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SUPERCONGRUENCES MOTIVATED BY e

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Abstract. In this paper we establish some new supercongruences motivated by the well-known fact $\lim_{n\to\infty}(1+1/n)^n=e$. Let $p>3$ be a prime. We prove that

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1}\equiv 0 \pmod{p^5} \text{ and } \sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1}\equiv \frac{2}{3}p^4B_{p-3} \pmod{p^5},$$

where $B_0,B_1,B_2,\ldots$ are Bernoulli numbers. We also show that for any $a\in\mathbb{Z}$ with $p\nmid a$ we have

$$\sum_{k=1}^{p-1}\frac{1}{k}\left(1+\frac{a}{k}\right)^k\equiv -1 \pmod p \quad\text{and}\quad \sum_{k=1}^{p-1}\frac{1}{k^2}\left(1+\frac{a}{k}\right)^k\equiv 1+\frac{1}{2a} \pmod p.$$

1. Introduction

A p-adic congruence (with p prime) is called a *supercongruence* if it happens to hold modulo higher powers of p. Here is a classical example due to J. Wolstenholme (cf. [W] or [HT]):

$$\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod{p^2} \quad\text{and}\quad \binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$$

for every prime $p>3$. In 1900 Glaisher [G1, G2] showed further that

$$\binom{2p-1}{p-1}\equiv 1-\frac{2}{3}p^3B_{p-3} \pmod{p^4}$$

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for any prime $p>3$, where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers. (See [IR, pp. 228–241] for an introduction to Bernoulli numbers.) The reader may consult [L, Su1, Su3, T] for some other known supercongruences.

In this paper we establish some new supercongruences modulo prime powers motivated by the well-known formula

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Now we state our main results.

Theorem 1.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \equiv 0 \pmod{p^5}. \tag{1.1}$$

Moreover, if $p>5$ then

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \equiv \frac{p^5}{18}B_{p-3} \pmod{p^6}. \tag{1.2}$$

Theorem 1.2. *Let $p>3$ be a prime and let m be an integer not divisible by p. Then we have*

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-1}{k}^m \equiv \frac{(m-1)(7m-5)}{36m^2}p^4B_{p-3} \pmod{p^5}. \tag{1.3}$$

In particular,

$$\sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1} \equiv \frac{2}{3}p^4B_{p-3} \pmod{p^5}. \tag{1.4}$$

Remark 1.1. Note that if p is a prime and m is an integer with $p\nmid m$ then

$$\binom{p/m-1}{k} \equiv \binom{-1}{k} = (-1)^k \not\equiv 0 \pmod{p} \quad \text{for all } k=0,1,\ldots,p-1.$$

(1.1)-(1.4) are interesting since supercongruences modulo p^5 are very rare. We conjecture that there are no composite numbers $p>1$ satisfying (1.1) or the congruence

$$\sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1} \equiv 0 \pmod{p^4}. \tag{1.5}$$

Theorem 1.3. *Let $p>3$ be a prime and let m be an integer not divisible by p.*

(i) *If $p>5$, then*

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^2}\binom{p/m-1}{k}^m\equiv\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{k}\pmod{p^3}.\tag{1.6}$$

Also, for any $n=1,\ldots,(p-3)/2$ we have

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n}}\binom{p/m-1}{k}^m\equiv-\frac{p}{2n+1}B_{p-1-2n}\pmod{p^2}.\tag{1.7}$$

(ii) *For every $n=1,\ldots,(p-3)/2$, we have*

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n-1}}\binom{p/m-1}{k}^m\equiv\left(1+\frac{1-m}{2m}(2n+1)\right)\frac{p^2n}{2n+1}B_{p-1-2n}\pmod{p^3}.\tag{1.8}$$

Remark 1.2. If n is a positive integer and $p>2n+1$ is a prime, then (1.8) with $m=p\pm1$ yields the congruences

$$\sum_{k=1}^{p-1}\frac{1}{k^{2n-1}}\binom{1/(p-1)}{k}^{p-1}\equiv-\frac{2p^2n^2}{2n+1}B_{p-1-2n}\pmod{p^3}\tag{1.9}$$

and

$$\sum_{k=1}^{p-1}\frac{1}{k^{2n-1}}\binom{-1/(p+1)}{k}^{p+1}\equiv\frac{p^2n}{2n+1}B_{p-1-2n}\pmod{p^3}.\tag{1.10}$$

Theorem 1.4. *Let p be an odd prime and let $a\in\mathbb{Z}$ with $p\nmid a$. Then*

$$\sum_{k=1}^{p-1}\frac{1}{k}\left(1+\frac{a}{k}\right)^k\equiv-1\pmod{p}.\tag{1.11}$$

If $p>3$, then

$$\sum_{k=1}^{p-1}\frac{1}{k^2}\left(1+\frac{a}{k}\right)^k\equiv1+\frac{1}{2a}\pmod{p}.\tag{1.12}$$

Remark 1.3. (1.11) with $a=1$ yields the congruence

$$\sum_{k=1}^{p-1}\frac{(k+1)^k}{k^{k+1}}\equiv-1\pmod{p}.\tag{1.13}$$

We will show Theorems 1.1-1.2 in the next section. Theorems 1.3 and 1.4 will be proved in Sections 3 and 4 respectively.

