ON MONOTONICITY OF SOME COMBINATORIAL SEQUENCES

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ABSTRACT. We confirm Sun's conjecture that $({}^{n+1}\!\sqrt{F_{n+1}}/\sqrt[n]{F_n})_{n\geqslant 4}$ is strictly decreasing to the limit 1, where $(F_n)_{n\geqslant 0}$ is the Fibonacci sequence. We also prove that the sequence $({}^{n+1}\!\sqrt{D_{n+1}}/\sqrt[n]{D_n})_{n\geqslant 3}$ is strictly decreasing with limit 1, where D_n is the n-th derangement number. For m-th order harmonic numbers $H_n^{(m)} = \sum_{k=1}^n 1/k^m$ $(n = 1, 2, 3, \ldots)$, we show that $({}^{n+1}\!\sqrt{H_{n+1}^{(m)}}/\sqrt[n]{H_n^{(m)}})_{n\geqslant 3}$ is strictly increasing.

1. INTRODUCTION

A challenging conjecture of Firoozbakht states that

$$\sqrt[n]{p_n} > {}^{n+1}\!\sqrt{p_{n+1}} \quad \text{for every } n = 1, 2, 3, \ldots,$$

where p_n denotes the n-th prime. Note that $\lim_{n\to\infty} \sqrt[n]{p_n} = 1$ by the Prime Number Theorem. In [4] the second author conjectured further that for any integer $n > 4$ we have the inequality

$$\frac{{}^{n+1}\!\sqrt{p_{n+1}}}{\sqrt[n]{p_n}} < 1 - \frac{\log\log n}{2n^2},$$

which has been verified for all $n \leqslant 3.5 \times 10^6$. Motivated by this and [3], Sun [4, Conj. 2.12] conjectured that the sequence $({}^{n+1}\!\sqrt{S_{n+1}}/\sqrt[n]{S_n})_{n\geqslant 7}$ is strictly increasing, where S_n is the sum of the first n positive squarefree numbers. Moreover, he also posed many conjectures on monotonicity of sequences of the type $({}^{n+1}\!\sqrt{a_{n+1}}/\sqrt[n]{a_n})_{n\geqslant N}$ with $(a_n)_{n\geqslant 1}$ a familiar combinatorial sequence of positive integers.

Throughout this paper, we set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

Let A and B be integers with $\Delta = A^2 - 4B \neq 0$. The Lucas sequence $u_n = u_n(A, B)$ $(n \in \mathbb{N})$ is defined as follows:

$$u_0 = 0, \ u_1 = 1, \text{ and } u_{n+1} = Au_n - Bu_{n-1} \text{ for } n = 1, 2, 3, \ldots.$$

It is well known that $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ for all $n \in \mathbb{N}$, where

$$\alpha = \frac{A + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{\Delta}}{2}$$

are the two roots of the characteristic equation $x^2 - Ax + B = 0$. The sequence $F_n = u_n(1, -1)$ $(n \in \mathbb{N})$ is the famous Fibonacci sequence, see [1, p. 46] for combinatorial interpretations of Fibonacci numbers.

Our first result is as follows.

Theorem 1.1. *Let $A > 0$ and $B \neq 0$ be integers with $\Delta = A^2 - 4B > 0$, and set $u_n = u_n(A, B)$ for $n \in \mathbb{N}$. Then there exists an integer $N > 0$ such that the sequence $\left(\sqrt[n+1]{u_{n+1}} / \sqrt[n]{u_n} \right)_{n \geqslant N}$ is strictly decreasing with limit 1. In the case $A = 1$ and $B = -1$ we may take $N = 4$.*

Remark 1.1. Under the condition of Theorem 1.1, by [2, Lemma 4] we have $u_n < u_{n+1}$ unless $A = n = 1$. Note that the second assertion in Theorem 1.1 confirms a conjecture of the second author [4, Conj. 3.1] on the Fibonacci sequence.

For $n \in \mathbb{Z}^+$ the nth derangement number D_n denotes the number of permutations σ of $\{1, \ldots, n\}$ with $\sigma(i) = i$ for no $i = 1, \ldots, k$. It has the following explicit expression (cf. [1, p. 67]):

$$D_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

Our second theorem is the following result conjectured by the second author [4, Conj. 3.3].

