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p-ADIC CONGRUENCES MOTIVATED BY SERIES

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Abstract. Let $p>5$ be a prime. Motivated by the known formulae

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3\binom{2k}{k}}=-\frac{2}{5}\zeta(3)\quad\text{and}\quad\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{(2k+1)16^k}=\frac{4G}{\pi}$$

(where $G=\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$ is the Catalan constant), we show that

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^3\binom{2k}{k}}\equiv-2B_{p-3}\pmod{p},$$
$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{(2k+1)16^k}\equiv-\frac{7}{4}p^2B_{p-3}\pmod{p^3}$$

and

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}\equiv-2q_p(2)-p\,q_p(2)^2+\frac{5}{12}p^2B_{p-3}\pmod{p^3},$$

where $B_0,B_1,B_2,\ldots$ are Bernoulli numbers and $q_p(2)$ is the Fermat quotient $(2^{p-1}-1)/p$.

1. Introduction

Let $p>3$ be a prime. In 2010 the author and R. Tauraso [ST] proved that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3},$$

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where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers. Note that

$$\binom{2k}{k} = \frac{(2k)!}{k!^2} \equiv 0 \pmod p \quad \text{for } k = \frac{p+1}{2}, \ldots, p-1.$$

The author [S11c] managed to show that

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} \frac{4}{3} E_{p-3} \pmod p,$$

where $E_0, E_1, E_2, \ldots$ are Euler numbers. It is interesting to compare the last congruence with the known formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} = \frac{\zeta(2)}{3} = \frac{\pi^2}{18}.$$

Note that van Hamme [vH] and his followers ever considered p-adic analogues of some hypergeometric series related to the Gamma function or $\pi = \Gamma(1/2)^2$ but Bernoulli numbers or Euler numbers never appeared in their work.

In 1979 Apéry (cf. [Ap] and [vP]) proved the irrationality of $\zeta(3) = \sum_{k=1}^{\infty} 1/n^3$ and his following formula

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5} \zeta(3)$$

plays an important role in the proof. Motivated by this, Tauraso [T10] showed that if $p > 5$ is a prime then

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3 \binom{2k}{k}} \equiv -\frac{2}{5} \cdot \frac{H_{p-1}}{p^2} \pmod{p^3}$$

and

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv \frac{4}{5} \cdot \frac{H_{p-1}}{p} \pmod{p^3},$$

where $H_n = \sum_{0<k\leqslant n} 1/k$ ($n = 0, 1, 2, \ldots$) are harmonic numbers. It is well known (cf. [G1] or [Su, Theorem 5.1(a)]) that

$$H_{p-1} \equiv -\frac{p^2}{3} B_{p-3} \pmod{p^3} \quad \text{for any prime } p > 3.$$

Actually Tauraso obtained $\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}$ mod p^4 for each prime $p>5$ via putting $n=p$ in the following identity

$$\sum_{k=1}^{n}\binom{2k}{k}\frac{k^2}{4n^4+k^4}\prod_{j=1}^{k-1}\frac{n^4-j^4}{4n^4+j^4}=\frac{2}{5n^2}$$

conjectured by J. M. Borwein and D. M. Bradley [BB] and proved by G. Almkvist and A. Granville [AG].

Let $p>3$ be a prime. The author [S11c] proved that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^{(p-1)/2}-p^2E_{p-3}\pmod{p^3}.$$

Recently Tauraso [T12] showed that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}\equiv-2H_{(p-1)/2}\ (\mathrm{mod}\ p^3).$$

Inputting the command

```
FullSimplify[Sum[Binomial[2k,k]^2/(k*16^k),{k,1,Infty}]],
```

we get from `Mathematica` (version 7) the identity

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}^2}{k16^k}=4\log 2-\frac{8G}{\pi},$$

where G is the Catalan constant given by

$$G:=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}.$$

Now we state our first theorem.

