}{46^{2k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{2}{p})=(\frac{p}{5})=(\frac{p}{7})=1,\ p=x^2+70y^2,\\ 8x^2-2p\pmod{p^2}&\text{if }(\frac{p}{7})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1,\ p=2x^2+35y^2,\\ 20x^2-2p\pmod{p^2}&\text{if }(\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{7})=-1,\ p=5x^2+14y^2,\\ 2p-28x^2\pmod{p^2}&\text{if }(\frac{2}{p})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1,\ p=7x^2+10y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-70}{p})=-1.\end{cases}$$

Provided $p\ne 3,7,11,17,31$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(73,576)}{434^{2k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{2}{p})=(\frac{p}{3})=(\frac{p}{17})=1,\ p=x^2+102y^2,\\ 8x^2-2p\pmod{p^2}&\text{if }(\frac{p}{17})=1,(\frac{2}{p})=(\frac{p}{3})=-1,\ p=2x^2+51y^2,\\ 12x^2-2p\pmod{p^2}&\text{if }(\frac{p}{3})=1,(\frac{2}{p})=(\frac{p}{17})=-1,\ p=3x^2+34y^2,\\ 24x^2-2p\pmod{p^2}&\text{if }(\frac{2}{p})=1,(\frac{p}{3})=(\frac{p}{17})=-1,\ p=6x^2+17y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-102}{p})=-1.\end{cases}$$

Conjecture 4.21. *Let $p>3$ be a prime. If $p\neq 23$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(23,7^4)}{46^{2k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1,\ p=x^2+78y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,(\frac{p}{3})=(\frac{p}{13})=-1,\ p=2x^2+39y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{p}{13})=1,(\frac{2}{p})=(\frac{p}{3})=-1,\ p=3x^2+26y^2,\\ 2p-24x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,(\frac{2}{p})=(\frac{p}{13})=-1,\ p=6x^2+13y^2,\\ p\delta_{p,7}\pmod{p^2} & \text{if } (\frac{-78}{p})=-1,\end{cases}$$

where $\delta_{m,n}$ takes 1 or 0 according as $m=n$ or not. If $p\neq 5$, then

$$\sum_{k=0}^{p-1}\frac{T_k^3(1298,651^2)}{(-100)^{3k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1,\ p=x^2+78y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } (\frac{2}{p})=1,(\frac{p}{3})=(\frac{p}{13})=-1,\ p=2x^2+39y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{13})=1,(\frac{2}{p})=(\frac{p}{3})=-1,\ p=3x^2+26y^2,\\ 2p-24x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,(\frac{2}{p})=(\frac{p}{13})=-1,\ p=6x^2+13y^2,\\ p(\delta_{p,7}+\delta_{p,31})\pmod{p^2} & \text{if } (\frac{-78}{p})=-1.\end{cases}$$

Conjecture 4.22. *Let $p\neq 2,7,11$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(7,81)}{14^{2k}}\equiv\left(\frac{p}{11}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(20,1)}{28^{2k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-11}{p})=(\frac{2}{p})=1\ \&\ p=x^2+22y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{-11}{p})=(\frac{2}{p})=-1\ \&\ p=2x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-11}{p})=-(\frac{2}{p}).\end{cases}$$

Conjecture 4.23. *Let $p\neq 2,7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2(6,2)}{450^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1,\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{221k+28}{450^k}\binom{2k}{k}T_k^2(6,2)\equiv\frac{4p}{7}\left(72\left(\frac{-1}{p}\right)-23\right)\pmod{p^2}.$$

Conjecture 4.24. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k^2(7,12)}{4^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_{2k}^2(3,3)}{36^k}$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \ \&\ p = x^2 + 9y^2, \\ 4xy \pmod{p^2} & \text{if } p \equiv 5 \pmod{12} \ \&\ p = x^2 + y^2 \ (3 \mid x - y), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Conjecture 4.25. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k^2(19,-20)}{22^{2k}} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_{2k}^2(9,20)}{4^k}$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{-1}{p}) = (\frac{p}{5}) = 1 \ \&\ p = x^2 + y^2 \ (5 \nmid x), \\ 4xy \pmod{p^2} & \text{if } (\frac{-1}{p}) = -(\frac{p}{5}) = 1 \ \&\ p = x^2 + y^2 \ (5 \mid x - y), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4} \ \&\ p \neq 11. \end{cases}$$