Theorem 1.2. *The sequence $\left(\sqrt[n+1]{D_{n+1}} / \sqrt[n]{D_n} \right)_{n \geqslant 3}$ is strictly decreasing with limit 1.*

Remark 1.2. It follows from Theorem 1.2 that the sequence $\left(\sqrt[n]{D_n} \right)_{n \geqslant 2}$ is strictly increasing.

For each $m \in \mathbb{Z}^+$ those $H_n^{(m)} = \sum_{k=1}^{n} 1/k^m$ $(n \in \mathbb{Z}^+)$ are called harmonic numbers of order m. The usual harmonic numbers (of order 1) are those rational numbers $H_n = H_n^{(1)}$ $(n = 1, 2, 3, \ldots)$.

Our following theorem confirms Conjecture 2.16 of Sun [4].

Theorem 1.3. *For any positive integer m, the sequence $\left(\sqrt[n+1]{H_{n+1}^{(m)}} / \sqrt[n]{H_n^{(m)}} \right)_{n \geqslant 3}$ is strictly increasing.*

We will prove Theorems 1.1–1.3 in Sections 2–4 respectively. It seems that there is no simple form for the generating function $\sum_{n=0}^{\infty} \sqrt[n]{a_n} x^n$ with $a_n = u_n, D_n, H_n^{(m)}$. Note also that the set of those sequences $(a_n)_{n \geqslant 1}$ of positive numbers with $(\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n})_{n \geqslant 1}$ decreasing (or increasing) is closed under multiplication.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Set

$$\alpha = \frac{A + \sqrt{\Delta}}{2}, \ \beta = \frac{A - \sqrt{\Delta}}{2}, \ \text{and } \gamma = \frac{\beta}{\alpha} = \frac{A - \sqrt{\Delta}}{A + \sqrt{\Delta}}.$$

Then

$$\log u_n = \log \frac{\alpha^n (1 - \gamma^n)}{\alpha - \beta} = n \log \alpha + \log(1 - \gamma^n) - \log \sqrt{\Delta}$$

for any $n \in \mathbb{Z}^+$. Note that

$$\log \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = \frac{\log u_{n+1}}{n+1} - \frac{\log u_n}{n} = \frac{\log(1 - \gamma^{n+1})}{n+1} - \frac{\log(1 - \gamma^n)}{n} + \frac{\log \sqrt{\Delta}}{n(n+1)}.$$

Since

$$\lim_{n \to \infty} \frac{\log(1 - \gamma^n)}{n} = \lim_{n \to \infty} \frac{-\gamma^n}{n} = 0 \ \text{and} \ \lim_{n \to \infty} \frac{1}{n(n+1)} = 0,$$

we deduce that

$$\lim_{n \to \infty} \log \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = 0, \ \text{i.e.,} \ \lim_{n \to \infty} \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = 1.$$

For any $n \in \mathbb{Z}^+$, clearly

$$\frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} > \frac{\sqrt[n+2]{u_{n+2}}}{\sqrt[n+1]{u_{n+1}}} \iff \frac{\log u_{n+1}}{n+1} - \frac{\log u_n}{n} > \frac{\log u_{n+2}}{n+2} - \frac{\log u_{n+1}}{n+1}$$

$$\iff \Delta_n := \frac{2 \log u_{n+1}}{n+1} - \frac{\log u_n}{n} - \frac{\log u_{n+2}}{n+2} > 0.$$

Observe that

$$\Delta_n = 2 \log \alpha + \frac{2 \log(1 - \gamma^{n+1})}{n+1} - \frac{2 \log \sqrt{\Delta}}{n+1}$$

$$- \left(2 \log \alpha + \frac{\log(1 - \gamma^n)}{n} + \frac{\log(1 - \gamma^{n+2})}{n+2} - \frac{\log \sqrt{\Delta}}{n} - \frac{\log \sqrt{\Delta}}{n+2} \right)$$

$$= \frac{\log \Delta}{n(n+1)(n+2)} + \frac{2}{n+1} \log(1 - \gamma^{n+1}) - \frac{\log(1 - \gamma^n)}{n} - \frac{\log(1 - \gamma^{n+2})}{n+2}.$$