Theorem 1.1. *Let $p>5$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^3\binom{2k}{k}}\equiv-\,2B_{p-3}\ (\mathrm{mod}\ p),\tag{1.1}$$

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv\frac{56}{15}pB_{p-3}\ (\mathrm{mod}\ p^2),\tag{1.2}$$

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{k16^k}\equiv-\,2H_{(p-1)/2}-\frac{7}{2}p^2B_{p-3}\ (\mathrm{mod}\ p^3),\tag{1.3}$$

$$-\frac{4}{p^2}\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}\equiv\sum_{k=1}^{(p-1)/2}\frac{16^k}{k^3\binom{2k}{k}^2}\equiv-14B_{p-3}\ (\mathrm{mod}\ p).\tag{1.4}$$

Remark 1.1. Both (1.1) and (1.2) were conjectured in [S11c, Conjecture 1.1]. The reader may consult [S10] and [S11a] for $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ modulo powers of p, where p is an odd prime and m is an integer not divisible by p.

Motivated by the formulae

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)16^k}=\frac{\pi}{3}\quad\text{and}\quad\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}=\frac{\pi^2}{10},$$

the author [S11b] showed that

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv 0 \pmod{p^2}\ \text{and}\ \sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv\frac{p}{3}E_{p-3} \pmod{p^2},$$

and conjectured that

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}\equiv\frac{H_{p-1}}{5p}\pmod{p^3}$$

(which was recently confirmed by K. Hessami Pilehrood, T. Hessami Pilehrood and R. Tauraso [HPT]) and

$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}\equiv-\frac{p}{4}B_{p-3}\pmod{p^2},$$

where p is any prime greater than 5.

Theorem 1.1 has the following consequence.

Corollary 1.1. *Let $p>5$ be a prime. Then*

$$\frac{1}{p}\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}\equiv-\sum_{k=0}^{(p-3)/2}\frac{(-16)^k}{(2k+1)^3\binom{2k}{k}}\equiv-\frac{B_{p-3}}{4}\pmod{p}.\tag{1.5}$$

Since

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(2k-1)16^k}=-\frac{2n+1}{16^n}\binom{2n}{n}^2\ \text{and}\ \sum_{k=0}^{n}\frac{(1-4k)\binom{2k}{k}^4}{(2k-1)^4256^k}=(8n^2+4n+1)\frac{\binom{2n}{n}^4}{256^n}$$

by induction, we find that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{(2k-1)16^k}=-\frac{2}{\pi}\quad\text{and}\quad\sum_{k=0}^{\infty}\frac{(4k-1)\binom{2k}{k}^4}{(2k-1)^4256^k}=-\frac{8}{\pi^2}$$

by Stirling's formula $n!\sim\sqrt{2\pi n}(n/e)^n$. (The latter was first obtained by J. W. L. Glaisher [G2].) Via `Mathematica` (version 7) we find that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{(2k+1)16^k}=\frac{4G}{\pi}\quad\text{and}\quad\sum_{k=0}^{\infty}\frac{16^k}{(2k+1)^3\binom{2k}{k}^2}=\frac{7}{2}\zeta(3)-G\pi.$$

Motivated by this we establish the following theorem.

Theorem 1.2. *Let $p>3$ be a prime. Then*

$$\frac{1}{p^2}\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{(2k+1)16^k}\equiv-\sum_{k=0}^{(p-3)/2}\frac{16^k}{(2k+1)^3\binom{2k}{k}^2}\equiv-\frac{7}{4}B_{p-3}\pmod{p}, \tag{1.6}$$

and

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}\equiv-2q_p(2)-p\,q_p(2)^2+\frac{5}{12}p^2B_{p-3}\pmod{p^3}, \tag{1.7}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Now we pose two conjectures for further research.

Conjecture 1.1. (i) *If $p>5$ is a prime, then*

$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}\equiv-\frac{21}{2}H_{p-1}\pmod{p^4},$$
$$\sum_{k=0}^{(p-3)/2}\frac{(-16)^k}{(2k+1)^3\binom{2k}{k}}\equiv-\frac{3}{4}\cdot\frac{H_{p-1}}{p^2}-\frac{47}{400}p^2B_{p-5}\pmod{p^3}.$$

(ii) *If $p>3$ is a prime, then*

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2H_{2k}}{k16^k}\equiv(-1)^{(p-1)/2}4E_{p-3}\pmod{p},$$
$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{k16^k}(H_{2k}-H_k)\equiv-\frac{7}{3}pB_{p-3}\pmod{p^2},$$
$$p^2\sum_{k=1}^{p-1}\frac{16^{k-1}H_{k-1}}{k^2\binom{2k}{k}^2}\equiv\frac{(-1)^{(p-1)/2}}{2}H_{(p-1)/2}+pE_{p-3}\pmod{p^2}.$$