To conclude this section, we pose two related conjectures for further research.

Conjecture 1.1. *Let $m>2$ and $q>0$ be integers with m even or q odd. Then, for any prime $p>mq$ we have the supercongruence*

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-q}{k}^m\equiv 0 \pmod{p^3}. \tag{1.14}$$

Remark 1.4. Clearly (1.14) with $q=1$ follows from (1.3).

For a prime p and a p-adic number x, as usual we let $\nu_p(x)$ denote the p-adic valuation (i.e., p-adic order) of x.

Conjecture 1.2. *Let p be a prime and let n be a positive integer. Then*

$$\nu_p\bigg(\sum_{k=0}^{n-1}\binom{-1/(p+1)}{k}^{p+1}\bigg)\geqslant c_p\left\lfloor\frac{\nu_p(n)+1}{2}\right\rfloor, \tag{1.15}$$

where

$$c_p=\begin{cases}1 & \text{if } p=2,\\ 3 & \text{if } p=3,\\ 5 & \text{if } p\geqslant 5.\end{cases}$$

If $p>3$, then we have

$$\nu_p\bigg(\sum_{k=0}^{n-1}\binom{1/(p-1)}{k}^{p-1}\bigg)\geqslant 4\left\lfloor\frac{\nu_p(n)+1}{2}\right\rfloor. \tag{1.16}$$

2. Proofs of Theorems 1.1 and 1.2

For $m=1,2,3,\ldots$ and $n=0,1,2,\ldots$, we define

$$H_n^{(m)}:=\sum_{0<k\leqslant n}\frac{1}{k^m}$$

and call it a harmonic number of order m. Those $H_n=H_n^{(1)}$ $(n=0,1,2,\ldots)$ are usually called *harmonic numbers*.

Lemma 2.1. *Let $p>3$ be a prime. Then*

$$H_{p-1}\equiv-\frac{p^2}{3}B_{p-3}\pmod{p^3}\quad\text{and}\quad H_{p-1}^{(2)}\equiv\frac{2}{3}pB_{p-3}\pmod{p^2}. \tag{2.1}$$

Also,

$$\sum_{k=1}^{p-1}H_k^{(2)}\equiv p^2B_{p-3}\pmod{p^3},\quad \sum_{k=1}^{p-1}H_k^{(3)}\equiv-\frac{2}{3}pB_{p-3}\pmod{p^2}, \tag{2.2}$$

and

$$\sum_{k=1}^{p-1} H_k^{(4)} \equiv H_{p-1}^{(3)} \equiv 0 \pmod p. \tag{2.3}$$

Proof. The two congruences in (2.1) are known results due to Glaisher [G2], see also Theorem 5.1 and Corollary 5.1 of [S].

For $m = 2, 3, 4$ we have

$$\sum_{k=1}^{p-1} H_k^{(m)} = \sum_{k=1}^{p-1}\sum_{j=1}^{k} \frac{1}{j^m} = \sum_{j=1}^{p-1} \frac{\sum_{k=j}^{p-1} 1}{j^m} = \sum_{j=1}^{p-1} \frac{p-j}{j^m} = pH_{p-1}^{(m)} - H_{p-1}^{(m-1)}.$$

Note also that

$$H_{p-1}^{(3)} = \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k^3} + \frac{1}{(p-k)^3}\right) \equiv 0 \pmod p.$$

Combining these with (2.1), we immediately get (2.2) and (2.3). $\square$

Lemma 2.2. *Let $p > 3$ be a prime. Set*

$$\Sigma_1 = \sum_{k=1}^{p-1} \sum_{1\leqslant i<j\leqslant k} \left(\frac{1}{ij^2} + \frac{1}{i^2 j}\right), \quad \Sigma_2 = \sum_{k=1}^{p-1} \sum_{1\leqslant i<j\leqslant k} \frac{1}{i^2 j^2},$$

$$\Sigma_3 = \sum_{k=1}^{p-1} \sum_{1\leqslant i<j\leqslant k} \left(\frac{1}{ij^3} + \frac{1}{i^3 j}\right) \quad \textit{and} \quad \Sigma_4 = \sum_{k=1}^{p-1} \sum_{1\leqslant i<j\leqslant k} \frac{H_k^{(2)}}{ij}.$$