The function $f(x) = \log(1 + x)$ on the interval $(-1, +\infty)$ is concave since $f''(x) = -1/(x + 1)^2 < 0$. Note that $|\gamma| < 1$. If $-|\gamma| \leqslant x \leqslant 0$, then $t = -x/|\gamma| \in [0, 1]$ and hence

$$f(x) = f(t(-|\gamma|) + (1 - t)0) \geqslant tf(-|\gamma|) + (1 - t)f(0) = qx,$$

where $q = -\log(1 - |\gamma|)/|\gamma| > 0$. Note also that $\log(1 + x) < x$ for $x > 0$. So we have

$$\log\left(1 - \gamma^{n+1}\right) \geqslant \log\left(1 - |\gamma|^{n+1}\right) \geqslant -q|\gamma|^{n+1},$$
$$\log\left(1 - \gamma^n\right) \leqslant \log\left(1 + |\gamma|^n\right) < |\gamma|^n,$$
$$\log\left(1 - \gamma^{n+2}\right) \leqslant \log\left(1 + |\gamma|^{n+2}\right) < |\gamma|^{n+2}.$$

Therefore

$$\Delta_n > \frac{\log \Delta}{n(n + 1)(n + 2)} - |\gamma|^n \left(\frac{2q|\gamma|}{n + 1} + \frac{1}{n} + \frac{|\gamma|^2}{n + 2}\right)$$

and hence

$$n(n + 1)(n + 2)\Delta_n$$
$$> \log \Delta - |\gamma|^n \left(2q|\gamma|n(n + 2) + (n + 1)(n + 2) + |\gamma|^2 n(n + 1)\right). \quad (1)$$

Since $\lim_{n\to\infty} n^2|\gamma|^n = 0$, when $\Delta > 1$ we have $\Delta_n > 0$ for large n.

Now it remains to consider the case $\Delta = 1$. Clearly $\gamma = (A - 1)/(A + 1) > 0$. Recall that $\log(1 - x) < -x$ for $x \in (0, 1)$. As

$$\frac{d}{dx}(\log(1 - x) + x + x^2) = -\frac{1}{1 - x} + 1 + 2x = \frac{x(1 - 2x)}{1 - x} > 0 \quad \text{for } x \in (0, 0.5),$$

we have $\log(1 - x) + x + x^2 > \log 1 + 0 + 0^2 = 0$ for $x \in (0, 0.5)$. If n is large enough, then $\gamma^n < 0.5$ and hence

$$\Delta_n = \frac{2}{n + 1}\log(1 - \gamma^{n+1}) - \frac{\log(1 - \gamma^n)}{n} - \frac{\log(1 - \gamma^{n+2})}{n + 2} > w_n,$$

where

$$w_n := \frac{2}{n + 1}(-\gamma^{n+1} - \gamma^{2n+2}) + \frac{\gamma^n}{n} + \frac{\gamma^{n+2}}{n + 2}.$$

Note that

$$\lim_{n\to\infty} \frac{nw_n}{\gamma^n} = -2\gamma + 1 + \gamma^2 = (1 - \gamma)^2 > 0.$$

So, for sufficiently large n we have $\Delta_n > w_n > 0$.

Now we show that $n \geqslant 4$ suffices in the case $A = 1$ and $B = -1$. Note that $\Delta = 5$ and $\gamma \approx -0.382$. The sequence $(|\gamma|^n(n + 1)(n + 2))_{n\geqslant 1}$ is

decreasing since

$$|\gamma|\frac{(n+2)(n+3)}{(n+1)(n+2)} < \frac{1}{2}\left(1 + \frac{2}{n+1}\right) \leqslant 1$$

for $n \geqslant 1$. It follows that $|\gamma|^n(n+1)(n+2) \leqslant \gamma^6 \times 7 \times 8 < 1/3$ for $n \geqslant 6$. In view of (1), if $n \geqslant 6$ then

$$n(n+1)(n+2)\Delta_n > \log 5 - |\gamma|^n(n+1)(n+2)\left(2q|\gamma| + 1 + |\gamma|^2\right)$$

$$> \log 5 - \frac{1+1+\gamma^2}{3} > \log 5 - 1 > 0.$$

It is easy to verify that Δ_4 and Δ_5 are positive. So $(\sqrt[n+1]{F_{n+1}}/\sqrt[n]{F_n})_{n\geqslant 4}$ is strictly decreasing.