We also have the identity

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}^2}{k16^k}(H_{2k}-H_k)=\frac{2}{3}\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2H_{2k}}{(2k+1)16^k}. \tag{1.8}$$

Conjecture 1.2. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{k64^k}\equiv -3H_{(p-1)/2}+\frac{7}{4}p^2B_{p-3}\ (\mathrm{mod}\ p^3),$$

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}\binom{4k}{2k}}{k64^k}\equiv -3H_{(p-1)/2}+(-1)^{(p+1)/2}\,2pE_{p-3}\ (\mathrm{mod}\ p^2),$$

$$p\sum_{k=1}^{(p-1)/2}\frac{64^{k-1}}{k^3\binom{2k}{k}\binom{4k}{2k}}\equiv\frac{(-1)^{(p-1)/2}}{2}E_{p-3}\pmod{p}.$$

If $p>3$, then

$$p\sum_{k=1}^{(p-1)/2}\frac{64^{k-1}}{(2k-1)k^2\binom{2k}{k}\binom{4k}{2k}}\equiv\frac{(-1)^{(p+1)/2}q_p(2)+pE_{p-3}}{4}\pmod{p^2}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}\binom{4k}{2k}}{k64^k}(H_{2k}-H_k)\equiv(-1)^{(p+1)/2}4E_{p-3}\pmod{p}.$$

In the next section we are going to show Theorem 1.1 and Corollary 1.1. Theorem 1.2 will be proved in Section 3. Our proofs involve certain combinatorial identities and harmonic numbers of higher orders given by

$$H_n^{(m)}=\sum_{0<k\leqslant n}\frac{1}{k^m}\quad(n=0,1,2,\ldots).$$

2. Proof of Theorem 1.1

Lemma 2.1. *Let $p=2n+1$ be an odd prime. For any $k=1,\ldots,p-1$, we have*

$$k\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv(-1)^{\lfloor 2k/p\rfloor-1}2p\pmod{p^2}\tag{2.1}$$

and

$$\binom{n}{k}\binom{n+k}{k}\equiv\frac{\binom{2k}{k}^2}{(-16)^k}\pmod{p^2}.\tag{2.2}$$

Proof. Congruence (2.1) was formulated in [S11c, Lemma 2.1]; see also [T10] for such a trick. Congruence (2.2) is known and easy (see p. 231 of [vH, §3.4]); in fact,

$$\binom{n}{k}\binom{n+k}{k}(-1)^k=\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k}\equiv\binom{-1/2}{k}^2=\frac{\binom{2k}{k}^2}{16^k}\ (\mathrm{mod}\,p^2).$$

We are done. □

Lemma 2.2. *Let $p > 3$ be a prime. Then*

$$(-1)^{(p-1)/2}\binom{p-1}{(p-1)/2} \equiv 4^{p-1} + \frac{p^3}{12}B_{p-3} \pmod{p^4} \tag{2.3}$$

and

$$H_{(p-1)/2} \equiv -2q_p(2) + p\,q_p(2)^2 - p^2\left(\frac{2}{3}q_p(2)^3 + \frac{7}{12}B_{p-3}\right) \pmod{p^3}. \tag{2.4}$$

Also,

$$H^{(2)}_{(p-1)/2} \equiv \frac{7}{3}pB_{p-3} \pmod{p^2} \quad \textit{and} \quad H^{(3)}_{(p-1)/2} \equiv -2B_{p-3} \pmod{p}. \tag{2.5}$$

Remark 2.1. (2.3) and (2.4) are refinements of Morley's congruence [Mo] and Lehmer's congruence [L] given by L. Carlitz [C] and Z.-H. Sun [Su, Theorem 5.2(c)] respectively. (2.5) follows from [Su, Corollary 5.2].