We have many conjectures similar to Conjectures 4.18-4.25. For example, we find that $\sum_{k=0}^{p-1} \binom{2k}{k} T_k^2(b,c)/m^k \bmod p^2$ is related to the representation $p = x^2 + dy^2$ if $(b,c,m;d)$ is among

$$(5,4,4;10),\ (3,-4,36;13),\ (5,4,14^2;30),$$
$$(7,1,14^2;30),\ (7,28,14^2;21),\ (11,49,22^2;42).$$

Though we will not list many other conjectures similar to Conjectures 4.4-4.25, the above conjectures should convince the reader that our conjectural series for $1/\pi$ in the next section are indeed reasonable in view of the corresponding congruences.

5. Dualities and new series for $1/\pi$

As mentioned in Section 1, for $b > 0$ and $c > 0$ the main term of $T_n(b,c)$ as $n \to +\infty$ is

$$f_n(b,c) := \frac{(b + 2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}.$$

Here we formulate a further refinement of this.

Conjecture 5.1. *For any positive real numbers b and c, we have*

$$T_n(b,c) = f_n(b,c)\left(1 + \frac{b - 4\sqrt{c}}{16n\sqrt{c}} + O\left(\frac{1}{n^2}\right)\right)$$

as $n\to+\infty$. If $c>0$ and $b=4\sqrt{c}$, then

$$\frac{T_n(b,c)}{\sqrt{c}^n}=T_n(4,1)=\frac{3\times 6^n}{\sqrt{6n\pi}}\left(1+\frac{1}{8n^2}+\frac{15}{64n^3}+\frac{21}{32n^4}+O\left(\frac{1}{n^5}\right)\right).$$

If $c<0$ and $b\in\mathbb{R}$ then

$$\lim_{n\to\infty}\sqrt[n]{|T_n(b,c)|}=\sqrt{b^2-4c}.$$

Let p be an odd prime. Z.-H. Sun [S1] proved the congruence

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}x^k\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}(1-x)^k \pmod{p^2} \tag{38}$$

via Legendre polynomials; in fact this follows from the well-known identity $P_n(-x)=(-1)^nP_n(x)$ with $n=(p-1)/2$. In [Su8] the author managed to show the following congruences via the Zeilberger algorithm:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}x^k\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}(1-x)^k \pmod{p^2}, \tag{39}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}x^k\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}(1-x)^k \pmod{p^2}\quad (p\neq 3), \tag{40}$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}x^k\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}(1-x)^k \pmod{p^2}\quad (p\neq 3). \tag{41}$$

Our following result on dualities was motivated by (38)-(41).

Theorem 5.1. *Let p be an odd prime and let b,c and $m\not\equiv 0 \pmod p$ be rational p-adic integers. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(16m)^k}T_k(b,c)\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(16m)^k}T_k(m-b,c) \pmod{p^2}, \tag{42}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}T_k(b,c)\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}T_k(m-b,c) \pmod{p^2}, \tag{43}$$

Provided $p>3$ we also have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(27m)^k}T_k(b,c)\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(27m)^k}T_k(m-b,c)\pmod{p^2},\tag{44}$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{(432m)^k}T_k(b,c)\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{(432m)^k}T_k(m-b,c)\pmod{p^2}.\tag{45}$$

Proof. Since the proofs of (42)-(45) are very similar, we just show (43) in detail.