In view of the above, we have completed the proof of Theorem 1.1.   □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $n \geqslant 3$. It is well known that $|D_n - n!/e| \leqslant 1/2$ (cf. [1, p. 67]). Applying the Intermediate Value Theorem in calculus, we obtain

$$\left|\log D_n - \log\left(\frac{n!}{e}\right)\right| \leqslant \left|D_n - \frac{n!}{e}\right| \leqslant 0.5.$$

Set $R_0(n) = \log D_n - \log n!$. Then $|R_0(n)| \leqslant 1.5$.

Since $\lim_{n\to\infty} R_0(n)/n = 0$, we have

$$\lim_{n\to\infty}\left(\frac{\log D_{n+1}}{n+1} - \frac{\log D_n}{n}\right) = \lim_{n\to\infty}\left(\frac{\log(n+1)!}{n+1} - \frac{\log n!}{n}\right)$$

$$= \lim_{n\to\infty}\frac{n\log(n+1) + n\log n! - (n+1)\log n!}{n(n+1)}$$

$$= \lim_{n\to\infty}\frac{n\log n + n\log(1+1/n) - \log n!}{n(n+1)}$$

$$= \lim_{n\to\infty}\frac{\log(n^n/n!)}{n(n+1)}.$$

As $n! \sim \sqrt{2\pi n}(n/e)^n$ (i.e., $\lim_{n\to\infty} n!/(\sqrt{2\pi n}(n/e)^n) = 1$) by Stirling's formula, we have $\log(n^n/n!) \sim n$ and hence

$$\lim_{n\to\infty}\left(\frac{\log D_{n+1}}{n+1} - \frac{\log D_n}{n}\right) = 0.$$

Thus $\lim_{n\to\infty} \sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n} = 1$.

From the known identity $D_n/n! = \sum_{k=0}^{n}(-1)^k/k!$, we have the recurrence $D_n = nD_{n-1} + (-1)^n$ for $n > 1$. Thus, if $n \geqslant 3$ then

$$R_0(n) - R_0(n-1) = \log \frac{D_n}{n!} - \log \frac{D_{n-1}}{(n-1)!} = \log \frac{D_n}{nD_{n-1}} = \log\left(1 + \frac{(-1)^n}{nD_{n-1}}\right).$$

Fix $n \geqslant 4$. If n is even, then

$$0 < R_0(n) - R_0(n-1) = \log\left(1 + \frac{1}{nD_{n-1}}\right) < \frac{1}{nD_{n-1}} = \frac{1}{D_n - 1} \leqslant \frac{3}{D_n + 0.5}.$$

If n is odd, then

$$0 > R_0(n) - R_0(n-1) = \log\left(1 - \frac{1}{nD_{n-1}}\right) > \frac{-2}{nD_{n-1}} = \frac{-2}{D_n + 1} \geqslant \frac{-3}{D_n + 0.5}$$

since $\log(1-x) + 2x > 0$ for $x \in (0, 0.5)$. So

$$|R_0(n) - R_0(n-1)| < \frac{3}{D_n + 0.5} \leqslant \frac{3e}{n!}$$

and hence

$$\left|\frac{R_0(n-1) - R_0(n)}{n-1}\right| < \frac{3e}{n!(n-1)} \leqslant \frac{3e}{n(n-1)(n+1)}.$$

Similarly, we also have

$$\left|\frac{R_0(n+1) - R_0(n)}{n+1}\right| < \frac{3e}{n!(n+1)} \leqslant \frac{3e}{n(n-1)(n+1)}.$$

Therefore,

$$\left|\frac{R_0(n+1)}{n+1} - \frac{2R_0(n)}{n} + \frac{R_0(n-1)}{n-1} - \frac{2R_0(n)}{n(n-1)(n+1)}\right|$$
$$= \left|\frac{R_0(n+1) - R_0(n)}{n+1} + \frac{R_0(n-1) - R_0(n)}{n-1}\right| \leqslant \frac{6e}{n(n-1)(n+1)}$$

and hence

$$\left|\frac{R_0(n+1)}{n+1} - \frac{2R_0(n)}{n} + \frac{R_0(n-1)}{n-1}\right| \leqslant \frac{2|R_0(n)| + 6e}{n(n-1)(n+1)} \leqslant \frac{6e+3}{n(n-1)(n+1)}.$$

Thus $|R_1(n)| \leqslant 6e + 3$, where

$$R_1(n) := n(n-1)(n+1)\left(\frac{R_0(n+1)}{n+1} - \frac{2R_0(n)}{n} + \frac{R_0(n-1)}{n-1}\right).$$

Since

$$\log((n-1)!) = \sum_{k=1}^{n-1} \int_k^{k+1} (\log k) dx$$

$$< \sum_{k=1}^{n-1} \int_k^{k+1} \log x \, dx = \int_1^n \log x \, dx = n \log n - n + 1$$

$$< \sum_{k=1}^{n-1} \int_k^{k+1} (\log(k+1)) dx = \log(n!),$$

we have

$$n \log n - n < \log(n!) = \log((n-1)!) + \log n < n \log n - n + \log n + 1$$

and so $\log(n!) = n \log n - n + R_2(n)$ with $|R_2(n)| < \log n + 1$.