Lemma 2.3. *For each $n = 1, 2, 3, \ldots$ we have*

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{n}{k}\binom{n+k}{k}} = 5\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{2k}{k}} + 2H_n^{(3)} \tag{2.6}$$

and

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k}(H_{n+k}-H_{n-k}) = \frac{5}{2}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{2k}{k} + 2H_n^{(2)}. \tag{2.7}$$

Proof. (2.6) is due to Apéry [Ap] (see also [vP]). The author found (2.7) via the math. software `Sigma`. (The reader may consult [OS, §5] for how to use `Sigma` to produce combinatorial identities.) In fact, if we let s_n denote the left-hand side or the right-hand side of (2.7) then `Sigma` yields the recurrence relation

$$(n+1)^2(s_{n+1}-s_n) = 2 - 5(-1)^n\binom{2n+1}{n} \quad (n = 1, 2, 3, \ldots).$$

So (2.7) can be proved by induction. $\square$

Lemma 2.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k^2 16^k} \equiv -2H^2_{(p-1)/2} \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k^3 16^k} \equiv -\frac{4}{3}H^3_{(p-1)/2} - \frac{2}{3}H^{(3)}_{(p-1)/2} \pmod{p}.$$

Remark 2.2. Lemma 2.4 is known, see [T12, Theorem 7] and its proof.

Lemma 2.5. *We have the new identity*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(2(n+k)+1)16^k}=\frac{\binom{2n}{n}^2}{16^n}\sum_{k=0}^{2n}\frac{1}{2k+1}. \tag{2.8}$$

Proof. Let u_n denote the left-hand side or the right-hand side of (2.8). Applying the Zeilberger algorithm (cf. [PWZ, pp. 101-119]) via `Mathematica` (version 7), we find the recurrence relation

$$(2n+1)^2u_n-4(n+1)^2u_{n+1}=-\frac{8(4n^3+8n^2+5n+1)}{(4n+3)(4n+5)16^n}\binom{2n}{n}^2\quad (n=0,1,2,\ldots).$$

So (2.8) holds by induction. $\square$

Proof of Theorem 1.1. For convenience we set $n=(p-1)/2$ and $S:=\sum_{k=1}^{n}(-1)^k/(k^3\binom{2k}{k})$. Below we divide the proof into two parts.

(i) In light of Lemma 2.1,

$$\frac{1}{p}\sum_{k=n+1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv\sum_{k=n+1}^{p-1}\frac{2(-1)^k}{k^3\binom{2(p-k)}{p-k}}=\sum_{k=1}^{n}\frac{2(-1)^{p-k}}{(p-k)^3\binom{2k}{k}}\equiv 2S\pmod p.$$

Hence (1.1) and (1.2) are equivalent since

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv-\frac{4}{15}pB_{p-3}\pmod{p^2} \tag{2.9}$$

by [T10].

By [S11c, (3.2)], for $k=1,\ldots,n$ we have

$$\binom{n}{k}\binom{n+k}{k}(-1)^k\left(1-\frac{p}{4}(H_{n+k}-H_{n-k})\right)\equiv\frac{\binom{2k}{k}^2}{16^k}\pmod{p^4}.$$

This, together with (2.7) and the known identity

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k}=-2H_n$$

(cf. [Pr, §2.1]), yields that

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}^2}{k16^k}+2H_n\equiv-\frac{p}{4}\left(\frac{5}{2}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{2k}{k}+2H_n^{(2)}\right)\pmod{p^4}.$$

Thus, in view of the first congruence in (2.5), we have

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}^2}{k16^k}+2H_n\equiv-\frac{5}{8}p\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{2k}{k}-\frac{7}{6}p^2B_{p-3}\pmod{p^3}.\tag{2.10}$$

Therefore (1.2) and (1.3) are equivalent.

In light of Lemma 2.1,

$$\begin{aligned}\frac{1}{p^2}\sum_{k=n+1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}&\equiv\sum_{k=n+1}^{p-1}\frac{4}{k^3 16^k\binom{2(p-k)}{p-k}^2}=\sum_{k=1}^{n}\frac{4}{(p-k)^3 16^{p-k}\binom{2k}{k}^2}\\&\equiv-\frac{1}{4}\sum_{k=1}^{n}\frac{16^k}{k^3\binom{2k}{k}^2}\pmod{p}.\end{aligned}$$

With the help of (2.2) and (2.5), we obtain from (2.6) the congruence

$$\sum_{k=1}^{n}\frac{16^k}{k^3\binom{2k}{k}^2}\equiv 5S-4B_{p-3}\pmod{p}.$$

Therefore (1.1) and (1.4) are equivalent.