For $d=0,\ldots,p-1$, by taking differentiations of both sides (39) d times we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\binom{k}{d}x^{k-d}\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}(-1)^d\binom{k}{d}(1-x)^{k-d}\pmod{p^2}.$$

In view of this, we have

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}T_k(b,c)\\
=&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}\binom{2j}{j}b^{k-2j}c^j\\
=&\sum_{j=0}^{p-1}\binom{2j}{j}\frac{c^j}{m^{2j}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\binom{k}{2j}\left(\frac{b}{m}\right)^{k-2j}\\
\equiv&\sum_{j=0}^{p-1}\binom{2j}{j}\frac{c^j}{m^{2j}}\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\binom{k}{2j}\left(1-\frac{b}{m}\right)^{k-2j}\\
=&\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}\binom{2j}{j}(m-b)^{k-2j}c^j\\
=&\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}T_k(m-b,c)\pmod{p^2}.\end{aligned}$$

The proof of Theorem 5.1 is now complete. □

Example 1. Let p be an odd prime. By (42) we have

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(5,4)}{16^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(-4,4)}{16^k}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\pmod{p^2}.$$

The author [Su4] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4),\end{cases}$$

and this was recently confirmed by Z.-H. Sun [S2]. When $p>3$, by (44) we have

$$\begin{aligned}&\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(3,1)}{27^k}\\ \equiv&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(-2,1)}{27^k}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k} \pmod{p^2};\end{aligned}$$

the reader may consult [Su4, Conjecture 5.6] for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/(-27)^k$ mod p^2.

Based on our investigations of congruences on sums of central binomial coefficients and central trinomial coefficients, and the author's philosophy about series for $1/\pi$ stated in [Su7], we raise 61 conjectural series for $1/\pi$ of the following seven new types with a,b,c,d,m integers and $abcd(b^2-4c)m$ nonzero.

`Type I.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}^2T_k(b,c)/m^k$.
`Type II.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}\binom{3k}{k}T_k(b,c)/m^k$.
`Type III.` $\sum_{k=0}^\infty(a+dk)\binom{4k}{2k}\binom{2k}{k}T_k(b,c)/m^k$.
`Type IV.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}^2T_{2k}(b,c)/m^k$.
`Type V.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}\binom{3k}{k}T_{3k}(b,c)/m^k$.
`Type VI.` $\sum_{k=0}^\infty(a+dk)T_k(b,c)^3/m^k$.
`Type VII.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}T_k(b,c)^2/m^k$.

Recall that a series $\sum_{k=0}^\infty a_k$ is said to converge at a geometric rate with ratio r if $\lim_{k\to+\infty}a_{k+1}/a_k=r\in(0,1)$. All the series in Conjectures I-VII below converge at geometrical rates, and they were found by the author in 2011 except that (IV19)-(IV21) were discovered in 2012.

Conjecture I. We have the following identities:

$$\sum_{k=0}^\infty\frac{30k+7}{(-256)^k}\binom{2k}{k}^2T_k(1,16)=\frac{24}{\pi},\tag{I1}$$

$$\sum_{k=0}^\infty\frac{30k+7}{(-1024)^k}\binom{2k}{k}^2T_k(34,1)=\frac{12}{\pi},\tag{I2}$$

$$\sum_{k=0}^{\infty}\frac{30k-1}{4096^k}\binom{2k}{k}^2T_k(194,1)=\frac{80}{\pi},\tag{I3}$$

$$\sum_{k=0}^{\infty}\frac{42k+5}{4096^k}\binom{2k}{k}^2T_k(62,1)=\frac{16\sqrt{3}}{\pi}.\tag{I4}$$

Remark 5.1. (I1) was the first identity for $1/\pi$ involving generalized central trinomial coefficients; it was discovered by the author on Jan. 2, 2011. Different from classical Ramanujan-type series for $1/\pi$ (cf. N. D. Baruah and B. C. Berndt [BB], and Berndt [Be, pp. 353-354]) and their known generalizations (see, e.g., S. Cooper [C]), the two coefficients in the linear part $30k-1$ of (I3) have *different signs*, and also its corresponding p-adic congruence (with $p>3$ a prime) involves *two* Legendre symbols:

$$\sum_{k=0}^{p-1}(30k-1)\frac{\binom{2k}{k}^2T_k(194,1)}{4096^k}\equiv p\left(5\left(\frac{-1}{p}\right)-6\left(\frac{3}{p}\right)\right)\pmod{p^2}.$$