Observe that

$$\frac{\log D_{n+1}}{n+1} - \frac{2}{n} \log D_n + \frac{\log D_{n-1}}{n-1}$$

$$= \frac{\log(n+1)!}{n+1} - \frac{2 \log n!}{n} + \frac{\log(n-1)!}{n-1} + \frac{R_1(n)}{(n-1)n(n+1)}$$

$$= \frac{2 \log n!}{(n-1)n(n+1)} - \frac{\log n}{n-1} + \frac{\log(n+1)}{n+1} + \frac{R_1(n)}{(n-1)n(n+1)}$$

$$= -\frac{2n}{(n-1)n(n+1)} + \frac{\log(n+1) - \log(n)}{n+1} + \frac{2R_2(n) + R_1(n)}{(n-1)n(n+1)}$$

$$\leqslant -\frac{2n}{(n-1)n(n+1)} + \frac{n-1}{(n-1)n(n+1)} + \frac{2R_2(n) + R_1(n)}{(n-1)n(n+1)}$$

$$= -\frac{n+1 - 2R_2(n) - R_1(n)}{(n-1)n(n+1)}.$$

If $n \geqslant 27$, then $n + 1 - 2R_2(n) - R_1(n) > n - 2 \log n - 1 - 6e - 3 > 0$, and hence we get

$$\log \frac{\sqrt[n]{D_n}}{\sqrt[n-1]{D_{n-1}}} > \log \frac{\sqrt[n+1]{D_{n+1}}}{\sqrt[n]{D_n}}.$$

By a direct check via computer, the last inequality also holds for $n = 4, \ldots, 26$. Therefore, the sequence $(\sqrt[n+1]{D_{n+1}} / \sqrt[n]{D_n})_{n \geqslant 3}$ is strictly decreasing. This ends the proof. $\square$

4. Proof of Theorem 1.3

Lemma 4.1. *For $x > 0$ we have*

$$\log(1+x) > x - \frac{x^2}{2}. \tag{2}$$

Proof. As

$$\frac{d}{dx}\left(\log(1+x) - x + \frac{x^2}{2}\right) = \frac{x^2}{1+x},$$

we see that $\log(1+x) - x + x^2/2 > \log 1 - 0 + 0^2/2 = 0$ for any $x > 0$. $\quad\square$

Lemma 4.2. *Let $m, n \in \mathbb{Z}^+$ with $n \geqslant 3$. If $m \geqslant 11$ or $n \geqslant 30$, then*

$$H_n^{(m)} \log H_n^{(m)} > 4\left(\frac{2}{n+2}\right)^{m-1}. \tag{3}$$

Proof. Recall that H_n refers to $H_n^{(1)}$. If $n \geqslant 30$, then $H_n \log H_n \geqslant H_{30} \log H_{30} > 4$ and hence (3) holds for $m = 1$.

Below we assume that $m \geqslant 2$. As $n \geqslant 3$, we have $H_n^{(m)} \log H_n^{(m)} \geqslant H_3^{(m)} \log H_3^{(m)}$. So it suffices to show that

$$\left(\frac{n+2}{2}\right)^{m-1} H_3^{(m)} \log H_3^{(m)} > 4 \tag{4}$$

whenever $m \geqslant 11$ or $n \geqslant 30$. By Lemma 4.1,

$$\log H_3^{(m)} = \log(1 + 2^{-m} + 3^{-m}) > 2^{-m} + 3^{-m} - \frac{(2^{-m} + 3^{-m})^2}{2}$$

$$> 2^{-m} + 3^{-m} - \frac{(2^{1-m})^2}{2} = \frac{1}{2^m} + \frac{1}{3^m} - \frac{2}{4^m}.$$