Note that

$$\sum_{k=n+1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}\equiv-\frac{p^2}{4}(5S-4B_{p-3})\equiv-\frac{5}{8}p\sum_{k=n+1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}+p^2B_{p-3}\pmod{p^3}.$$

Combining this with (2.10) we get

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}+2H_n\equiv-\frac{5}{8}p\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}-\frac{p^2}{6}B_{p-3}\pmod{p^3}.\tag{2.11}$$

So Tauraso's result $\sum_{k=1}^{p-1}\binom{2k}{k}^2/(k16^k)\equiv-2H_n\pmod{p^3}$ in [T12] actually follows from his earlier result (2.9).

(ii) By the above, it suffices to show (1.3). Note that

$$\sum_{k=0}^{2n}\frac{1}{2k+1}=H_{4n+1}-\frac{H_{2n}}{2}=H_{p-1}+\frac{1}{p}+\sum_{k=1}^{p-1}\frac{1}{p+k}-\frac{H_{p-1}}{2}$$

and

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{1}{p+k}&=\sum_{k=1}^{p-1}\frac{k^2-kp+p^2}{k^3+p^3}\\&\equiv\sum_{k=1}^{p-1}\frac{k^2-kp+p^2}{k^3}=H_{p-1}-pH_{p-1}^{(2)}+p^2H_{p-1}^{(3)}\pmod{p^3}.\end{aligned}$$

As
$$H_{p-1}^{(3)}=\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^3}+\frac{1}{(p-k)^3}\right)\equiv 0\pmod p,$$
we see that
$$\sum_{k=0}^{2n}\frac{1}{2k+1}-\frac1p\equiv\frac32H_{p-1}-pH_{p-1}^{(2)}\equiv-\frac{p^2}{2}B_{p-3}-\frac23p^2B_{p-3}=-\frac76p^2B_{p-3}\pmod{p^3}$$
since $H_{p-1}\equiv -p^2B_{p-3}/3\ (\mathrm{mod}\ p^3)$ and $H_{p-1}^{(2)}\equiv 2pB_{p-3}/3\ (\mathrm{mod}\ p^2)$ by Glaisher [G1] (see also [Su, Theorem 5.1(a)]). In view of (2.3),
$$\begin{aligned}\frac{\binom{2n}{n}^2}{16^n}\equiv&\frac{(4^{p-1}(1+p^3B_{p-3}/12))^2}{4^{p-1}}\equiv 4^{p-1}\left(1+\frac{p^3}{6}B_{p-3}\right)\\ \equiv&(1+p\,q_p(2))^2+\frac{p^3}{6}B_{p-3}\pmod{p^4}.\end{aligned}$$
Therefore
$$\begin{aligned}\frac{\binom{2n}{n}^2}{16^n}\sum_{k=0}^{2n}\frac{1}{2k+1}-\frac1p=&\frac{\binom{2n}{n}^2}{16^n}\left(\sum_{k=0}^{2n}\frac{1}{2k+1}-\frac1p\right)+\frac{\binom{2n}{n}^2/16^n-1}{p}\\ \equiv&-\frac76p^2B_{p-3}+\frac{(1+p\,q_p(2))^2+p^3B_{p-3}/6-1}{p}\\ =&2q_p(2)+p\,q_p(2)^2-p^2B_{p-3}\pmod{p^3}\end{aligned}$$
and hence (2.8) yields the congruence
$$\sum_{k=1}^n\frac{\binom{2k}{k}^2}{(2k+p)16^k}\equiv 2q_p(2)+p\,q_p(2)^2-p^2B_{p-3}\pmod{p^3}.\tag{2.12}$$

Note that
$$\begin{aligned}\sum_{k=1}^n\frac{\binom{2k}{k}^2}{(2k+p)16^k}=&\sum_{k=1}^n\frac{(4k^2-2kp+p^2)\binom{2k}{k}^2}{((2k)^3+p^3)16^k}\\ \equiv&\frac12\sum_{k=1}^n\frac{\binom{2k}{k}^2}{k16^k}-\frac p4\sum_{k=1}^n\frac{\binom{2k}{k}^2}{k^216^k}+\frac{p^2}{8}\sum_{k=1}^n\frac{\binom{2k}{k}^2}{k^316^k}\pmod{p^3}.\end{aligned}\tag{2.13}$$
By Lemma 2.4 and (2.4)-(2.5),
$$\sum_{k=1}^n\frac{\binom{2k}{k}^2}{k^216^k}\equiv-2(-2q_p(2)+p\,q_p(2)^2)^2\equiv-8q_p(2)^2+8p\,q_p(2)^3\pmod{p^2}\tag{2.14}$$

and

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}^2}{k^3 16^k}\equiv-\frac{4}{3}(-2q_p(2))^3-\frac{2}{3}(-2B_{p-3})=\frac{32}{3}q_p(2)^3+\frac{4}{3}B_{p-3}\pmod{p}. \tag{2.15}$$