Then we have

$$\Sigma_1 \equiv pB_{p-3} \pmod{p^2}, \qquad \Sigma_2 \equiv B_{p-3} \pmod p \tag{2.4}$$

and

$$\Sigma_3 \equiv \Sigma_4 \equiv -B_{p-3} \pmod p. \tag{2.5}$$

Proof. With the help of Lemma 2.1,

$$\begin{aligned}
\Sigma_1 =& \sum_{1\leqslant i<j\leqslant p-1} \left(\frac{1}{ij^2} + \frac{1}{i^2 j}\right) \sum_{k=j}^{p-1} 1 \\
=& p \sum_{1\leqslant i<j\leqslant p-1} \left(\frac{1}{ij^2} + \frac{1}{i^2 j}\right) - \sum_{1\leqslant i<j\leqslant p-1} \left(\frac{1}{ij} + \frac{1}{i^2}\right) \\
=& p\left(H_{p-1}H_{p-1}^{(2)} - H_{p-1}^{(3)}\right) - \sum_{1\leqslant i<j\leqslant p-1} \frac{1}{ij} - \sum_{i=1}^{p-1} \frac{p-1-i}{i^2} \\
\equiv& 0 - \frac{1}{2}\left(H_{p-1}^2 - H_{p-1}^{(2)}\right) - (p-1)H_{p-1}^{(2)} + H_{p-1} \equiv pB_{p-3} \pmod{p^2}.
\end{aligned}$$

Recall the congruence $\sum_{k=1}^{p-1} H_k/k^2 \equiv B_{p-3} \pmod p$ (cf. [ST, (5.4)]). Note that

$$\begin{aligned}\Sigma_2 &= \sum_{1\leqslant i<j\leqslant p-1} \frac{1}{i^2j^2}\sum_{k=j}^{p-1} 1 = p\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j^2} - \sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j}\\ &\equiv -\sum_{i=1}^{p-1}\frac{H_{p-1}-H_i}{i^2} \equiv \sum_{k=1}^{p-1}\frac{H_k}{k^2} \equiv B_{p-3} \pmod p\end{aligned}$$

and

$$\begin{aligned}\Sigma_3 &= \sum_{1\leqslant i<j\leqslant p-1}\left(\frac{1}{ij^3}+\frac{1}{i^3j}\right)\sum_{k=j}^{p-1} 1 = \sum_{1\leqslant i<j\leqslant p-1}\left(\frac{1}{ij^3}+\frac{1}{i^3j}\right)(p-j)\\ &\equiv -\sum_{1\leqslant i<j\leqslant p-1}\left(\frac{1}{ij^2}+\frac{1}{i^3}\right) = -\sum_{j=1}^{p-1}\frac{H_j-1/j}{j^2} - \sum_{i=1}^{p-1}\frac{p-1-i}{i^3}\\ &\equiv -\sum_{k=1}^{p-1}\frac{H_k}{k^2} + 2H_{p-1}^{(3)} + H_{p-1}^{(2)} \equiv -B_{p-3} \pmod p.\end{aligned}$$

In light of (2.2),

$$\begin{aligned}\Sigma_4 &= \sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}\left(\sum_{k=1}^{p-1}H_k^{(2)} - \sum_{s=1}^{j-1}H_s^{(2)}\right)\\ &\equiv -\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}\sum_{1\leqslant t\leqslant s<j}\frac{1}{t^2} = -\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}\sum_{t=1}^{j}\frac{j-t}{t^2}\\ &= -\sum_{j=1}^{p-1}H_{j-1}H_j^{(2)} + \sum_{j=1}^{p-1}\frac{H_j}{j}\left(H_j-\frac{1}{j}\right) \pmod{p^2}.\end{aligned}$$

For every $k = 1,\ldots,p-1$, we have

$$H_{p-k} = H_{p-1} - \sum_{0<j<k}\frac{1}{p-j} \equiv H_{k-1} \pmod p$$

and

$$H_{p-k}^{(2)} = H_{p-1}^{(2)} - \sum_{0<j<k}\frac{1}{(p-j)^2} \equiv -H_{k-1}^{(2)} \pmod p.$$

Thus

$$\begin{aligned}\sum_{k=1}^{p-1}H_{k-1}H_k^{(2)} &\equiv \sum_{k=1}^{p-1}H_{p-k}H_k^{(2)} = \sum_{k=1}^{p-1}H_kH_{p-k}^{(2)}\\ &\equiv -\sum_{k=1}^{p-1}H_kH_{k-1}^{(2)} = -\sum_{k=1}^{p-1}H_k\left(H_k^{(2)}-\frac{1}{k^2}\right) \pmod p.\end{aligned}$$

and hence

$$\Sigma_4 \equiv \sum_{k=1}^{p-1} H_k H_k^{(2)} - 2\sum_{k=1}^{p-1}\frac{H_k}{k^2} + \sum_{k=1}^{p-1}\frac{H_k^2}{k} \pmod{p}. \tag{2.6}$$