Conjecture II. We have

$$\sum_{k=0}^{\infty}\frac{15k+2}{972^k}\binom{2k}{k}\binom{3k}{k}T_k(18,6)=\frac{45\sqrt{3}}{4\pi},\tag{II1}$$

$$\sum_{k=0}^{\infty}\frac{91k+12}{10^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(10,1)=\frac{75\sqrt{3}}{2\pi},\tag{II2}$$

$$\sum_{k=0}^{\infty}\frac{15k-4}{18^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(198,1)=\frac{135\sqrt{3}}{2\pi},\tag{II3}$$

$$\sum_{k=0}^{\infty}\frac{42k-41}{30^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(970,1)=\frac{525\sqrt{3}}{\pi},\tag{II4}$$

$$\sum_{k=0}^{\infty}\frac{18k+1}{30^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(730,729)=\frac{25\sqrt{3}}{\pi},\tag{II5}$$

$$\sum_{k=0}^{\infty}\frac{6930k+559}{102^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(102,1)=\frac{1445\sqrt{6}}{2\pi},\tag{II6}$$

and

$$\sum_{k=0}^{\infty}\frac{222105k+15724}{198^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(198,1)=\frac{114345\sqrt{3}}{4\pi}, \tag{II7}$$

$$\sum_{k=0}^{\infty}\frac{390k-3967}{102^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(39202,1)=\frac{56355\sqrt{3}}{\pi}, \tag{II8}$$

$$\sum_{k=0}^{\infty}\frac{210k-7157}{198^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(287298,1)=\frac{114345\sqrt{3}}{\pi}, \tag{II9}$$

$$\sum_{k=0}^{\infty}\frac{45k+7}{24^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(26,729)=\frac{8}{3\pi}(3\sqrt{3}+\sqrt{15}), \tag{II10}$$

$$\sum_{k=0}^{\infty}\frac{9k+2}{(-5400)^k}\binom{2k}{k}\binom{3k}{k}T_k(70,3645)=\frac{15\sqrt{3}+\sqrt{15}}{6\pi}, \tag{II11}$$

$$\sum_{k=0}^{\infty}\frac{63k+11}{(-13500)^k}\binom{2k}{k}\binom{3k}{k}T_k(40,1458)=\frac{25}{12\pi}(3\sqrt{3}+4\sqrt{6}). \tag{II12}$$

Remark 5.2. . In view of (44), we may view (II9) as the dual of (II7) since $198^3/27-198=287298$. The series in (II7) converges rapidly at a geometric rate with ratio $25/35937$, but the series in (II9) converges very slow at a geometric rate with ratio $71825/71874$. (II2), (II9) and (II10) were motivated by the following congruences (with $p>3$ a prime):

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(10,1)}{10^{3k}}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8 \ \&\ p=x^2+2y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-2}{p})=-1,\ \text{i.e.,}\ p\equiv 5,7 \pmod 8,\end{cases}$$

$$\sum_{k=0}^{p-1}(91k+12)\frac{\binom{2k}{k}\binom{3k}{k}T_k(10,1)}{10^{3k}}$$
$$\equiv\frac{3p}{2}\left(9\left(\frac{-3}{p}\right)-1\right) \pmod{p^2}\quad (p\neq 5);$$

$$\sum_{k=0}^{p-1}(210k-7157)\frac{\binom{2k}{k}\binom{3k}{k}T_k(287298,1)}{198^{3k}}$$
$$\equiv p\left(35\left(\frac{-3}{p}\right)-7192\right)\pmod{p^2}\quad(p\neq 11);$$

$$\sum_{k=0}^{p-1}\frac{45k+7}{24^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(26,729)$$
$$\equiv\frac{p}{2}\left(9\left(\frac{-3}{p}\right)+5\left(\frac{-15}{p}\right)\right)\pmod{p^2}.$$