Combining (2.12)-(2.15) we obtain

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{n}\frac{\binom{2k}{k}^2}{k16^k}\equiv&2q_p(2)+p\,q_p(2)^2-p^2B_{p-3}+\frac{p}{4}(-8q_p(2)^2+8p\,q_p(2)^3)\\&-\frac{p^2}{8}\left(\frac{32}{3}q_p(2)^3+\frac{4}{3}B_{p-3}\right)\pmod{p^3}.\end{aligned}$$

In view of (2.4), this yields the desired (1.3).

The proof of Theorem 1.1 is now complete. □

Proof of Corollary 1.1. Write $p=2n+1$. In view of Lemma 2.1,

$$\begin{aligned}&\frac{1}{p}\sum_{k=n+1}^{p-1}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}\\=&\sum_{k=1}^{n}\frac{\binom{2(p-k)}{p-k}/p}{(2(p-k)+1)^2(-16)^{p-k}}\\\equiv&\sum_{k=1}^{n}\frac{-2/(k\binom{2k}{k})}{(2(k-1)+1)^2(-16)^{1-k}}=-\sum_{k=0}^{n-1}\frac{(-16)^k}{(2k+1)^3\binom{2k}{k}}\pmod{p}.\end{aligned}$$

Since

$$\binom{-1/2}{k}\equiv\binom{n}{k}=\binom{n}{n-k}\equiv\binom{-1/2}{n-k}\pmod{p}\quad\text{for all } k=0,\ldots,n,$$

we have

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{(-16)^k}{(2k+1)^3\binom{2k}{k}}=&\sum_{k=0}^{n-1}\frac{4^k}{(2k+1)^3\binom{-1/2}{k}}\\\equiv&\sum_{k=0}^{n-1}\frac{4^k}{(2k+1)^3\binom{-1/2}{n-k}}=\sum_{k=1}^{n}\frac{4^{n-k}}{(2(n-k)+1)^3\binom{-1/2}{k}}\\\equiv&-\frac{1}{8}\sum_{k=1}^{n}\frac{1}{k^3\binom{-1/2}{k}4^k}=-\frac{1}{8}\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{2k}{k}}\\\equiv&\frac{B_{p-3}}{4}\pmod{p}\quad(\text{by (1.1)}).\end{aligned}$$

Therefore (1.5) holds. □

3. Proof of Theorem 1.2

Lemma 3.1. *For any positive integer n we have*

$$\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(n-k)16^k}=\frac{\binom{2n}{n}^2}{4^{2n-1}}\sum_{k=0}^{n-1}\frac{1}{2k+1}\tag{3.1}$$

and

$$\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)^2}\binom{n}{k}\binom{n+k}{k}=\frac{1}{(2n+1)^2}+\frac{2}{2n+1}\sum_{k=0}^{n-1}\frac{1}{2k+1}.\tag{3.2}$$

Proof. Let $(x)_0=1$ and $(x)_k=\prod_{j=0}^{k-1}(x+j)$ for $k=1,2,3,\ldots$. Then $(1/2)_k^2/(1)_k^2=\binom{2k}{k}^2/16^k$ for $k=0,1,2,\ldots$. Thus (3.1) is just (21) of [Lu, Ch. 5.2] with $x=1/2$ (see also [T12, (1)]).

Identity (3.2) is new and it can be proved via the Zeilberger algorithm (cf. [PWZ, pp. 101–119]). It is easy to verify (3.2) for $n=1,2$. Applying the Zeilberger algorithm via `Mathematica` (version 7), we find that if a_n denotes the left-hand side or the right-hand side of (3.2) then

$$(n+1)(2n+5)^2a_{n+2}=(2n+3)(4n^2+12n+7)a_{n+1}-(n+2)(2n+1)^2a_n$$

for all $n=1,2,3,\ldots$. So (3.2) follows by induction. □

Lemma 3.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)^3 16^k}\equiv-\frac{4}{3}q_p(2)^3-\frac{B_{p-3}}{6}\pmod{p}.\tag{3.3}$$