If $m \geqslant 3$, then $(4/3)^m \geqslant (4/3)^3 > 2$ and hence $\log H_3^{(m)} > 1/2^m$. Note also that $H_3^{(2)} \log H_3^{(2)} > 1/4$. So we always have

$$H_3^{(m)} \log H_3^{(m)} > \frac{1}{2^m}.$$

If $m \geqslant 11$, then $1.25^m \geqslant 1.25^{11} > 10$ and hence

$$\frac{1}{2^m} > \frac{4}{2.5^{m-1}} \geqslant \frac{4}{((n+2)/2)^{m-1}},$$

therefore (4) holds. When $n \geqslant 30$, we have

$$\frac{1}{2^m} \geqslant \frac{1}{2^{4m-6}} = \frac{4}{16^{m-1}} \geqslant 4\left(\frac{2}{n+2}\right)^{m-1}$$

and hence (4) also holds. $\quad\square$

Proof of Theorem 1.3. Let $m \geqslant 1$ and $n \geqslant 3$. Set

$$\Delta_n(m) := \log \frac{\sqrt[n+1]{H_{n+1}^{(m)}}}{\sqrt[n]{H_n^{(m)}}} - \log \frac{\sqrt[n+2]{H_{n+2}^{(m)}}}{\sqrt[n+1]{H_{n+1}^{(m)}}} = \frac{2}{n+1} \log H_{n+1}^{(m)} - \frac{\log H_n^{(m)}}{n} - \frac{\log H_{n+2}^{(m)}}{n+2}.$$

It suffices to show that $\Delta_n(m) < 0$. This can be easily verified by computer if $m \in \{1, \ldots, 10\}$ and $n \in \{3, \ldots, 29\}$.

Below we assume that $m \geqslant 11$ or $n \geqslant 30$. Recall (2) and the known fact that $\log(1 + x) < x$ for $x > 0$. We clearly have

$$\log \frac{H_{n+1}^{(m)}}{H_n^{(m)}} = \log\left(1 + \frac{1}{(n+1)^m H_n^{(m)}}\right) < \frac{1}{(n+1)^m H_n^{(m)}}$$

and

$$\log \frac{H_{n+2}^{(m)}}{H_n^{(m)}} > \log\left(1 + \frac{2}{(n+2)^m H_n^{(m)}}\right) > \frac{2}{(n+2)^m H_n^{(m)}} - \frac{2}{(n+2)^{2m}(H_n^{(m)})^2}.$$

It follows that

$$\Delta_n(m) = \left(\frac{2}{n+1} - \frac{1}{n} - \frac{1}{n+2}\right) \log H_n^{(m)}$$

$$+ \frac{2}{n+1} \log \frac{H_{n+1}^{(m)}}{H_n^{(m)}} - \frac{1}{n+2} \log \frac{H_{n+2}^{(m)}}{H_n^{(m)}}$$

$$< \frac{-2\log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2}{(n+1)^{m+1} H_n^{(m)}}$$

$$- \frac{2}{(n+2)^{m+1} H_n^{(m)}} + \frac{2}{(n+2)^{2m+1}(H_n^{(m)})^2}.$$

Since $(n+2)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k}(n+1)^k$ by the binomial theorem, we obtain

$$\Delta_n(m) \leqslant \frac{-2\log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2\sum_{k=0}^m \binom{m+1}{k}(n+1)^k}{(n+1)^{m+1}(n+2)^{m+1} H_n^{(m)}} + \frac{2}{(n+2)^{m+2} H_n^{(m)}}$$

$$< \frac{-2\log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2(n+1)^m \sum_{k=0}^m \binom{m+1}{k}}{(n+1)^{m+1}(n+2)^{m+1} H_n^{(m)}} + \frac{2}{(n+1)(n+2)^{m+1} H_n^{(m)}}$$

$$= \frac{-2\log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2(2^{m+1} - 1) + 2}{(n+1)(n+2)^{m+1} H_n^{(m)}}.$$

Thus

$$n(n+1)(n+2)\Delta_n(m) \frac{H_n^{(m)}}{2} < -H_n^{(m)} \log H_n^{(m)} + \frac{2^{m+1} n}{(n+2)^m}$$

$$< 4\left(\frac{2}{n+2}\right)^{m-1} - H_n^{(m)} \log H_n^{(m)}.$$

Applying (3) we find that $\Delta_n(m) < 0$. This completes the proof.   □

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