Conjecture III. We have the following formulae:

$$\sum_{k=0}^{\infty}\frac{85k+2}{66^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(52,1)=\frac{33\sqrt{33}}{\pi},\tag{III1}$$

$$\sum_{k=0}^{\infty}\frac{28k+5}{(-96^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(110,1)=\frac{3\sqrt{6}}{\pi},\tag{III2}$$

$$\sum_{k=0}^{\infty}\frac{40k+3}{112^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(98,1)=\frac{70\sqrt{21}}{9\pi},\tag{III3}$$

$$\sum_{k=0}^{\infty}\frac{80k+9}{264^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(257,256)=\frac{11\sqrt{66}}{2\pi},\tag{III4}$$

$$\sum_{k=0}^{\infty}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi},\tag{III5}$$

$$\sum_{k=0}^{\infty}\frac{760k+71}{336^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(322,1)=\frac{126\sqrt{7}}{\pi},\tag{III6}$$

$$\sum_{k=0}^{\infty}\frac{10k-1}{336^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(1442,1)=\frac{7\sqrt{210}}{4\pi},\tag{III7}$$

$$\sum_{k=0}^{\infty}\frac{770k+69}{912^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(898,1)=\frac{95\sqrt{114}}{4\pi},\tag{III8}$$

$$\sum_{k=0}^{\infty}\frac{280k-139}{912^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(12098,1)=\frac{95\sqrt{399}}{\pi},\tag{III9}$$

$$\sum_{k=0}^{\infty}\frac{84370k+6011}{10416^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(10402,1)=\frac{3689\sqrt{434}}{4\pi},\tag{III10}$$

$$\sum_{k=0}^{\infty}\frac{8840k-50087}{10416^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(1684802,1)=\frac{7378\sqrt{8463}}{\pi},\tag{III11}$$

$$\sum_{k=0}^{\infty}\frac{11657240k+732103}{39216^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(39202,1)=\frac{80883\sqrt{817}}{\pi},\tag{III12}$$

$$\sum_{k=0}^{\infty}\frac{3080k-58871}{39216^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(23990402,1)=\frac{17974\sqrt{2451}}{\pi}.\tag{III13}$$

Remark 5.3. (III12) and (III13) are dual in view of (43). Other dual pairs include (III6) and (III7), (III8) and (III9), (III10) and (III11). Below are the corresponding p-adic congruences for (III1) and (III13) (with $p>3$ a prime):

$$\sum_{k=0}^{p-1}(85k+2)\frac{\binom{4k}{2k}\binom{2k}{k}T_k(52,1)}{66^{2k}}$$
$$\equiv p\left(12\left(\frac{-33}{p}\right)-10\left(\frac{33}{p}\right)\right)\pmod{p^2}\quad(p\neq 11),$$
$$\sum_{k=0}^{p-1}(3080k-58871)\frac{\binom{4k}{2k}\binom{2k}{k}T_k(23990402,1)}{39216^{2k}}$$
$$\equiv p\left(385\left(\frac{-2451}{p}\right)-59256\left(\frac{1634}{p}\right)\right)\pmod{p^2}\quad(p\neq 19,43).$$

Conjecture IV. We have

$$\sum_{k=0}^{\infty}\frac{26k+5}{(-48^2)^k}\binom{2k}{k}^2T_{2k}(7,1)=\frac{48}{5\pi},\tag{IV1}$$

$$\sum_{k=0}^{\infty}\frac{340k+59}{(-480^2)^k}\binom{2k}{k}^2T_{2k}(62,1)=\frac{120}{\pi},\tag{IV2}$$

$$\sum_{k=0}^{\infty}\frac{13940k+1559}{(-5760^2)^k}\binom{2k}{k}^2T_{2k}(322,1)=\frac{4320}{\pi},\tag{IV3}$$

$$\sum_{k=0}^{\infty}\frac{8k+1}{96^{2k}}\binom{2k}{k}^2T_{2k}(10,1)=\frac{10\sqrt2}{3\pi},\tag{IV4}$$

$$\sum_{k=0}^{\infty}\frac{10k+1}{240^{2k}}\binom{2k}{k}^2T_{2k}(38,1)=\frac{15\sqrt6}{4\pi},\tag{IV5}$$