Proof. Set $n=(p-1)/2$. For $k=0,\ldots,n$ we clearly have

$$\binom{n+k}{k}(-1)^k=\binom{-n-1}{k}\equiv\binom{n}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod{p}.$$

Thus

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(2k+1)^3 16^k}&\equiv\sum_{k=0}^{n-1}\frac{\binom{n}{k}^2}{(2k+1)^3}=\sum_{k=1}^{n}\frac{\binom{n}{k}^2}{(2(n-k)+1)^3}\\&\equiv-\frac{1}{8}\sum_{k=1}^{n}\frac{\binom{2k}{k}^2}{k^3 16^k}\pmod{p}.\end{aligned}$$

Combining this with (2.15) we obtain the desired (3.3). □

Lemma 3.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)^2 16^k}\equiv -2q_p(2)^2+\frac{2}{3}p\,q_p(2)^3-\frac{p}{6}B_{p-3}\pmod{p^2}. \tag{3.4}$$

Proof. Write $p=2n+1$. By (3.2) we have

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{(-1)^k}{(2k+1)^2}\binom{n}{k}\binom{n+k}{k}=&\frac{1-(-1)^n\binom{2n}{n}}{p^2}+\frac{2}{p}\left(H_{p-1}-\frac{H_n}{2}\right)\\ =&\frac{1-(-1)^n\binom{2n}{n}-pH_n}{p^2}+\frac{2}{p}H_{p-1}.\end{aligned}$$

In light of Lemma 2.2,

$$\begin{aligned}1-(-1)^n\binom{2n}{n}-pH_n\equiv&1-(1+p\,q_p(2))^2-\frac{p^3}{12}B_{p-3}\\ &+2p\,q_p(2)-p^2q_p(2)^2+p^3\left(\frac{2}{3}q_p(2)^3+\frac{7}{12}B_{p-3}\right)\\ =&-2p^2q_p(2)^2+p^3\left(\frac{2}{3}q_p(2)^3+\frac{B_{p-3}}{2}\right)\pmod{p^4}.\end{aligned}$$

So, with the help of (2.2), we get

$$\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(2k+1)^2 16^k}\equiv -2q_p(2)^2+p\left(\frac{2}{3}q_p(2)^3+\frac{B_{p-3}}{2}\right)+\frac{2}{p}H_{p-1}\pmod{p^2},$$

which gives (3.4) since $H_{p-1}/p\equiv -pB_{p-3}/3 \pmod{p^2}$. $\square$

Proof of Theorem 1.2. For convenience we set $n=(p-1)/2$.

We first prove (1.6). By Lemma 2.1,

$$\begin{aligned}&\frac{1}{p^2}\sum_{k=n+1}^{p-1}\frac{\binom{2k}{k}^2}{(2k+1)16^k}=\sum_{k=1}^{n}\frac{(\binom{2(p-k)}{p-k}/p)^2}{(2(p-k)+1)16^{p-k}}\\ \equiv&\sum_{k=1}^{n}\frac{(-2/(k\binom{2k}{k}))^2 16^{k-1}}{1-2k}=-\sum_{k=0}^{n-1}\frac{16^k}{(2k+1)^3\binom{2k}{k}^2}\pmod{p}.\end{aligned}$$

Note also that

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{16^k}{(2k+1)^3\binom{2k}{k}^2}\equiv&\sum_{k=0}^{n-1}\frac{1}{(2k+1)^3\binom{n}{k}^2}=\sum_{k=1}^{n}\frac{1}{(2(n-k)+1)^3\binom{n}{k}^2}\\ \equiv&-\frac{1}{8}\sum_{k=1}^{n}\frac{1}{k^3\binom{n}{k}^2}\equiv-\frac{1}{8}\sum_{k=1}^{n}\frac{16^k}{k^3\binom{2k}{k}^2}\pmod{p}.\end{aligned}$$

Thus, with the help of (1.4) we get (1.6). (In the case $p=5$, (1.6) can be verified directly.)