Since

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k} = \sum_{k=1}^{p-1}\frac{H_{p-k}^2}{p-k} \equiv -\sum_{k=1}^{p-1}\frac{1}{k}\left(H_k - \frac{1}{k}\right)^2 = -\sum_{k=1}^{p-1}\frac{H_k^2}{k} + 2\sum_{k=1}^{p-1}\frac{H_k}{k^2} - H_{p-1}^{(3)} \pmod{p}$$

and $H_{p-1}^{(3)} \equiv 0 \pmod{p}$, we have

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k} \equiv \sum_{k=1}^{p-1}\frac{H_k}{k^2} \equiv B_{p-3} \pmod{p}. \tag{2.7}$$

Combining this with (2.6) and the congruence $\sum_{k=1}^{p-1} H_k H_k^{(2)} \equiv 0 \pmod{p}$ (cf. [Su2, (2.8)]), we obtain $\Sigma_4 \equiv -B_{p-3} \pmod{p}$. This concludes the proof. $\square$

Proof of Theorem 1.1. (1.1) in the case $p = 5$ can be verified directly; in fact,

$$\sum_{k=0}^{5-1}\binom{-1/(5+1)}{k}^{5+1} \equiv 5^5 \pmod{5^6}.$$

Below we assume $p > 5$. As (1.2) implies (1.1), we only need to prove (1.2).

For any p-adic integer x, we can write $(1+px)^m$ as the p-adic series $\sum_{n=0}^{\infty}\binom{m}{n}p^n x^n$. Thus, for each $k = 1, \ldots, p-1$, we have

$$\begin{aligned}
&\binom{-1/(p+1)}{k}^{p+1} = \binom{p/(p+1)-1}{k}^{p+1} = \prod_{j=1}^{k}\left(1 - \frac{p}{(p+1)j}\right)^{p+1}\\
&\equiv \prod_{j=1}^{k}\left(1 - \frac{p}{j} + \binom{p+1}{2}\frac{p^2}{(p+1)^2 j^2} - \binom{p+1}{3}\frac{p^3}{(p+1)^3 j^3} + \binom{p+1}{4}\frac{p^4}{(p+1)^4 j^4}\right)\\
&= \prod_{j=1}^{k}\left(1 - \frac{p}{j} + \frac{p^3}{2(p+1)j^2} - \frac{p^4(p-1)}{6(p+1)^2 j^3} + \frac{p^5(p-1)(p-2)}{24(p+1)^3 j^4}\right)\\
&\equiv \prod_{j=1}^{k}\left(1 - \frac{p}{j} + \frac{p^3(p^2-p+1)}{2j^2} - \frac{p^4}{6j^3}(p-1)(1-2p) + \frac{2p^5}{24j^4}\right)\\
&\equiv \prod_{j=1}^{k}\left(1 - \frac{p}{j} + \frac{p^5-p^4+p^3}{2j^2} + \frac{p^4-3p^5}{6j^3} + \frac{p^5}{12j^4}\right) \pmod{p^6}
\end{aligned}$$

and hence

$$\begin{aligned}&\binom{-1/(p+1)}{k}^{p+1}-\prod_{j=1}^{k}\left(1-\frac{p}{j}\right)\\ \equiv&\frac{p^5-p^4+p^3}{2}H_k^{(2)}+\frac{p^4-3p^5}{6}H_k^{(3)}+\frac{p^5}{12}H_k^{(4)}\\ &+\frac{p(p^4-p^3)}{2}\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right)-\frac{p^5}{6}\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^3}+\frac{1}{i^3j}\right)\\ &+\frac{p^5}{2}\sum_{1\leqslant i_1<i_2\leqslant k}\frac{1}{i_1i_2}\left(\sum_{j=1}^{k}\frac{1}{j^2}-\frac{1}{i_1^2}-\frac{1}{i_2^2}\right)\pmod{p^6}.\end{aligned}$$

Thus, in view of Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}&\sum_{k=1}^{p-1}\binom{-1/(p+1)}{k}^{p+1}-\sum_{k=1}^{p-1}(-1)^k\binom{p-1}{k}\\ \equiv&\frac{p^5-p^4+p^3}{2}p^2B_{p-3}+\frac{p^4-3p^5}{6}\left(-\frac{2}{3}pB_{p-3}\right)+\frac{p^5}{12}\times 0\\ &+\frac{p^5-p^4}{2}pB_{p-3}-\frac{p^5}{6}(-B_{p-3})+\frac{p^5}{2}\left(-B_{p-3}-(-B_{p-3})\right)\\ \equiv&\frac{p^5}{18}B_{p-3}\pmod{p^6}\end{aligned}$$

and hence (1.2) follows since $\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}=(1-1)^{p-1}=0$.