$$\sum_{k=0}^{\infty}\frac{14280k+899}{39200^{2k}}\binom{2k}{k}^2T_{2k}(198,1)=\frac{1155\sqrt6}{\pi},\tag{IV6}$$

$$\sum_{k=0}^{\infty}\frac{120k+13}{320^{2k}}\binom{2k}{k}^2T_{2k}(18,1)=\frac{12\sqrt{15}}{\pi},\tag{IV7}$$

$$\sum_{k=0}^{\infty}\frac{21k+2}{896^{2k}}\binom{2k}{k}^2T_{2k}(30,1)=\frac{5\sqrt7}{2\pi},\tag{IV8}$$

$$\sum_{k=0}^{\infty}\frac{56k+3}{24^{4k}}\binom{2k}{k}^2T_{2k}(110,1)=\frac{30\sqrt7}{\pi},\tag{IV9}$$

$$\sum_{k=0}^{\infty}\frac{56k+5}{48^{4k}}\binom{2k}{k}^2T_{2k}(322,1)=\frac{72\sqrt7}{5\pi},\tag{IV10}$$

$$\sum_{k=0}^{\infty}\frac{10k+1}{2800^{2k}}\binom{2k}{k}^2T_{2k}(198,1)=\frac{25\sqrt{14}}{24\pi},\tag{IV11}$$

$$\sum_{k=0}^{\infty}\frac{195k+14}{10400^{2k}}\binom{2k}{k}^2T_{2k}(102,1)=\frac{85\sqrt{39}}{12\pi},\tag{IV12}$$

$$\sum_{k=0}^{\infty}\frac{3230k+263}{46800^{2k}}\binom{2k}{k}^2T_{2k}(1298,1)=\frac{675\sqrt{26}}{4\pi},\tag{IV13}$$

$$\sum_{k=0}^{\infty}\frac{520k-111}{5616^{2k}}\binom{2k}{k}^2T_{2k}(1298,1)=\frac{1326\sqrt3}{\pi},\tag{IV14}$$

$$\sum_{k=0}^{\infty}\frac{280k-149}{20400^{2k}}\binom{2k}{k}^2T_{2k}(4898,1)=\frac{330\sqrt{51}}{\pi},\tag{IV15}$$

$$\sum_{k=0}^{\infty}\frac{78k-1}{28880^{2k}}\binom{2k}{k}^2T_{2k}(5778,1)=\frac{741\sqrt{10}}{20\pi},\tag{IV16}$$

$$\sum_{k=0}^{\infty}\frac{57720k+3967}{439280^{2k}}\binom{2k}{k}^2T_{2k}(5778,1)=\frac{2890\sqrt{19}}{\pi},\tag{IV17}$$

$$\sum_{k=0}^{\infty}\frac{1615k-314}{243360^{2k}}\binom{2k}{k}^2T_{2k}(54758,1)=\frac{1989\sqrt{95}}{4\pi},\tag{IV18}$$

and

$$\sum_{k=0}^{\infty}\frac{34k+5}{4608^k}\binom{2k}{k}^2 T_{2k}(10,-2)=\frac{12\sqrt{6}}{\pi}, \tag{IV19}$$

$$\sum_{k=0}^{\infty}\frac{130k+1}{1161216^k}\binom{2k}{k}^2 T_{2k}(238,-14)=\frac{288\sqrt{2}}{\pi}, \tag{IV20}$$

$$\sum_{k=0}^{\infty}\frac{2380k+299}{(-16629048064)^k}\binom{2k}{k}^2 T_{2k}(9918,-19)=\frac{860\sqrt{7}}{3\pi}. \tag{IV21}$$

Remark 5.4. For (IV6), `Mathematica` indicates that if we set

$$s(n):=\sum_{k=0}^{n}\frac{14280k+899}{39200^{2k}}\binom{2k}{k}^2 T_{2k}(198,1)$$

then

$$\left|s(15)\times\frac{\pi}{1155\sqrt{6}}-1\right|<\frac{1}{10^{50}} \quad\text{and}\quad \left|s(30)\times\frac{\pi}{1155\sqrt{6}}-1\right|<\frac{1}{10^{100}}.$$