Since $\binom{2n}{n}^2\equiv 4^{4n} \pmod{p^3}$ by (2.3), we have

$$\begin{aligned}\frac{\binom{2n}{n}^2}{4^{2n-1}}\sum_{k=0}^{n-1}\frac{1}{2k+1}\equiv&4^p\left(H_{p-1}-\frac{H_n}{2}\right)=4^pH_{p-1}-2(1+p\,q_p(2))^2H_n\\ \equiv&-\frac{4}{3}p^2B_{p-3}-2(1+2p\,q_p(2)+p^2q_p(2)^2)H_n\\ \equiv&4q_p(2)+6p\,q_p(2)^2+p^2\left(\frac{4}{3}q_p(2)^3-\frac{B_{p-3}}{6}\right)\pmod{p^3}\end{aligned}$$

with the help of (2.4). Note also that

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(n-k)16^k}=\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(p-(2k+1))16^k}\\ =&\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2(p^2+p(2k+1)+(2k+1)^2)}{(p^3-(2k+1)^3)16^k}\\ \equiv&-p^2\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(2k+1)^316^k}-p\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(2k+1)^216^k}-\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(2k+1)16^k}\pmod{p^3}.\end{aligned}$$

Combining these with (3.1), (3.3) and (3.4) we finally obtain the desired (1.7). □

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References

[AG] G. Almkvist and A. Granville, *Borwein and Bradley's Apéry-like formulae for* $\zeta(4n+3)$, Experiment. Math. **8** (1999), 197–203.

[Ap] R. Apéry, *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$, Astérisque **61** (1979), 11–13.

[BB] J. M. Borwein and D. M. Bradley, *Empirically determined Apéry-like formulae for* $\zeta(4n+3)$, Experiment. Math. **6** (1997), 181–194.

[C] L. Carlitz, *A theorem of Glaisher*, Canad. J. Math. **5** (1953), 306–316.

[G1] J. W. L. Glaisher, *On the residues of the sums of products of the first* $p-1$ *numbers, and their powers, to modulus* p^2 *or* p^3, Quart. J. Pure Appl. Math. **31** (1900), 321–353.

[G2] J. W. L. Glaisher, *On series for* $1/\pi$ *and* $1/\pi^2$, Quart. J. Pure Appl. Math. **37** (1905), 173–198.

[HPT] K. Hessami Pilehrood, T. Hessami Pilehrood and R. Tauraso, *Congruences concerning Jacobi polynomials and Apéry-like formulae*, Int. J. Number Theory **8** (2012), 1789–1811.

[L] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. Math. **39** (1938), 350–360.

[Lu] Y. L. Luke, *Mathematical Functions and Their Approximations*, Academic Press, New York, 1975.

[Mo] F. Morley, *Note on the congruence* $2^{4n} \equiv (-1)^n(2n)!/(n!)^2$, *where* $2n+1$ *is a prime*, Ann. Math. **9** (1895), 168–170.

[OS] R. Osburn and C. Schneider, *Gaussian hypergeometric series and supercongruences*, Math. Comp. **78** (2009), 275–292.

[PWZ] M. Petkovšek, H. S. Wilf and D. Zeilberger, $A = B$, A K Peters, Wellesley, 1996.

[Pr] H. Prodinger, *Human proofs of identities by Osburn and Schneider*, Integers **8** (2008), #A10, 8pp (electronic).

[Su] Z.-H. Sun, *Congruences concerning Bernoulli numbers and Bernoulli polynomials*, Discrete Appl. Math. **105** (2000), 193–223.

[S10] Z.-W. Sun, *Binomial coefficients, Catalan numbers and Lucas quotients*, Sci. China Math. **53** (2010), 2473–2488.

[S11a] Z.-W. Sun, *p-adic valuations of some sums of multinomial coefficients*, Acta Arith. **148** (2011), 63–76.

[S11b] Z.-W. Sun, *On congruences related to central binomial coefficients*, J. Number Theory **131** (2011), 2219–2238.

[S11c] Z.-W. Sun, *Super congruences and Euler numbers*, Sci. China Math. **54** (2011), 2509–2535.

[ST] Z.-W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math. **45** (2010), 125–148.

[T10] R. Tauraso, *More congruences for central binomial coefficients*, J. Number Theory **130** (2010), 2639–2649.

[T12] R. Tauraso, *Supercongruences for a truncated hypergeometric series*, Integers **12** (2012), #A45, 12pp (electronic).

[vH] L. van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, in: p-adic Functional Analysis (Nijmegen, 1996), pp. 223–236, Lecture Notes in Pure and Appl. Math., Vol. 192, Dekker, 1997.

[vP] A. van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of* $\zeta(3)$, Math. Intelligencer **1** (1978/79), 195–203.