The proof of Theorem 1.1 is now complete. □

Proof of Theorem 1.2. For each $k\in\{1,\ldots,p-1\}$, obviously

$$(-1)^{km}\binom{p/m-1}{k}^m=\prod_{j=1}^{k}\left(1-\frac{p}{jm}\right)^m$$

is congruent to

$$\begin{aligned}&\prod_{j=1}^{k}\left(1-\frac{pm}{jm}+\frac{m(m-1)}{2}\cdot\frac{p^2}{j^2m^2}-\binom{m}{3}\frac{p^3}{j^3m^3}+\binom{m}{4}\frac{p^4}{j^4m^4}\right)\\ =&\prod_{j=1}^{k}\left(1-\frac{p}{j}+\frac{m-1}{2m}\cdot\frac{p^2}{j^2}-\frac{(m-1)(m-2)}{6m^2}\cdot\frac{p^3}{j^3}+\frac{(m-1)(m-2)(m-3)}{24m^3}\cdot\frac{p^4}{j^4}\right)\end{aligned}$$

modulo p^5, and hence

$$\begin{aligned}
&(-1)^{km}\binom{p/m-1}{k}^m-\prod_{j=1}^k\left(1-\frac{p}{j}\right)\\
\equiv&\frac{m-1}{2m}p^2H_k^{(2)}-\frac{(m-1)(m-2)}{6m^2}p^3H_k^{(3)}+\frac{(m-1)(m-2)(m-3)}{24m^3}p^4H_k^{(4)}\\
&-\frac{m-1}{2m}p^3\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right)+\frac{(m-1)^2}{4m^2}p^4\sum_{1\leqslant i<j\leqslant k}\frac{1}{i^2j^2}\\
&+\frac{(m-1)(m-2)}{6m^2}p^4\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^3}+\frac{1}{i^3j}\right)\\
&+\frac{m-1}{2m}p^4\sum_{1\leqslant i_1<i_2\leqslant k}\frac{1}{i_1i_2}\sum_{\substack{j=1\\ j\neq i_1,i_2}}^k\frac{1}{j^2}\pmod{p^5}.
\end{aligned}$$

Therefore, applying (2.2) and (2.3) we get

$$\begin{aligned}
&\sum_{k=1}^{p-1}(-1)^{km}\binom{p/m-1}{k}^m-\sum_{k=1}^{p-1}(-1)^k\binom{p-1}{k}\\
\equiv&\frac{m-1}{2m}p^2(p^2B_{p-3})-\frac{(m-1)(m-2)}{6m^2}p^3\left(-\frac{2}{3}pB_{p-3}\right)\\
&-\frac{m-1}{2m}p^3\Sigma_1+\frac{(m-1)^2}{4m^2}p^4\Sigma_2\\
&+\frac{(m-1)(m-2)}{6m^2}p^4\Sigma_3+\frac{m-1}{2m}p^4(\Sigma_4-\Sigma_3)\pmod{p^5},
\end{aligned}$$

where $\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4$ are defined in Lemma 2.2. Combining this with Lemma 2.2, we finally obtain

$$\begin{aligned}
&\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-1}{k}^m-(1-1)^{p-1}\\
\equiv&\frac{m-1}{2m}p^4B_{p-3}+\frac{(m-1)(m-2)}{9m^2}p^4B_{p-3}\\
&-\frac{m-1}{2m}p^3(pB_{p-3})+\frac{(m-1)^2}{4m^2}p^4B_{p-3}+\frac{(m-1)(m-2)}{6m^2}p^4(-B_{p-3})\\
=&\left(\frac{(m-1)^2}{4m^2}-\frac{(m-1)(m-2)}{18m^2}\right)p^4B_{p-3}\pmod{p^5},
\end{aligned}$$

which gives (1.3). Putting $m=p-1$ in (1.3) we immediately get (1.4). This concludes the proof. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let m and n be positive integers with $m \leqslant 2n$, and let $p > 2n+1$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{H_k^{(m)}}{k^{2n+1-m}} \equiv \frac{(-1)^{m-1}}{2n+1}\binom{2n+1}{m} B_{p-1-2n} \pmod{p}. \tag{3.1}$$

When $m < 2n$ we have

$$\sum_{k=1}^{p-1} \frac{H_k^{(m)}}{k^{2n-m}} \equiv \frac{pB_{p-1-2n}}{2n+1}\left(n + (-1)^m \frac{n-m}{m+1}\binom{2n+1}{m}\right) \pmod{p^2}. \tag{3.2}$$