Here are corresponding p-adic congruences of (IV9)-(IV11) and (IV18) with $p>5$ a prime:

$$\sum_{k=0}^{p-1}(56k+3)\frac{\binom{2k}{k}^2 T_{2k}(110,1)}{24^{4k}}\equiv\frac{p}{4}\left(35\left(\frac{p}{7}\right)-23\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}(56k+5)\frac{\binom{2k}{k}^2 T_{2k}(322,1)}{48^{4k}}\equiv\frac{p}{20}\left(147\left(\frac{p}{7}\right)-47\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}(10k+1)\frac{\binom{2k}{k}^2 T_{2k}(198,1)}{2800^{2k}} \equiv\frac{p}{12}\left(\frac{2}{p}\right)\left(13\left(\frac{p}{7}\right)-1\right)\pmod{p^2}\quad(p\neq 7),$$

and

$$\sum_{k=0}^{p-1}(1615k-314)\frac{\binom{2k}{k}^2 T_{2k}(54758,1)}{243360^{2k}} \equiv\frac{p}{26}\left(6137\left(\frac{p}{95}\right)-14301\right)\pmod{p^2}\quad(p\neq 13).$$

For any prime $p>3$, the corresponding p-adic congruence of (IV19) is

$$\sum_{k=0}^{p-1}\frac{34k+5}{4608^k}\binom{2k}{k}^2 T_{2k}(10,-2)\equiv p\left(6\left(\frac{-6}{p}\right)-\left(\frac{6}{p}\right)\right)\pmod{p^2}.$$

Conjecture V. We have the formula

$$\sum_{k=0}^{\infty}\frac{1638k+277}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=\frac{44\sqrt{105}}{\pi}. \tag{V1}$$

Remark 5.5. (V1) was motivated by Conjecture 4.10; the series converges at a geometric rate with ratio $-64/125$.

We conjecture that (IV1)-(IV18) have exhausted all identities of the form

$$\sum_{k=0}^{\infty}(a+dk)\frac{\binom{2k}{k}^2T_{2k}(b,1)}{m^k}=\frac{C}{\pi}$$

with $a,d,m\in\mathbb{Z}$, $b\in\{1,3,4,\ldots\}$, $d>0$, and C^2 rational. This comes from our following hypothesis motivated by (16) in the case $h=2$ and the author's philosophy about series for $1/\pi$ stated in [Su7]. We have applied the hypothesis to seek for series for $1/\pi$ of type IV and checked all those $b=1,\ldots,10^6$ via computer.

Hypothesis 5.1 (i) Suppose that

$$\sum_{k=0}^{\infty}\frac{a+dk}{m^k}\binom{2k}{k}^2T_{2k}(b,1)=\frac{C}{\pi}$$

with $a,d,m\in\mathbb{Z}$, $b\in\mathbb{Z}^+$ and $C^2\in\mathbb{Q}\setminus\{0\}$. Then $\sqrt{|m|}$ is an integer dividing $16(b^2-4)$. Also, $b=7$ or $b\equiv 2\pmod 4$.

(ii) Let $\varepsilon\in\{\pm1\}$, $b,m\in\mathbb{Z}^+$ and $m\mid 16(b^2-4)$. Then, there are $a,d\in\mathbb{Z}$ such that

$$\sum_{k=0}^{\infty}\frac{a+dk}{(\varepsilon m^2)^k}\binom{2k}{k}^2T_{2k}(b,1)=\frac{C}{\pi}$$

for some $C\neq 0$ with C^2 rational, if and only if $m>4(b+2)$ and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(b,1)}{(\varepsilon m^2)^k}\equiv\left(\frac{\varepsilon(b^2-4)}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(b,1)}{(\varepsilon\bar{m}^2)^k}\pmod{p^2}$$

for all odd primes $p\nmid b^2-4$, where $\bar{m}=16(b^2-4)/m$.