Proof. Since $\sum_{k=1}^{p-1} k^s \equiv 0 \pmod{p}$ for any integer $s \not\equiv 0 \pmod{p-1}$ (see, e.g., [IR, p. 235]), we have $\sum_{k=1}^{p-1} 1/k^{2n+1} \equiv 0 \pmod{p}$. Hence

$$\begin{aligned}
&\sum_{k=1}^{p-1} \frac{H_k^{(m)}}{k^{2n+1-m}} \\
\equiv &\sum_{k=1}^{p-1} \frac{1}{k^{2n+1}} + \sum_{k=1}^{p-1} \frac{1}{k^{2n+1-m}} \sum_{0<j<k} j^{p-1-m} \\
\equiv &\sum_{k=1}^{p-1} \frac{1}{k^{2n+1-m}(p-m)} \sum_{i=0}^{p-1-m} \binom{p-m}{i} B_i k^{p-m-i} \qquad \text{(by [IR, p. 230])} \\
\equiv &-\frac{1}{m} \sum_{i=0}^{p-1-m} \binom{p-m}{i} B_i \sum_{k=1}^{p-1} k^{p-1-2n-i} \\
\equiv &\frac{1}{m} \sum_{\substack{0 \leqslant i \leqslant p-1-m \\ p-1 \mid 2n+i}} \binom{p-m}{i} B_i = \frac{1}{m}\binom{p-m}{2n+1-m} B_{p-1-2n} \\
\equiv &\frac{1}{m}\binom{-m}{2n+1-m} B_{p-1-2n} = \frac{(-1)^{m-1}}{2n+1}\binom{2n+1}{m} B_{p-1-2n} \pmod{p}.
\end{aligned}$$

This proves (3.1).

Now assume that $m < 2n$. As $m, 2n-m \in \{1, \ldots, p-2\}$, we have

$$\sum_{j=1}^{p-1} \frac{1}{j^m} \equiv \sum_{k=1}^{p-1} \frac{1}{k^{2n-m}} \equiv 0 \pmod{p}.$$

It is known that

$$\sum_{k=1}^{p-1} \frac{1}{k^s} \equiv \frac{ps}{s+1} B_{p-1-s} \pmod{p^2} \quad \text{for each } s = 1, \ldots, p-2. \tag{3.3}$$

(See, e.g., [G2] or [S, Corollary 5.1].) Thus

$$\sum_{j=1}^{p-1}\frac{1}{j^m}\sum_{k=1}^{p-1}\frac{1}{k^{2n-m}}+\sum_{k=1}^{p-1}\frac{1}{k^{2n}}\equiv\sum_{k=1}^{p-1}\frac{1}{k^{2n}}\equiv\frac{2n}{2n+1}pB_{p-1-2n}\pmod{p^2}.$$

On the other hand,

$$\begin{aligned}&\sum_{j=1}^{p-1}\frac{1}{j^m}\sum_{k=1}^{p-1}\frac{1}{k^{2n-m}}+\sum_{k=1}^{p-1}\frac{1}{k^{2n}}-\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{j^mk^{2n-m}}\\ =&\sum_{1\leqslant k\leqslant j\leqslant p-1}\frac{1}{j^mk^{2n-m}}=\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{(p-j)^m(p-k)^{2n-m}}\\ =&\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{(p+j)^m(p+k)^{2n-m}}{(p^2-j^2)^m(p^2-k^2)^{2n-m}}\\ \equiv&\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{(j^m+pmj^{m-1})(k^{2n-m}+p(2n-m)k^{2n-m-1})}{j^{2m}k^{2(2n-m)}}\\ \equiv&\sum_{1\leqslant j\leqslant k\leqslant p-1}\left(\frac{1}{j^mk^{2n-m}}+\frac{pm}{j^{m+1}k^{2n-m}}+\frac{p(2n-m)}{j^mk^{2n-m+1}}\right)\pmod{p^2}.\end{aligned}$$

Therefore, with the help of (3.1) we have

$$\begin{aligned}&\frac{2n}{2n+1}pB_{p-1-2n}-2\sum_{k=1}^{p-1}\frac{H_k^{(m)}}{k^{2n-m}}\\ \equiv&pm\sum_{k=1}^{p-1}\frac{H_k^{(m+1)}}{k^{2n-m}}+p(2n-m)\sum_{k=1}^{p-1}\frac{H_k^{(m)}}{k^{2n-m+1}}\\ \equiv&pm\frac{(-1)^m}{2n+1}\binom{2n+1}{m+1}B_{p-1-2n}+p(2n-m)\frac{(-1)^{m-1}}{2n+1}\binom{2n+1}{m}B_{p-1-2n}\\ =&(-1)^m\frac{2(m-n)}{m+1}\binom{2n+1}{m}\frac{pB_{p-1-2n}}{2n+1}\pmod{p^2}\end{aligned}$$

and hence (3.2) holds. □

Remark 3.1. By [ST, (5.4)], $\sum_{k=1}^{p-1}H_k/k^2\equiv B_{p-3}\pmod p$ for any prime $p>3$. By [M, (5)], $\sum_{k=1}^{p-1}H_k/k^3\equiv -pB_{p-5}/10\pmod{p^2}$ for any prime $p>5$. Obviously these two results are particular cases of Lemma 3.1.

Lemma 3.2. *Let $p>5$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{1-pH_k}{k^2}\equiv\frac{H_{p-1}}{p}\pmod{p^3}.\tag{3.4}$$

Proof. In view of Theorem 5.1(a) and Remark 5.1 of [S],

$$\frac{H_{p-1}^{(2)}}{2} \equiv p\left(\frac{B_{2p-4}}{2p-4} - 2\frac{B_{p-3}}{p-3}\right) \equiv -\frac{H_{p-1}}{p} \pmod{p^3}$$

and $H_{p-1}^{(3)} \equiv 0 \pmod{p^2}$. Also,

$$\sum_{k=1}^{p-1} \frac{H_{k-1}}{k^2} = \sum_{1\leqslant j<k\leqslant p-1} \frac{1}{jk^2} \equiv -3\frac{H_{p-1}}{p^2} \pmod{p^2}$$

by [T, Theorem 2.3]. So we have

$$\sum_{k=1}^{p-1} \frac{1-pH_k}{k^2} = H_{p-1}^{(2)} - pH_{p-1}^{(3)} - p\sum_{k=1}^{p-1}\frac{H_{k-1}}{k^2} \equiv \frac{H_{p-1}}{p} \pmod{p^3}.$$

This concludes the proof. $\square$

Proof of Theorem 1.3. Let $k \in \{1, \ldots, p-1\}$. Then

$$\begin{aligned}(-1)^{km}\binom{p/m-1}{k}^m =& \prod_{j=1}^{k}\left(1-\frac{p}{jm}\right)^m \\ \equiv& \prod_{j=1}^{k}\left(1-\frac{p}{j}+\frac{m(m-1)}{2}\cdot\frac{p^2}{j^2m^2}\right) \\ \equiv& 1-pH_k+\frac{m-1}{2m}p^2H_k^{(2)}+p^2\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij} \pmod{p^3}.\end{aligned}$$

Thus, for every $r = 1, \ldots, p-3$ we have

$$\begin{aligned}&\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^r}\binom{p/m-1}{k}^m \\ \equiv&\sum_{k=1}^{p-1}\frac{1-pH_k}{k^r}+\frac{m-1}{2m}p^2\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^r}+\frac{p^2}{2}\sum_{k=1}^{p-1}\frac{H_k^2-H_k^{(2)}}{k^r} \pmod{p^3}.\end{aligned} \tag{3.5}$$

(i) Now suppose that $n \in \{1, \ldots, (p-3)/2\}$. Then (3.5) with $r = 2n$ yields

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n}}\binom{p/m-1}{k}^m \equiv \sum_{k=1}^{p-1}\frac{1-pH_k}{k^{2n}} \pmod{p^2}. \tag{3.6}$$

By (3.3) and Lemma 3.1,

$$\sum_{k=1}^{p-1}\frac{1-pH_k}{k^{2n}}\equiv\left(\frac{2n}{2n+1}-1\right)pB_{p-1-2n}=-\frac{pB_{p-1-2n}}{2n+1}\pmod{p^2}.$$

So (1.7) follows from (3.6). If $n<(p-3)/2$, then

$$2\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^{2n}}\equiv\sum_{k=1}^{p-1}\left(\frac{H_k^{(2)}}{k^{2n}}+\frac{H_{p-k}^{(2)}}{(p-k)^{2n}}\right)\equiv\sum_{k=1}^{p-1}\frac{1/k^2}{k^{2n}}\equiv 0\pmod p,$$

and hence by (3.5) with $r=2n$ we obtain

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n}}\binom{p/m-1}{k}^m\equiv\sum_{k=1}^{p-1}\frac{1-pH_k}{k^{2n}}+\frac{p^2}{2}\sum_{k=1}^{p-1}\frac{H_k^2}{k^{2n}}\ (\mathrm{mod}\ p^3). \tag{3.7}$$

When $p>5$, (3.7) in the case $n=1$, together with (3.4) and the subtle congruence

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^2}\equiv 0\ (\mathrm{mod}\ p)$$

of Sun [Su2, (1.5)], yields (1.6).