Conjecture VI. We have the following formulae:

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi}, \tag{VI1}$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi}, \tag{VI2}$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}). \tag{VI3}$$

Remark 5.6. The series (VI1)-(VI3) converge at geometric rates with ratios

$$\frac{16}{27},\ -\frac{64}{125},\ -\frac{343}{512}$$

respectively. The author would like to offer \$300 as the prize for the person (not joint authors) who can provide first rigorous proofs of all the three identities (VI1)-(VI3). (VI1) and (VI3) were motivated by the author's following conjectural congruences for any prime $p>3$:

$$\sum_{k=0}^{p-1}\frac{T_k^3(10,11^2)}{(2^{11}3^3)^k}\equiv\begin{cases}(\frac{2}{p})(4x^2-2p)\ (\mathrm{mod}\ p^2) & \text{if } p=x^2+3y^2,\\ 0\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 2\ (\mathrm{mod}\ 3),\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)\equiv\frac{p}{11}\left(195\left(\frac{-2}{p}\right)-8\left(\frac{-6}{p}\right)\right)\ (\mathrm{mod}\ p^2);$$

$$\sum_{k=0}^{p-1}\frac{T_k^3(62,95^2)}{(-288)^{3k}}\equiv\begin{cases}(\frac{-2}{p})(4x^2-2p)\ (\mathrm{mod}\ p^2) & \text{if } p=x^2+7y^2,\\ 0\ (\mathrm{mod}\ p^2) & \text{if } (\frac{p}{7})=-1,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)$$
$$\equiv\frac{p}{19}\left(17563\left(\frac{-14}{p}\right)+4230\left(\frac{-2}{p}\right)\right)\ (\mathrm{mod}\ p^2).$$

Conjecture VII. We have the following formulae:

$$\sum_{k=0}^{\infty}\frac{221k+28}{450^k}\binom{2k}{k}T_k^2(6,2)=\frac{2700}{7\pi},\tag{VII1}$$

$$\sum_{k=0}^{\infty}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k^2(4,9)=\frac{49}{9\pi}(\sqrt{3}+\sqrt{6}),\tag{VII2}$$

$$\sum_{k=0}^{\infty}\frac{560k+71}{22^{2k}}\binom{2k}{k}T_k^2(5,1)=\frac{605\sqrt{7}}{3\pi},\tag{VII3}$$

$$\sum_{k=0}^{\infty}\frac{3696k+445}{46^{2k}}\binom{2k}{k}T_k^2(7,1)=\frac{1587\sqrt{7}}{2\pi},\tag{VII4}$$

$$\sum_{k=0}^{\infty}\frac{56k+19}{(-108)^k}\binom{2k}{k}T_k^2(3,-3)=\frac{9\sqrt{7}}{\pi},\tag{VII5}$$

$$\sum_{k=0}^{\infty}\frac{450296k+53323}{(-5177196)^k}\binom{2k}{k}T_k^2(171,-171)=\frac{113535\sqrt{7}}{2\pi},\tag{VII6}$$

$$\sum_{k=0}^{\infty}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k^2(73,576)=\frac{10406669}{2\sqrt{6}\,\pi}.\tag{VII7}$$

Remark 5.7. The series (VII1)-(VII7) converge at geometric rates with ratios

$$\frac{88+48\sqrt{2}}{225},\ \frac{25}{49},\ \frac{49}{121},\ \frac{81}{529},\ -\frac{7}{9},\ -\frac{175}{7569},\ \frac{14641}{47089}.$$

respectively. The author found (VII2) and (VII3) in light of Conjecture 4.18. Similarly, (VII6)-(VII7) were motivated by Conjectures 4.19-4.20.

Concerning the new identities in Conjectures I-VII, actually we first discovered congruences without linear parts related to binary quadratic forms (like many congruences in Section 4), then found corresponding p-adic congruences with linear parts, and finally figured out the series for $1/\pi$.

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Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

E-mail address: `zwsun@nju.edu.cn`