(ii) Fix $n\in\{1,\ldots,(p-3)/2\}$. Putting $r=2n-1$ in (3.5) we get

$$\begin{aligned}&\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n-1}}\binom{p/m-1}{k}^m\\ \equiv&\sum_{k=1}^{p-1}\frac{1-pH_k}{k^{2n-1}}+\frac{m-1}{2m}p^2\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^{2n-1}}+\frac{p^2}{2}\sum_{k=1}^{p-1}\frac{H_k^2-H_k^{(2)}}{k^{2n-1}}\ (\mathrm{mod}\ p^3).\end{aligned} \tag{3.8}$$

In view of a known result (cf. [G2] or [S, Theorem 5.1(a)]),

$$\sum_{k=1}^{p-1}\frac{1}{k^{2n-1}}\equiv\frac{n-2n^2}{2n+1}p^2B_{p-1-2n}\pmod{p^3}.$$

By Lemma 3.1,

$$\sum_{k=1}^{p-1}\frac{H_k}{k^{2n-1}}\equiv\frac{1+3n-2n^2}{2(2n+1)}pB_{p-1-2n}\pmod{p^2}$$

and

$$\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^{2n-1}}\equiv -nB_{p-1-2n}\pmod p.$$

Note also that

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^{2n-1}}=\sum_{k=1}^{p-1}\frac{H_{p-k}^2}{(p-k)^{2n-1}}\equiv-\sum_{k=1}^{p-1}\frac{(H_k-1/k)^2}{k^{2n-1}}\pmod p$$

and hence

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^{2n-1}}\equiv\sum_{k=1}^{p-1}\frac{H_k}{k^{2n}}-\frac{1}{2}\sum_{k=1}^{p-1}\frac{1}{k^{2n+1}}\equiv\sum_{k=1}^{p-1}\frac{H_k}{k^{2n}}\equiv B_{p-1-2n}\pmod p$$

with the help of (3.1) in the case $m=1$. Combining all these we obtain (1.8) from (3.8).

The proof of Theorem 1.3 is now complete. □

4. Proof of Theorem 1.4

Lemma 4.1. *Let p be an odd prime. Then, for any positive integers d and r with $d+r<p$, we have*

$$\sum_{k=r}^{p-1}\frac{\binom{k}{r}}{k^{r+d}}\equiv 0\pmod p). \tag{4.1}$$

Proof. Observe that

$$\begin{aligned}\sum_{k=r}^{p-1}\binom{k}{r}k^{-r-d}=&\sum_{s=0}^{p-1-r}\binom{r+s}{s}(r+s)^{-r-d}=\sum_{s=0}^{p-1-r}(-1)^s\binom{-r-1}{s}(r+s)^{-r-d}\\ \equiv&\sum_{s=0}^{p-1-r}\binom{p-1-r}{s}(-1)^s(s-1-(p-1-r))^{p-1-r-d}\\ =&\sum_{k=0}^{p-1-r}\binom{p-1-r}{k}(-1)^{p-1-r-k}(-1-k)^{p-1-r-d}\\ =&(-1)^d\sum_{k=0}^{p-1-r}\binom{p-1-r}{k}(-1)^k(k+1)^{p-1-r-d}\pmod p).\end{aligned}$$

It is known that for any positive integer n we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^m=0\quad\text{for all } m=0,1,\dots,n-1.$$

(See, e,g., [vLW, pp. 125-126].) Therefore, (4.1) follows from the above. □

Proof of Theorem 1.4. Let $d\in\{1,2\}$. With the help of Lemma 4.1, we have

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{1}{k^d}\left(1+\frac{a}{k}\right)^k=&\sum_{k=1}^{p-1}\frac{(k+a)^k}{k^{k+d}}\\ =&\sum_{k=1}^{p-1}\frac{1}{k^{k+d}}\left(k^k+\sum_{r=1}^{k}\binom{k}{r}k^{k-r}a^r\right)\\ =&\sum_{k=1}^{p-1}\frac{1}{k^d}+\sum_{r=1}^{p-1}a^r\sum_{k=r}^{p-1}\frac{\binom{k}{r}}{k^{r+d}}\\ \equiv&\sum_{k=1}^{p-1}\frac{1}{k^d}+\sum_{r=p-d}^{p-1}a^r\sum_{k=r}^{p-1}\frac{\binom{k}{r}}{k^{r+d}}\pmod p.\end{aligned}$$

Thus,

$$\sum_{k=1}^{p-1}\frac{1}{k}\left(1+\frac{a}{k}\right)^k\equiv\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k}+\frac{1}{p-k}\right)+\frac{a^{p-1}}{(p-1)^p}\equiv-1\pmod p$$

in view of Fermat's little theorem. When $p>3$, we have $\sum_{k=1}^{p-1}1/k^2\equiv0\pmod p$ by [W], and hence

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{1}{k^2}\left(1+\frac{a}{k}\right)^k\equiv&\sum_{k=1}^{p-1}\frac{1}{k^2}+\frac{a^{p-1}}{(p-1)^{p+1}}+a^{p-2}\left(\frac{\binom{p-2}{p-2}}{(p-2)^p}+\frac{\binom{p-1}{p-2}}{(p-1)^p}\right)\\ \equiv&0+1+\frac{1}{a}\left(\frac{1}{-2}+1\right)=1+\frac{1}{2a}\pmod p\end{aligned}$$

as desired. This concludes the proof. □

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